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Weak Hardy space and endpoint estimates for singular integrals on space of homogeneous type

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Abstract

We develop the theory of weak Hardy spaces $H^{1,\infty}$ on space of homogeneous type. As some applications, we show that certain singular integral operators and fractional integral operators are bounded from $H^{1,\infty}$ to $L^{1,\infty}$ and $L^{\frac{1}{1-\alpha},\infty}$, respectively. We give also the endpoint estimates for Nagel and Stein's singular integrals studied in [10].

Key Words: Weak Hardy spaces, singular integral, fractional integral, endpoint estimate.

1. Introduction and the main nesults

The theory of weak Hardy spaces on $\mathbb{R}^n$ was first studied in [3] as the special Hardy-Lorentz spaces, which are the intermediate spaces between two Hardy spaces. The atomic decomposition characterization of $H^{1,\infty}(\mathbb{R}^n)$ was given by R. Fefferman and Soria [4]. In 1991, Liu established weak H^p spaces on Homogeneous groups [9]. Recently, Ding and Lan [2] studied weak anisotropic Hardy spaces.

The theory of weak Hardy spaces is very important in Harmonic Analysis since it can sharpen the endpoint weak type estimate for variant important operators (see, for example, [4]). Recently, Nagel and Stein [10] studied certain singular integral operators on an unbounded model polynomial domains, which were applied to some problems in several complex variables (see [11]). Motivated by considering the endpoint weak type estimate for Nagel and Stein's singular integrals, in this paper, we want to develop the weak Hardy space $H^{1,\infty}$ on general space of homogeneous type satisfying certain reverse doubling condition. Our theory is so general that it can be applied to variant different settings such as Euclidean spaces with A_∞-weights, Ahlfors n-regular metric measure spaces, Lie groups of polynomial growth and Carnot-Carathéodory spaces with doubling measure (see [7]). We remark that the corresponding Hardy spaces in this setting were studied in [5, 6, 7].

First we recall the notions of spaces of homogeneous type in the sense of Coifman and Weiss [1].

Definition 1.1 *Let $(\mathcal{X}, d)$ be a metric space with a regular Borel measure μ such that all balls defined by d*

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have finite and positive measures. The quasi metric satisfies the triangle inequality

$$d(x,z) \le \tau(d(x,y)+d(y,z)). \tag{1.1}$$

For any $x \in \mathcal{X}$ *and* $r>0$, *set* $B(x,r)=\{y \in \mathcal{X} : d(x,y)<r\}$. (X,d,μ) *is called a space of homogeneous type if there exists a constant* $C_1 \ge 1$ *such that for all* $x \in \mathcal{X}$ *and* $r>0$,

$$\mu(B(x,2r)) \le C_1 \mu(B(x,r)). \tag{1.2}$$

We also assume that μ has the following *reverse doubling condition*: there exists $C>1$ such that for all $x \in \mathcal{X}$ and $r>0$

$$\mu(B(x,2r)) \ge C\mu(B(x,r)). \tag{1.3}$$

It can be shown from (1.2) and (1.3) that there exist constants $1<d\le D<\infty$ such that for all $x \in \mathcal{X}$ and $s>1$

$$s^d \mu(B(x,r)) \le \mu(B(x,sr)) \le s^D \mu(B(x,r)). \tag{1.4}$$

Denote

$$V(x,y)=\mu(B(x,d(x,y))).$$

It is easy to see

$$V(x,y) \approx V(y,x). \tag{1.5}$$

Now, let us recall some definitions. The first one is $(\epsilon_1,\epsilon_2,\epsilon_3)$-approximately of the identity (in short, $(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI), which was used to define Hardy spaces in [5].

Definition 1.2 ($(\epsilon_1,\epsilon_2,\epsilon_3)$-AOTI) *Let* $\epsilon_1 \in (0,1], \epsilon_2>0$ *and* $\epsilon_3>0$. *A sequence* $\{S_k\}_{k\in\mathbb{Z}}$ *of bounded linear integral operators on* $L^2(\mathcal{X})$ *is said to be an approximation of the identity of order* $(\epsilon_1,\epsilon_2,\epsilon_3)$ *(in short,* $(\epsilon_1,\epsilon_2,\epsilon_3)$*-AOTI), if there exists a constant* $C_4>0$ *such that for all* $k \in \mathbb{Z}$ *and all* x,x',y *and* $y' \in \mathcal{X}$, $S_k(x,y)$, *the integral kernel of* S_k *is a function from* $\mathcal{X}\times\mathcal{X}$ *into* $\mathbb{C}$ *satisfying*

(i) $|S_k(x,y)| \le C_4 \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$;

(ii) $|S_k(x,y)-S_k(x',y)| \le C_4 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_2}}{(2^{-k}+d(x,y))^{\epsilon_2}}$
for $d(x,x') \le (2^{-k}+d(x,y))/2$;

(iii) *Property* (ii) *holds with* x *and* y *interchanged;*

(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]| \le C_4 \frac{d(x,x')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}} \frac{d(y,y')^{\epsilon_1}}{(2^{-k}+d(x,y))^{\epsilon_1}}$
$\times \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon_3}}{(2^{-k}+d(x,y))^{\epsilon_3}}$ *for* $\max\{d(x,x'),d(y,y')\} \le (2^{-k}+d(x,y))/3$;

(v) $\int_{\mathcal{X}} S_k(x,y)dy = \int_{\mathcal{X}} S_k(x,y)dx = 1$.

Here and in the sequel, we write dx instead of $d\mu(x)$ for simplicity.

Definition 1.3 (Test function) *Let $x \in \mathcal{X}, r \in (0,\infty), \beta \in (0,1]$ and $\gamma \in (0,\infty)$. A function φ on $\mathcal{X}$ is said to be a test function of type (x_1, r, β, γ) if*

(i) $|\varphi(x)| \le C \frac{1}{\mu(B(x, r+d(x,x_1)))} \left(\frac{r}{r+d(x_1,x)}\right)^\gamma$ *for all $x \in \mathcal{X}$;*

(ii) $|\varphi(x) - \varphi(y)| \le C \left(\frac{d(x,y)}{r+d(x_1,x)}\right)^\beta \frac{1}{\mu(B(x,r+d(x,x_1)))} \left(\frac{r}{r+d(x_1,x)}\right)^\gamma$ *for all $x, y \in \mathcal{X}$ satisfying $d(x,y) \le (r + d(x_1,x))/2$.*

We denote by $\mathcal{G}(x_1, r, \beta, \gamma)$ the set of all test functions of type (x_1, r, β, γ). If $\varphi \in \mathcal{G}(x_1, r, \beta, \gamma)$ we define its norm by $\|\varphi\|_{\mathcal{G}(x_1,r,\beta,\gamma)} := \inf\{C :$ (i) and (ii) hold$\}$. The space $\mathcal{G}(x_1, r, \beta, \gamma)$ is called the space of test functions.

To give the definition of weak Hardy space $H^{1,\infty}(\mathcal{X})$, we recall the following definitions of maximal functions. Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$ and $0 < \epsilon < \min\{\epsilon_1, \epsilon_2\}$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI and

$$S_k(f)(x) = \int_{\mathcal{X}} S_k(x,y) f(y) dy.$$

For $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ and $\beta, \gamma \in (0,\epsilon)$, the *non-tangential maximal operator* $\mathcal{M}_\sigma$ is defined by

$$\mathcal{M}_\sigma(f)(x) := \sup_{k\in\mathbb{Z}} \sup_{d(x,y)\le\sigma 2^{-k}} |S_k(f)(y)|.$$

The *radial maximal operator* $\mathcal{M}_0$ is defined by

$$\mathcal{M}_0 f(x) := \sup_{k\in\mathbb{Z}} |S_k(f)(x)|.$$

The *grand maximal operator* $\mathcal{M}_g$ is defined by

$$\mathcal{M}_g f(x) := \sup\left\{|\langle f, \varphi\rangle| : \varphi \in \mathcal{G}_0^\epsilon(\beta,\gamma), \|\varphi\|_{\mathcal{G}(x,r,\beta,\gamma)} \le 1 \text{ for some } r > 0\right\}.$$

Definition 1.4 *Let $\epsilon_1 \in (0,1]$, $\epsilon_2 > 0$, $\epsilon_3 > 0$, $\epsilon \in (0, \min\{\epsilon_1, \epsilon_2\})$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an $(\epsilon_1, \epsilon_2, \epsilon_3)$-AOTI. Let $p \in (0,\infty]$, $\sigma \in (0,\infty)$ and $f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))'$ with some $\beta, \gamma \in (0,\epsilon)$. The weak Hardy spaces $H^{1,\infty}$ is defined by*

$$H^{1,\infty}(\mathcal{X}) = \{f \in (\mathcal{G}_0^\epsilon(\beta,\gamma))' : \mathcal{M}_\sigma f \in L^{1,\infty}(\mathcal{X})\}.$$

The $H^{1,\infty}$ norm of f is defined by

$$\|f\|_{H^{1,\infty}(\mathcal{X})} := \|\mathcal{M}_\sigma f\|_{L^{1,\infty}(\mathcal{X})}.$$

Remark 1.1 It has been proved in [5, 6, 7] that the $H^p(\mathcal{X})$ can equivalently be defined via Littlewood-Paley functions related to sub-Laplacians, or non-tangetial maximal functions and dyadic maximal functions. By interpolation theory, we can replace $\mathcal{M}_\alpha$ in the definition of $H^{1,\infty}$ by other maximal functions or Littlewood-Paley functions as above.

Remark 1.2 Let $\mathcal{M}$ denote the centered Hardy-Littlewood maximal operator on $\mathcal{X}$ defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y)dy \quad \text{for } f \in L^1_{loc}(\mathcal{X}).$$

$\mathcal{M}$ is proved to be weak type $(1,1)$ in [1]. Since $H^{1,\infty}(\mathcal{X})$ can be characterized by radial maximal function and $\mathcal{M}_0(f)(x) \lesssim \mathcal{M}(f)(x)$, we have the inclusion relationship

$$L^1(\mathcal{X}) \subset H^{1,\infty}(\mathcal{X}) \quad \text{and} \quad \|f\|_{H^{1,\infty}} \lesssim \|f\|_{L^1}. \tag{1.6}$$

The first result in the paper is the following theorem.

Theorem 1.1 *Given $f \in H^{1,\infty}$, there exists a sequence of bounded functions $\{f_k\}_{k=-\infty}^{\infty}$ with the following properties:*

(a) *$f - \sum_{|k|\le N} f_k \to 0$ in the sense of distributions and $|f_k| \le C2^k$.*

(b) *Each f_k may be further decomposed as $f_k = \sum_{l=1}^{\infty} h_{kl}$ in L^1, where the h_{kl} satisfies:*

(i) *h_{kl} is supported in a ball B_{kl} with $\{B_{kl}\}$ having bounded overlap for each k.*

(ii) $\int_{Q_{kl}} h_{kl} dx = 0.$

(iii) *$\|h_{kl}\|_{L^\infty} \le C2^k$ and $\sum_l \mu(B_{kl}) \le C_1 2^{-k}$. Moreover, C_1 is (up to an absolute constant) less than the $H^{1,\infty}$ norm of f.*

Conversely, if f is a distribution satisfying (a) *and* (i)-(iii) *in* (b)*, then $f \in H^{1,\infty}$ and $\|f\|_{H^{1,\infty}} \le cC_1$ (where c is some absolute constant).*

Using this atom decomposition characterizations of $H^{1,\infty}$, we can prove the endpoint weak type estimate for certain singular integrals, which generalizes the result in [4].

Theorem 1.2 *Suppose that $Tf(x) = \text{p.v.} \int_{\mathcal{X}} K(x,y)f(y)dy$ is a bounded operator on $L^2(\mathcal{X})$ with its kernel K satisfies Dini's condition, $\int_0^1 (\Gamma(\delta)/\delta)d\delta < \infty$, where*

$$\Gamma(\delta) = \sup_{d(y,z)\neq 0} \int_{d(x,y)>\delta^{-1}d(y,z)} |K(x,y) - K(x,z)|dx.$$

Then for $f \in H^{1,\infty}(\mathcal{X})$,

$$|\{x \in \mathcal{X} : |Tf(x)| > \alpha\}| \le C\|f\|_{H^{1,\infty}(\mathcal{X})}/\alpha.$$

In particular, T is of weak type $(1,1)$ *by* (1.6).

For the fractional integrals T_α with α between 0 and 1, we have the following conclusion:

Theorem 1.3 *Suppose that* $T_\alpha f(x)=\int_{\mathcal X}K_\alpha(x,y)f(y)dy$ *is a bounded operator from* $L^{p_0}(\mathcal X)$ *to* $L^{q_0}(\mathcal X)$ *for some* $1<p_0<q_0<\infty$ *satisfying* $\frac{1}{p_0}-\frac{1}{q_0}=\alpha$ *and* $0<\alpha<1$. *If* K_α *satisfies the following regularity in the second variable: there exists constants* C, $\epsilon>0$ *such that for all* $x,y,y'\in\mathcal X$ *with* $d(y,y')\le d(x,y)/2$ *and* $x\neq y$,

$$|K_\alpha(x,y)-K_\alpha(x,y')|\le C\frac{d(y,y')^\epsilon}{V(x,y)^{1-\alpha}d(x,y)^\epsilon}. \tag{1.7}$$

Then for $0<\alpha<\frac{\epsilon}{D}$, T_α *is bounded from* $H^{1,\infty}(\mathcal X)$ *to* $L^{\frac{1}{1-\alpha},\infty}(\mathcal X)$. *Moreover, there exists a constant* C *such that for each* $\lambda>0$,

$$\mu(\{x:|T_\alpha f(x)|>\lambda\})\le C\left(\frac{\|f\|_{H^{1,\infty}(\mathcal X)}}{\lambda}\right)^{\frac{1}{1-\alpha}}.$$

In particular, T_α *is of weak type* $(1,\frac{1}{1-\alpha})$ *by* (1.6).

We also get the following $H^1(\mathcal X)\to L^{\frac{1}{1-\alpha}}(\mathcal X)$ estimate for T_α.

Theorem 1.4 *Under the same conditions of Theorem 1.3, then*

(1) *there exists a constant* $C>0$ *such that for all* $(1,q_0)$*-atom* a, $\|T_\alpha a\|_{L^{\frac{1}{1-\alpha}}(\mathcal X)}\le C$;

(2) *if the kernel* K_α *satisfies regularity condition like* (1.7) *in the first variable, then* T_α *is bounded form* $H^1(\mathcal X)$ *to* $L^{1/(1-\alpha)}(\mathcal X)$.

Remark 1.3 If (1.7) holds for $d(y,y')\le d(x,y)/c$ with some $c>1$, then the conclusions of Theorems 1.3 and 1.4 remain true.

Finally, we give an application of Theorem 1.2. In [10], Nagel and Stein considered a class of singular integral operator $\widetilde T$ on an unbounded model polynomial domain M, which initially is given as a map from $C_0^\infty(M)$ to $C^\infty(M)$. The distribution kernel $\widetilde K(x,y)$ of $\widetilde T$ coincides with a C^∞ function away from the diagonal of $M\times M$, and the following four properties are supposed to hold:

(a) If φ, $\psi\in C_0^\infty(M)$ have disjoint supports, then

$$\langle\widetilde T\varphi,\psi\rangle\ =\ \int_{M\times M}\widetilde K(x,y)\,\varphi(y)\,\psi(x)\,dx\,dy.$$

(b) If φ is a normalized bump function associated to a ball of radius r, then $|\partial_X^a\widetilde T\varphi|\lesssim r^{-a}$. More precisely, for each integer $a\ge0$, there is another integer $b\ge0$ and a constant $M_{a,b}$ so that whenever φ is a C^∞ function supported in a ball $B(x_0,r)$, then

$$\sup_{x\in M}r^a|(\partial_X^a\widetilde T\varphi)(x)|\le M_{a,b}\sup_{c\le b}\sup_{x\in B(x_0,r)}r^c\,|\partial_X^c(\varphi)|.$$

(c) If $x \neq y$, then for every $a \geq 0$

$$|\partial^a_{X,Y} \widetilde{K}(x,y)| \lesssim d(x,y)^{-a} V(x,y)^{-1}, \tag{1.8}$$

where d denotes the quasi metric on M and $V(x,y)$ denotes the measure of ball $B(x, d(x,y))$.

(d) Properties (a) through (c) also hold with x and y interchanged. That is, these properties also hold for the adjoint operator $\widetilde{T}^t$ defined by

$$\langle \widetilde{T}^t \varphi, \psi \rangle = \langle \widetilde{T} \psi, \varphi \rangle.$$

Note that the measure on M is just the Lebesgue measure on $\mathbb{C} \times \mathbb{R}$ and the properties (1.2), (1.3) hold in this setting (see [10, Section 2.1]). The smoothness condition (1.8) guarantees the required Dini's condition in Theorem 1.2, so we have this corollary:

Corollary 1.1 *The Nagel and Stein's singular integral operator* $\widetilde{T}$ *is bounded from* $H^{1,\infty}(M)$ *to* $L^{1,\infty}(M)$*; in particular, it is of weak-type* $(1,1)$.

2. Proofs of theorems

In this section, we will give the proofs of Theorems 1.1–1.4.

Proof of Theorem 1.1 For k an integer we set $\Omega_k = \{x \in \mathcal{X} : \mathcal{M}_g f(x) > 2^k\}$. Let B_{kj} be the Whitney decomposition of Ω_k's and φ^k_j are the bump functions associated to B_{kj}. Let

$$m^k_j = \frac{1}{\int_{\mathcal{X}} \varphi^k_j} \int_{\mathcal{X}} f \varphi^k_j.$$

By Proposition 4.14 in [5], we have

$$\begin{aligned} f(x) &= \left(f(x)\chi_{\Omega^c_k}(x) + \sum_{j=1}^{\infty} m^k_j \varphi^k_j(x) \right) + \sum_{j=1}^{\infty} (f(x) - m^k_j)\varphi^k_j(x) \\ &:= g_k(x) + \sum_{j=1}^{\infty} (f(x) - m^k_j)\varphi^k_j(x). \end{aligned} \tag{2.1}$$

and $|m^k_j| \leq C2^k$. Thus we find

$$f = \sum_{k=1}^{\infty} g_{k+1} - g_k := \sum_{k=1}^{\infty} f_k \quad a.e.$$

One can check

$$
\begin{aligned}
f_k =& \sum_{i=1}^{\infty} \big[(f-m_i^k)\varphi_i^k - \sum_{j=1}^{\infty}(f-m_{ij}^{k+1})\varphi_i^k\varphi_j^{k+1}\big] \\
&+ \sum_{j=1}^{\infty}\big[\sum_{i=1}^{\infty}(f-m_{ij}^{k+1})\varphi_i^k\varphi_j^{k+1} - (f-m_j^{k+1})\varphi_j^{k+1}\big] \\
=& \sum_{i=1}^{\infty} h_i^k + \sum_{j+1}^{\infty}\gamma_j^k,
\end{aligned}
\tag{2.2}
$$

where

$$
m_{ij}^{k+1} = \frac{1}{\int \varphi_i^k \varphi_j^{k+1}} \int f\varphi_i^k\varphi_j^{k+1}.
$$

Now we have

$$
|h_j^k| \le C2^{k+1}, \quad |\gamma_j^k| \le C2^{k+1}, \quad \text{and} \int h_j^k = 0 = \int \gamma_j^k.
$$

Finally we observe that $\mu(\Omega_k) \le C2^{-k}$ since $f \in H^{1,\infty}(\mathcal{X})$. Thus we finish construction of the atom decomposition.

For the converse, we fix $\alpha > 0$, and choose k_0 so that $2^{k_0} \le \alpha < 2^{k_0+1}$. Write

$$
f = \sum_{k=-\infty}^{k_0-1} f_k + \sum_{k=k_0}^{\infty} f_k = F_1 + F_2.
$$

Now since

$$
\mathcal{M}_0(F_1)(x) \le \sum_{k=-\infty}^{k_0-1} \mathcal{M}_0(f_k)(x) \le C\sum_{k=-\infty}^{k_0-1} 2^k \le C_2\alpha,
$$

we have

$$
|\{\mathcal{M}_0(f)(x) > (C_2+1)\alpha\}| \le |\{\mathcal{M}_0(F_2)(x) > \alpha\}|.
$$

Set

$$
A_{k_0} = \bigcup_{k=k_0}^{\infty}\bigcup_{i\ge 1} 3\tau B_i^k,
$$

where $3\tau B_i^k$ denotes the ball with radii of $3r_i^k$ centered at x_i^k. By (1.2),

$$
|A_{k_0}| \le C_1^{\log_2 C_1 + 1} C_0 2^{-k_0} \le C/\alpha
$$

Therefore it suffices to estimate

$$
I = \mu(\{x \notin A_{k_0} : \mathcal{M}_0(F_2)(x) > \alpha\}).
$$

Note that, for $x \notin 3\tau B_i^k$ and $y \in B_i^k$, we have

$$
d(x,y) \ge \frac{1}{\tau} d(x, x_i^k) - d(y, x_i^k) \ge 2d(y, x_i^k).
$$

Thus by (ii) of Definition 1.2, in the same region, we have

$$|S_j(x,y) - S_j(x,x_i^k)| \lesssim \frac{d(y,x_i^k)^{\epsilon_1}}{d(x,y)^{\epsilon_1} V(x,y)}.$$

Hence by the cancellation condition of h_{ki},

$$\begin{aligned}\mathcal{M}_0(f_k)(x) =& \sup_j \left| \int [S_j(x,y) - S_j(x,x_i^k)] f_k(y) dy \right| \\ \le& C 2^k \frac{\mu(B_i^k) d(y,x_i^k)^{\epsilon_1}}{V(x,y) d(x,y)^{\epsilon_1}} \\ \le& C 2^k \frac{\mu(B_i^k)^{1+\frac{\epsilon_1}{D}}}{V(x,x_i^k)^{1+\frac{\epsilon_1}{D}}}.\end{aligned} \tag{2.3}$$

Now, we shall use the following result in measure theory which was independently founded by Stein-Taibleson-Weiss [12] and by Kalton [8].

Lemma 2.1 *Let g_k be a sequence of measurable functions and let $0 < p < 1$. Assume that $|\{|g_k| > \lambda\}| \le C/\lambda^p$ with C independent of k and λ. Then, for every numerical sequence $\{c_k\}$ in l^p we have*

$$\left| \left\{ x : \left| \sum_k c_k g_k \right| > \lambda \right\} \right| \le \frac{2-p}{1-p} \frac{C}{\lambda^p} \sum_k |c_k|^p.$$

Using this lemma with $g_{ki} = V(x,x_i^k)^{-1-\frac{\epsilon_1}{D}}$, $p = (1+\frac{\epsilon_1}{D})^{-1}$, and $c_{ki} = 2^k \mu(B_i^k)^{1+\frac{\epsilon_1}{D}}$, we obtain

$$I \le \frac{C_{\epsilon_1,D}}{\alpha^p} \sum_{k \ge k_0} \sum_i 2^{kp} \mu(B_i^k) \le C_1 \frac{C_{\epsilon_1,D}}{\alpha^p} 2^{k_0(p-1)} \le C_1 C_{\epsilon_1,D}/\alpha. \tag{2.4}$$

Hence, $f \in H^{1,\infty}(\mathcal{X})$ and $\|f\|_{H^{1,\infty}(\mathcal{X})} \le cC_1$.

Thus we complete the proof of Theorem 1.1. □

Proof of Theorem 1.2 For $\alpha > 0$, take $k_0 \in \mathbb{Z}$ such that $2^{k_0} \le \alpha < 2^{k_0+1}$. Let $f = \sum_{k=-\infty}^{\infty} f_k = \sum_{k=-\infty}^{\infty} \sum_i h_i^k$ be an atom decomposition and $\text{supp} h_i^k \subset B_i^k$. Write

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{\infty} f_k := F_1 + F_2.$$

Then $F_1 \in L^2(\mathcal{X})$ and

$$\|F_1\|_2 \le C \sum_{k=-\infty}^{k_0} 2^k (\sum_i \mu(B_i^k))^{1/2} \le C \|f\|_{H^{1,\infty}(\mathcal{X})}^{1/2} \sum_{k=-\infty}^{k_0} 2^{k/2} \le C \|f\|_{H^{1,\infty}(\mathcal{X})}^{1/2} \alpha^{1/2}.$$

Hence by the $L^2(\mathcal{X})$ boundedness of T,

$$\begin{aligned}|\{x\in\mathcal{X}:|TF_1(x)|>\alpha\}| \leq&\frac{1}{\alpha^2}\|TF_1\|^2_{L^2(\mathcal{X})}\\ \leq&\frac{1}{\alpha^2}\|T\|^2_{L^2(\mathcal{X})\to L^2(\mathcal{X})}\|F_1\|^2_{L^2(\mathcal{X})}\\ \leq&C\|T\|^2_{L^2(\mathcal{X})\to L^2(\mathcal{X})}\|f\|_{H^{1,\infty}(\mathcal{X})}/\alpha.\end{aligned}\tag{2.5}$$

Let $\bar{B}_i^k$ denote the dilation of B_i^k by the factor of $\left[(\frac{3}{2})^{(k-k_0)/D}+1\right]\tau$ and let

$$A_{k_0}=\bigcup_{k=k_0+1}^{\infty}\bigcup_i \bar{B}_i^k,$$

then,

$$\begin{aligned}|A_{k_0}|\leq&\sum_{k=k_0+1}^{\infty}\sum_i\left(\frac{3}{2}\right)^{k-k_0}\mu(B_i^k)\leq\sum_{k=k_0+1}^{\infty}\left(\frac{3}{2}\right)^{k-k_0}2^{-k}\|f\|_{H^{1,\infty}}\\ \leq&\sum_{k=k_0+1}^{\infty}\left(\frac{3}{4}\right)^{k-k_0}\|f\|_{H^{1,\infty}}/\alpha\leq C\|f\|_{H^{1,\infty}}/\alpha.\end{aligned}\tag{2.6}$$

It suffices to show $\int_{A_{k_0}^c}|TF_2(x)|dx\leq C\|f\|_{H^{1,\infty}}$. By Fubini's theorem and the cancellation conditions for h_i^k,

$$\int_{A_{k_0}^c}|TF_2(x)|dx\leq C\sum_{k=k_0+1}^{\infty}2^k\sum_i\int_{B_i^k}\int_{(\bar{B}_i^k)^c}|K(x,y)-K(x,x_i^k)|dxdy.\tag{2.7}$$

Since $x\in[(\bar{B}_i^k]^c$ and $y\in B_i^k$, we have

$$d(x,y)\geq\frac{1}{\tau}d(x,x_i^k)-d(y,x_i^k)\geq\left(\frac{3}{2}\right)^{(k-k_0)/D}\cdot d(y,x_i^k).$$

Thus,

$$\begin{aligned}\int_{A_{k_0}^c}|TF_2(x)|dx\leq&C\sum_{k=k_0+1}^{\infty}2^k\sum_i\mu(B_i^k)\cdot\Gamma\left((\frac{2}{3})^{(k-k_0)/D}\right)\\ \leq&C\|f\|_{H^{1,\infty}}\cdot\sum_{k=k_0+1}^{\infty}\Gamma\left((\frac{2}{3})^{(k-k_0)/D}\right)\\ \leq&C\int_0^1\frac{\Gamma(\delta)}{\delta}d\delta\cdot\|f\|_{H^{1,\infty}}.\end{aligned}\tag{2.8}$$

This ends the proof of Theorem 1.2. □

Proof of Theorem 1.3 Fix λ. Set $q=\frac{1}{1-\alpha}$ and $\eta=\lambda^q\|f\|_{H^{1,\infty}}^{1-q}$. Take $\bar{k}_0\in\mathbb{Z}$ such that $2^{\bar{k}_0}\leq\eta<2^{\bar{k}_0+1}$. Split f into two parts

$$f=\sum_{k=-\infty}^{\bar{k}_0}f_k+\sum_{k=\bar{k}_0+1}^{\infty}f_k:=F_3+F_4.$$

From atom decomposition for $H^{1,\infty}$ function f, it follows that

$$\begin{aligned}\|F_3\|_{L^{p_0}(\mathcal{X})}\leq&C\sum_{k=-\infty}^{\bar{k}_0}2^k\left(\sum_i\mu(B_i^k)\right)^{1/p_0}\\ \leq&C\|f\|_{H^{1,\infty}(\mathcal{X})}^{1/p_0}\sum_{k=-\infty}^{\bar{k}_0}2^{k(1-1/p_0)}\\ \leq&C\|f\|_{H^{1,\infty}(\mathcal{X})}^{1/p_0}\eta^{1-1/p_0}\\ =&C\lambda^{q(1-1/p_0)}\|f\|_{H^{1,\infty}(\mathcal{X})}^{1-q(1-1/p_0)}.\end{aligned}\tag{2.9}$$

By the $L^{p_0}(\mathcal{X})\to L^{q_0}(\mathcal{X})$ boundedness of T_α and (2.9),

$$\begin{aligned}\mu(\{x\in\mathcal{X}:|T_\alpha F_3(x)|>\lambda\})&\leq c\lambda^{q_0}\|T_\alpha F_3\|_{L^{q_0}(\mathcal{X})}^{q_0}\\ &\leq c\lambda^{q_0}\|F_3\|_{L^{p_0}(\mathcal{X})}^{q_0}\\ &\leq C\left(\frac{\|f\|_{H^{1,\infty}}}{\lambda}\right)^q.\end{aligned}\tag{2.10}$$

Let $\widetilde{B_i^k}=3\tau B_i^k$ and $E_{\bar{k}_0}=\bigcup_{k=\bar{k}_0+1}^{\infty}\bigcup_i\widetilde{B_i^k}$, By Theorem 1.1,

$$\begin{aligned}\mu(E_{\bar{k}_0})&\leq C\sum_{k=\bar{k}_0+1}^{\infty}\sum_i\mu(B_i^k)\\ &\leq C\|f\|_{H^{1,\infty}(\mathcal{X})}\sum_{k=\bar{k}_0+1}^{\infty}2^{-k}\\ &\leq C\|f\|_{H^{1,\infty}(\mathcal{X})}\eta^{-1}\\ &=C\left(\frac{\|f\|_{H^{1,\infty}}}{\lambda}\right)^q.\end{aligned}\tag{2.11}$$

To finish the proof, it suffices to show

$$\mu(\{x\in E_{\bar{k}_0}^c:|T_\alpha F_4(x)|>\lambda\})\leq C\left(\frac{\|f\|_{H^{1,\infty}}}{\lambda}\right)^q.\tag{2.12}$$

Note that if $x\in E_{\bar{k}_0}^c$ and $y\in B_i^k$, then by (1.1),

$$d(x,y)\geq\frac{1}{\tau}d(x,x_i^k)-d(x_i^k,y)\geq 2d(x_i^k,y).$$

Thus by the use of cancellation condition of h_i^k, Mincowski's inequality and (1.7),

$$\begin{aligned}
&\mu(\{x\in E^c_{\bar{k}_0}:|T_\alpha F_4(x)|>\lambda\})\\
&\le \lambda^{-1}\int_{E^c_{\bar{k}_0}}|T_\alpha F_4(x)|dx\\
&\le \lambda^{-1}\sum_{k=\bar{k}_0+1}^{\infty}\sum_{i=0}^{\infty}\int_{B_i^k}|h_i^k(y)|\int_{E^c_{\bar{k}_0}}|K_\alpha(x,y)-K_\alpha(x,x_i^k)|dxdy\\
&\le \lambda^{-1}\sum_{k=\bar{k}_0+1}^{\infty}\sum_{i=0}^{\infty}\int_{B_i^k}|h_i^k(y)|\int_{E^c_{\bar{k}_0}}\frac{d(y,x_i^k)^\epsilon}{V(x,y)^{1-\alpha}d(x,y)^\epsilon}dxdy.
\end{aligned} \tag{2.13}$$

By (1.2)–(1.5),

$$\begin{aligned}
&\int_{E^c_{\bar{k}_0}}\frac{d(y,x_i^k)^\epsilon}{V(x,y)^{1-\alpha}d(x,y)^\epsilon}dx\\
=&\sum_{j=1}^{\infty}\int_{2^jd(x_i^k,y)\le d(x,y)<2^{j+1}d(x_i^k,y)}\frac{d(y,x_i^k)^\epsilon}{V(x,y)^{1-\alpha}d(x,y)^\epsilon}dx\\
\lesssim&\sum_{j=1}^{\infty}\frac{d(y,x_i^k)^\epsilon}{\left[\mu(B(y,2^jd(x_i^k,y)))\right]^{1-\alpha}(2^jd(x_i^k,y))^\epsilon}\cdot\mu(B(y,2^{j+1}d(x_i^k,y)))\\
\lesssim&\sum_{j=1}^{\infty}2^{-j(\epsilon-\alpha D)}\left[\mu(B(y,d(x_i^k,y)))\right]^\alpha\\
\lesssim&V(x_i^k,y)^\alpha\lesssim\mu(B_i^k)^\alpha.
\end{aligned} \tag{2.14}$$

This estimate together with (2.13) yields,

$$\begin{aligned}
\mu(\{x\in E^c_{\bar{k}_0}:|T_\alpha F_2(x)|>\lambda\})\lesssim&\lambda^{-1}\sum_{k=\bar{k}_0+1}^{\infty}2^k\left(\sum_{i=0}^{\infty}\mu(B_i^k)\right)^{1+\alpha}\\
\lesssim&\lambda^{-1}\sum_{k=\bar{k}_0+1}^{\infty}2^{-k\alpha}\|f\|^{1+\alpha}_{H^{1,\infty}(\mathcal{X})}\\
\lesssim&\lambda^{-1}\eta^{-\alpha}\|f\|^{1+\alpha}_{H^{1,\infty}(\mathcal{X})}\\
=&\left(\frac{\|f\|_{H^{1,\infty}}}{\lambda}\right)^q,
\end{aligned} \tag{2.15}$$

which gives (2.12). Thus we complete the proof of Theorem 1.3. □

Proof of Theorem 1.4 The proof of (1) is rather standard. For the sake of completeness, we give the details. Let a be an $(1,q_0)$-atom supported on $B(x_0,r)$. Let $q_1=\frac{1}{1-\alpha}$. Write

$$\|T_\alpha a\|^{q_1}_{L^{q_1}(\mathcal{X})}=\int_{B(x_0,2r)}|T_\alpha a(x)|^{q_1}dx+\int_{\mathcal{X}\setminus B(x_0,2r)}|T_\alpha a(x)|^{q_1}dx:=I_1+I_2.$$

Since $q_0 > q_1$, Hölder inequality together with the $L^{p_0} \to L^{q_0}$ boundedness of T_α yields

$$I_1 \le \|T_\alpha a\|_{L^{q_0}}^{q_1} \mu(B(x_0,2r))^{1-q_1/q_0} \lesssim \|a\|_{p_0}^{q_1} \mu(B(x_0,2r))^{q_1-q_1/p_0} \lesssim 1.$$

Next, if $d(x,x_0) \ge 2r \ge 2(y,x_0)$, since $\int_{\mathcal{X}} a(x)d\mu(x) = 0$, by (1.7) we have

$$\begin{aligned} |T_\alpha a(x)| &= \left| \int_{\mathcal{X}} [K_\alpha(x,y) - K_\alpha(x,x_0)] a(y) dy \right| \\ &\lesssim \frac{r^\epsilon}{V(x,x_0)^{1-\alpha} d(x,x_0)^\epsilon} \|a\|_{L^1(\mathcal{X})} \\ &\lesssim \frac{r^\epsilon}{V(x,x_0)^{1-\alpha} d(x,x_0)^\epsilon}. \end{aligned} \tag{2.16}$$

From this we obtain

$$\begin{aligned} I_2 &\lesssim \int_{\mathcal{X} \setminus B(x_0,2r)} \frac{r^{q_1 \epsilon}}{V(x,x_0)^{q_1 - q_1\alpha} d(x,x_0)^{q_1 \epsilon}} dx \\ &= \sum_{j=1}^{\infty} \int_{2^j r \le d(x,x_0) < 2^{j+1} r} \frac{r^{q_1 \epsilon}}{V(x,x_0) d(x,x_0)^{q_1 \epsilon}} dx \\ &\lesssim \sum_{j=1}^{\infty} \frac{r^{q_1 \epsilon}}{\mu(B(x_0, 2^j r))(2^j r)^{q_1 \epsilon}} \int_{d(x,x_0) < 2^{j+1} r} dx \\ &\lesssim 1. \end{aligned} \tag{2.17}$$

The conclusion (2) is a direct consequence of the conclusion (1) and Theorem 1.4 follows. □

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