Turk J Math
34 (2010) , 305 – 315.
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doi:10.3906/mat-0810-19

On the codifferential of the Kähler form and cosymplectic metrics on maximal flag manifolds

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Abstract

Using moving frames we obtain a formula to calculate the codifferential of the Kähler form on a maximal flag manifold. We use this formula to obtain some differential type conditions so that a metric on the classical maximal flag manifold be cosymplectic.

Key Words: Codifferential, Kähler form, flag manifolds, differential forms.

1. Introduction

In this note we study the Kähler form on the classical maximal flag manifold $\mathbb{F}(n) = U(n)/(U(1) \times \cdots U(1))$. The geometry of this manifold has been studied in several papers. Burstall and Salamon [2] showed the existence of a bijective relation between almost complex structures on $\mathbb{F}(n)$ and tournaments with n vertices. This correspondence has been very important to study the geometry of the maximal complex manifold, see for example [5], [6], [9], [11], [12] and [13]. In [6], was showed the existence of a one to-one correspondence between $(1,2)$-symplectic metrics and locally transitive tournaments. In [4], this result was generalized for $(1,2)$-symplectic metrics defined using f-structures.

Mo and Negreiros [9], by using moving frames and tournaments, showed explicitly the existence of an n-dimensional family of invariant $(1,2)$-symplectic metrics on $\mathbb{F}(n)$. In order to do this, they obtained a formula to calculate the differential of the Kähler form by using the moving frames technique. In the present work we use a similar method in order to obtain a formula to calculate the codifferential of the Kähler form. An important reference to our calculations is the book by Griffiths and Harris [8]; we use definitions, results and notations contained in this book to differential forms of type (p,q).

Finally, we use such formula to find some differential type conditions in order for a metric on a maximal flag manifold be cosymplectic. We show that a metric on the classical flag manifold is cosymplectic if and only if the complex functions f_k^{ij} in the Kähler form (see (13)) satisfy different types of partial differential equations.

2000 *AMS Mathematics Subject Classification:* 14M15, 32M10, 58A10.

2. Flag manifolds

The usual manifold of full flags of subspaces of $\mathbb{C}^n$ is defined by

$$\mathbb{F}(n) = \{(V_1, \ldots, V_n) : V_i \subset V_{i+1}, \dim V_i = i\}. \tag{1}$$

The unitary group $U(n)$ acts transitively on $\mathbb{F}(n)$ turning this manifold into the homogeneous space

$$\mathbb{F}(n) = \frac{U(n)}{U(1) \times U(1) \times \cdots \times U(1)} = \frac{U(n)}{M}\,, \tag{2}$$

where $M = U(1) \times U(1) \times \cdots \times U(1)$ is any maximal torus of $U(n)$.

Let $\mathfrak{p}$ be the tangent space of $\mathbb{F}(n)$ at the point (M). It is known that $\mathfrak{u}(n)$, the Lie algebra of skew-hermitian matrices, decomposes as

$$\mathfrak{u}(n) = \mathfrak{p} \oplus \mathfrak{u}(1) \oplus \cdots \oplus \mathfrak{u}(1)\,,$$

where $\mathfrak{p} \subset \mathfrak{u}(n)$ is the subspace of zero-diagonal matrices.

In order to define any tensor on $\mathbb{F}(n)$ it is sufficient to give it on $\mathfrak{p}$, because the action of $U(n)$ on $\mathbb{F}(n)$ is transitive. An invariant almost complex structure on $\mathbb{F}(n)$ is determined by a linear map $J\colon \mathfrak{p} \to \mathfrak{p}$ such that $J^2 = -I$ and commutes with the adjoint representation of the torus M on $\mathfrak{p}$.

For each almost complex structure we assign a tournament, a special class of directed graph. A tournament or n-tournament $\mathcal{T}$, consists of a finite set $T = \{p_1, \ldots, p_n\}$ of n players together with a dominance relation, $\to$, which assigns to every pair of players a winner, that is, $p_i \to p_j$ or $p_j \to p_i$. A tournament $\mathcal{T}$ can be represented by a directed graph in which T is the set of vertices and any two vertices are joined by an oriented edge. If the dominance relation is transitive, then the tournament is called transitive. For a complete reference on tournaments see [10].

Given an invariant complex structure J, we define the associated tournament $\mathcal{T}(J)$ in the following way: if $J(a_{ij}) = (a'_{ij})$, then $\mathcal{T}(J)$ is such that for $i < j$

$$\Big(i \to j \;\Leftrightarrow\; a'_{ij} = \sqrt{-1}\, a_{ij}\Big) \quad \text{or} \quad \Big(i \leftarrow j \;\Leftrightarrow\; a'_{ij} = -\sqrt{-1}\, a_{ij}\Big);$$

see [9].

We consider $\mathbb{C}^n$ equipped with the standard Hermitian inner product, that is, for $V = (v_1, \ldots, v_n)$ and $W = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we have $\langle V, W\rangle = \sum_{i=1}^{n} v_i \overline{w_i}$. We use the convention $v_{\bar{\imath}} = \overline{v_i}$ and $f_{\bar{\imath}j} = \overline{f_{i\bar{\jmath}}}$.

A frame consists of an ordered set of n vectors $(Z_1, \ldots, Z_n)$ such that $Z_1 \wedge \ldots \wedge Z_n \neq 0$, and it is called unitary if $\langle Z_i, Z_j\rangle = \delta_{i\bar{\jmath}}$. The set of unitary frames can be identified with the unitary group $U(n)$.

If we write $dZ_i = \sum_j \omega_{i\bar{\jmath}} Z_j$, the coefficients $\omega_{i\bar{\jmath}}$ are the Maurer-Cartan forms of the unitary group $U(n)$. They are skew-Hermitian, this is, $\omega_{i\bar{\jmath}} + \omega_{\bar{\jmath}i} = 0$. For more details see [3].

We may define all left-invariant metrics on $(\mathbb{F}(n), J)$ by (see [1])

$$ds^2_\Lambda = \sum_{i,j} \lambda_{ij}\, \omega_{i\bar{\jmath}} \otimes \omega_{\bar{\imath}j}, \tag{3}$$

where $\Lambda = (\lambda_{ij})$ is a simetric real matrix such that

$$\begin{cases} \lambda_{ij} > 0, & \text{if} \quad i \neq j, \\ \lambda_{ij} = 0, & \text{if} \quad i = j, \end{cases} \tag{4}$$

and the Maurer-Cartan forms $\omega_{i\bar{j}}$ are such that

$$\omega_{i\bar{j}} \in \mathbb{C}^{1,0} \text{ (forms of type (1,0))} \quad \Longleftrightarrow \quad i \overset{\mathcal{T}(J)}{\longrightarrow} j. \tag{5}$$

The metrics (3) are called of Borel type and they are almost Hermitian for every invariant almost complex structure J, that is, $ds^2_\Lambda(JX, JY) = ds^2_\Lambda(X,Y)$ for all tangent vectors X, Y. When J is integrable, ds^2_Λ is said to be Hermitian.

Given J an invariant almost complex structure on $\mathbb{F}(n)$ and ds^2_Λ an invariant metric, the Kähler form with respect to J and ds^2_Λ is defined by

$$\Omega(X,Y) = ds^2_\Lambda(X, JY), \tag{6}$$

for any tangent vectors X, Y. For each permutation σ of n elements, this Kähler form can be written as (see [9])

$$\Omega = -2\sqrt{-1} \sum_{i<j} \mu_{\sigma(i)\sigma(j)} \omega_{\sigma(i)\overline{\sigma(j)}} \wedge \omega_{\overline{\sigma(i)}\sigma(j)}, \tag{7}$$

where $\mu_{\sigma(i)\sigma(j)} = \varepsilon_{\sigma(i)\sigma(j)} \lambda_{\sigma(i)\sigma(j)}$ and

$$\varepsilon_{ij} = \begin{cases} 1, & \text{if} \quad \sigma(i) \to \sigma(j), \\ -1, & \text{if} \quad \sigma(j) \to \sigma(i), \\ 0, & \text{if} \quad \sigma(i) = \sigma(j). \end{cases}$$

$\mathbb{F}(n)$ is said to be almost Kähler if and only if Ω is closed, that is, $d\Omega = 0$. If J is integrable and Ω is closed, then $\mathbb{F}(n)$ is said to be a Kähler manifold.

In [9], Mo and Negreiros proved the following result.

Theorem 2.1

$$d\Omega = 4 \sum_{i<j<k} C_{\sigma(i)\sigma(j)\sigma(k)} \Psi_{\sigma(i)\sigma(j)\sigma(k)}, \tag{8}$$

where

$$C_{ijk} = \mu_{ij} - \mu_{ik} + \mu_{jk}, \tag{9}$$

and

$$\Psi_{ijk} = \mathrm{Im}(\omega_{i\bar{j}} \wedge \omega_{\bar{i}k} \wedge \omega_{j\bar{k}}). \tag{10}$$

We denote by $\mathbb{C}^{p,q}$ the space of forms of type (p,q) on $\mathbb{F}(n)$. Then, for any i, j, k, we have either $\Psi_{ijk} \in \mathbb{C}^{0,3} \oplus \mathbb{C}^{3,0}$ or $\Psi_{ijk} \in \mathbb{C}^{1,2} \oplus \mathbb{C}^{2,1}$. An invariant almost Hermitian metric ds^2_Λ is said to be (1,2)-symplectic if and only if $(d\Omega)^{1,2} = 0$. If $\delta\Omega = 0$, the codifferential of the Kähler form is zero, then the metric is said to be cosymplectic.

3. The codifferential of the Kähler form

In order to calculate the codifferential of the Kähler form we need to use the Hodge star operator. Now, following the book by Griffiths and Harris [8], we give a precise definition of the star operator.

If $\eta \in \mathbb{C}^{p,q}$, we can write

$$\eta = \sum_{I,J} \eta_{IJ} \psi_I \wedge \overline{\psi_J}\,, \tag{11}$$

where η_{IJ} are complex functions, $I = \{i_1, \ldots, i_p\}$, $J = \{j_1, \ldots, j_q\}$ and ψ_I, ψ_J are forms of type $(p, 0)$ and $(0, q)$ respectively. The Hodge star operator

$$*\colon \mathbb{C}^{p,q} \longrightarrow \mathbb{C}^{N-p,N-q}$$

transforms forms of type (p, q) to forms of type $(N-p, N-q)$, where $N = n(n-1)/2$ is the complex dimension of $\mathbb{F}(n)$. Then the star operator is defined by

$$*\,\eta = 2^{p+q+N} \sum_{I,J} \epsilon_{IJ} \overline{\eta_{IJ}} \psi_{I^0} \wedge \overline{\psi_{J^0}}\,, \tag{12}$$

where $I^0 = \{1, \ldots, N\} - I$, $J^0 = \{1, \ldots, N\} - J$ and ϵ_{IJ} is the sign of the permutation

$$(1, \ldots, N, 1, \ldots, N) \to (i_1, \ldots, i_p, j_1, \ldots, j_q, i_1^0, \ldots, i_{N-p}^0, j_1^0, \ldots, j_{N-q}^0).$$

The codifferential operator transforms p-forms in $(p-1)$-forms. It is known that (see [8])

$$\delta = (-1)^{m(p+1)+1} * d *\,,$$

where m is the real dimension of the manifold and $*$ is the Hodge star operator. In our case, m is even, then

$$\delta = - * d * .$$

By (7), up to isomorphisms, the Kähler form can be written in the following way:

$$\Omega = -2\sqrt{-1} \sum_{i<j} \mu_{ij} \omega_{i\bar{j}} \wedge \omega_{\bar{i}j},$$

where $\mu_{ij} = \varepsilon_{ij} \lambda_{ij}$ and

$$\varepsilon_{ij} = \begin{cases} 1, & \text{if} \quad i \to j\,, \\ -1, & \text{if} \quad j \to i\,, \\ 0, & \text{if} \quad i = j\,. \end{cases}$$

Then,

$$\delta\Omega = -2\sqrt{-1} \sum_{i<j} \mu_{ij}\, \delta(\omega_{i\bar{j}} \wedge \omega_{\bar{i}j}).$$

We know that $\omega_{i\bar{j}} \in \mathbb{C}^{1,0}$ and using (11) we can write

$$\omega_{i\bar{j}} = \sum_k f_k^{ij} dz_k\,, \tag{13}$$

where f_k^{ij} are complex functions. It follows that

$$\omega_{\bar{i}j} = \overline{\omega_{i\bar{j}}} = \sum_k \overline{f_k^{ij}} d\bar{z}_k \,,$$

and

$$\omega_{i\bar{j}} \wedge \omega_{\bar{i}j} = \sum_{k,l} f_k^{ij} \overline{f_l^{ij}} dz_k \wedge d\bar{z}_l \,.$$

Now, we can prove the following result.

Theorem 3.1 *The codifferential of the Kähler form is given by*

$$\delta\Omega = -2^{2+2N}\sqrt{-1} \sum_{i<j} \mu_{ij} \left\{ \sum_{k,l} \frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial z_l} dz_k - \frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial \bar{z}_k} d\bar{z}_l \right\}. \tag{14}$$

Proof. At first we use (12) to calculate $*(\omega_{i\bar{j}} \wedge \omega_{\bar{i}j})$ and we obtain

$$\begin{aligned} *(\omega_{i\bar{j}} \wedge \omega_{\bar{i}j}) &= 2^{1+1-N} \sum_{k,l} \epsilon_{kl} \overline{f_k^{ij} \overline{f_l^{ij}}} \Big\{ (dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_N) \wedge \\ &\qquad \wedge \ (d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_l} \wedge \ldots \wedge d\bar{z}_N) \Big\}, \\ &= 2^{2-N} \sum_{k,l} \epsilon_{kl} \overline{f_k^{ij}} f_l^{ij} \Big\{ (dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_N) \wedge \\ &\qquad \wedge \ (d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_l} \wedge \ldots \wedge d\bar{z}_N) \Big\}, \end{aligned}$$

where $\widehat{dz_k}$ and $\widehat{d\bar{z}_l}$ mean that these terms are removed and ϵ_{kl} is the sign of the permutation

$$(1, \ldots, N, 1, \ldots, N) \to (1, \ldots, \widehat{k}, \ldots, N, 1, \ldots, \widehat{l}, \ldots, N, k, l).$$

We calculate ϵ_{kl} in the following way:

$$\begin{aligned} &(1, \ldots, \widehat{k}, \ldots, N, 1, \ldots, \widehat{l}, \ldots, N, k, l) \to \\ &\qquad \to (-1)^{N-1}(1, \ldots, \widehat{k}, \ldots, N, k, 1, \ldots, \widehat{l}, \ldots, N, l), \\ &\qquad \to (-1)^{N-1}(-1)^{N-k}(1, \ldots, k, \ldots, N, 1, \ldots, \widehat{l}, \ldots, N, l), \\ &\qquad \to (-1)^{2N-k-1}(-1)^{N-l}(1, \ldots, k, \ldots, N, 1, \ldots, l, \ldots, N), \\ &\qquad \to (-1)^{3N-k-l-1}(1, \ldots, N, 1, \ldots, N,), \end{aligned}$$

then, $\epsilon_{kl} = (-1)^{3N-k-l-1}$. It implies that

$$\begin{aligned} *(\omega_{i\bar{j}} \wedge \omega_{\bar{i}j}) &= 2^{2-N} \sum_{k,l} (-1)^{3N-k-l-1} \, \overline{f_k^{ij}} f_l^{ij} \Big\{ (dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_N) \wedge \\ &\qquad \wedge \ (d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_l} \wedge \ldots \wedge d\bar{z}_N) \Big\}. \end{aligned}$$

Now we calculate the exterior differential of this differential form,

$$d*(\omega_{i\bar{j}}\wedge\omega_{\bar{i}j}) = 2^{2-N}\sum_{k,l}(-1)^{3N-k-l-1}\, d\left(\overline{f_k^{ij}}f_l^{ij}\right)\wedge(dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_N)$$

$$\wedge(d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_l}\wedge\ldots\wedge d\overline{z}_N).$$

We know that

$$d\left(\overline{f_k^{ij}}f_l^{ij}\right) = \sum_m \frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial z_m}dz_m + \frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial\overline{z}_m}d\overline{z}_m\,,$$

then,

$$d*(\omega_{i\bar{j}}\wedge\omega_{\bar{i}j}) = 2^{2-N}\sum_{k,l}(-1)^{3N-k-l-1}\left\{\frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial z_k}dz_k\wedge(dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_N)\wedge\right.$$

$$\wedge\ (d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_l}\wedge\ldots\wedge d\overline{z}_N) + \frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial\overline{z}_l}\,d\overline{z}_l\wedge(dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_N)\wedge$$

$$\left.\wedge(d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_l}\wedge\ldots\wedge d\overline{z}_N)\right\}.$$

On the other hand,

$$\frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial z_k}dz_k\wedge(dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_N)\wedge(d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_l}\wedge\ldots\wedge d\overline{z}_N) =$$

$$(-1)^{k-1}\frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial z_k}(dz_1\wedge\ldots\wedge dz_k\wedge\ldots\wedge dz_N)\wedge(d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_l}\wedge\ldots\wedge d\overline{z}_N)$$

and

$$\frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial\overline{z}_l}d\overline{z}_l\wedge(dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_N)\wedge(d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_l}\wedge\ldots\wedge d\overline{z}_N) =$$

$$(-1)^{N-1+l-1}\frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial\overline{z}_l}(dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_N)\wedge(d\overline{z}_1\wedge\ldots\wedge d\overline{z}_l\wedge\ldots\wedge d\overline{z}_N)\,.$$

This implies

$$d*(\omega_{i\bar{j}}\wedge\omega_{\bar{i}j}) = 2^{2-N}\sum_{k,l}\left\{(-1)^{3N-l}\,\frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial z_k}(dz_1\wedge\ldots\wedge dz_k\wedge\ldots\wedge dz_N)\wedge\right.$$

$$\wedge(d\overline{z}_1\wedge\ldots\wedge\widehat{d\overline{z}_l}\wedge\ldots\wedge d\overline{z}_N) +\ (-1)^{k+1}\frac{\partial\left(\overline{f_k^{ij}}f_l^{ij}\right)}{\partial\overline{z}_l}(dz_1\wedge\ldots\wedge\widehat{dz_k}\wedge\ldots\wedge dz_N)\wedge$$

$$\left.\wedge(d\overline{z}_1\wedge\ldots\wedge d\overline{z}_l\wedge\ldots\wedge d\overline{z}_N)\right\}.$$

Finally, we apply the star operator to the last form to calculate

$$* \, d * (\omega_{i\bar{j}} \wedge \omega_{\bar{i}j}).$$

In order to better understand the procedure we calculate separately the following expressions:

$$2^{2-N} \sum_{k,l} (-1)^{3N-l} * \left\{ \frac{\partial \left(\overline{f_k^{ij}} f_l^{ij} \right)}{\partial z_k} (dz_1 \wedge \ldots \wedge dz_k \wedge \ldots \wedge dz_N) \wedge (d\overline{z}_1 \wedge \ldots \wedge \widehat{d\overline{z}_l} \wedge \ldots \wedge d\overline{z}_N) \right\}$$

and

$$2^{2-N} \sum_{k,l} (-1)^{k+1} * \left\{ \frac{\partial \left(\overline{f_k^{ij}} f_l^{ij} \right)}{\partial \overline{z}_l} (dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_N) \wedge (d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_l \wedge \ldots \wedge d\overline{z}_N) \right\}.$$

To the first, by using the formula (12), we have

$$2^{2-N} \sum_{k,l} (-1)^{3N-l} * \left\{ \frac{\partial \left(\overline{f_k^{ij}} f_l^{ij} \right)}{\partial z_k} (dz_1 \wedge \ldots \wedge dz_k \wedge \ldots \wedge dz_N) \wedge (d\overline{z}_1 \wedge \ldots \wedge \widehat{d\overline{z}_l} \wedge \ldots \wedge d\overline{z}_N) \right\} =$$

$$= 2^{2-N} \sum_{k,l} (-1)^{3N-l} 2^{3N-1} \epsilon_{IJ} \overline{\frac{\partial \left(\overline{f_k^{ij}} f_l^{ij} \right)}{\partial z_k}} d\overline{z}_l,$$

where $I = \{1, \ldots, N\}$ and $J = \{1, \ldots, \widehat{l}, \ldots, N\}$. ϵ_{IJ} is the sign of the permutation

$$(1, \ldots, N, 1, \ldots, N) \to (1, \ldots, N, 1, \ldots, \widehat{l}, \ldots, N, l),$$

to which it is easy to see that $\epsilon_{IJ} = (-1)^{N-l}$. Then,

$$2^{2-N} \sum_{k,l} (-1)^{3N-l} * \left\{ \frac{\partial \left(\overline{f_k^{ij}} f_l^{ij} \right)}{\partial z_k} (dz_1 \wedge \ldots \wedge dz_k \wedge \ldots \wedge dz_N) \wedge (d\overline{z}_1 \wedge \ldots \wedge \widehat{d\overline{z}_l} \wedge \ldots \wedge d\overline{z}_N) \right\} =$$

$$= 2^{1+2N} \sum_{k,l} (-1)^{4N-2l} \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial \overline{z}_k} d\overline{z}_l \;=\; 2^{1+2N} \sum_{k,l} \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial \overline{z}_k} d\overline{z}_l \,.$$

Similarly, we can prove

$$2^{2-N} \sum_{k,l} (-1)^{k+1} * \left\{ \frac{\partial \left(\overline{f_k^{ij}} f_l^{ij} \right)}{\partial \overline{z}_l} (dz_1 \wedge \ldots \wedge \widehat{dz_k} \wedge \ldots \wedge dz_N) \wedge (d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_l \wedge \ldots \wedge d\overline{z}_N) \right\} =$$

$$= -2^{1+2N} \sum_{k,l} \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial z_l} dz_k \,.$$

So, we obtain

$$* \, d * (\omega_{i\bar{j}} \wedge \omega_{\bar{i}j}) = 2^{1+2N} \sum_{k,l} \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial \overline{z}_k} d\overline{z}_l - \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial z_l} dz_k \,,$$

and it implies that

$$\delta(\omega_{i\bar{j}} \wedge \omega_{\bar{i}j}) = 2^{1+2N} \sum_{k,l} \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial z_l} dz_k - \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial \overline{z}_k} d\overline{z}_l \, .$$

Therefore, we arrive to the desired formula:

$$\delta\Omega = -2^{2+2N}\sqrt{-1} \sum_{i<j} \mu_{ij} \left\{ \sum_{k,l} \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial z_l} dz_k - \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial \overline{z}_k} d\overline{z}_l \right\}.$$

□

4. Cosymplectic metrics

San Martin and Negreiros [14], proved that the metrics ds^2_Λ in (3) are cosymplectic. The condition for ds^2_Λ to be cosymplectic is the codifferential of the Kähler form $\delta\Omega$ be zero, however, they did not calculate this codifferential because they used another equivalent condition due to Gray and Hervella [7].

By Theorem 3.1 we can write the following proposition.

Proposition 4.1 *A metric on $(\mathbb{F}(n), J)$ is cosymplectic if and only if the functions f_k^{ij} in the Kähler form satisfy the partial differential equation*

$$\sum_{i<j} \mu_{ij} \left\{ \sum_{k,l} \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial z_l} dz_k - \frac{\partial \left(f_k^{ij} \overline{f_l^{ij}} \right)}{\partial \overline{z}_k} d\overline{z}_l \right\} = 0. \tag{15}$$

Expanding the sums over k and l in (15) and reordering, we obtain the following system of partial differential equations:

$$\sum_{i<j} \mu_{ij} \left(\frac{\partial \left(f_k^{ij} \overline{f_1^{ij}} \right)}{\partial z_1} + \cdots + \frac{\partial \left(f_k^{ij} \overline{f_N^{ij}} \right)}{\partial z_N} \right) = 0, \qquad k = 1, \ldots, N, \tag{16}$$

$$\sum_{i<j} \mu_{ij} \left(\frac{\partial \left(\overline{f_1^{ij}} f_1^{ij} \right)}{\partial \overline{z_1}} + \cdots + \frac{\partial \left(\overline{f_1^{ij}} f_N^{ij} \right)}{\partial \overline{z_N}} \right) = 0, \qquad k = 1, \ldots, N. \tag{17}$$

Actually, equation (17) is the conjugate of equation (16); then we have the following result.

Proposition 4.2 *A metric on $(\mathbb{F}(n), J)$ is cosymplectic if and only if the functions f_k^{ij} in the Kähler form satisfy the system of partial differential equations*

$$\sum_{i<j} \mu_{ij} \left(\frac{\partial \left(f_k^{ij} \overline{f_1^{ij}} \right)}{\partial z_1} + \cdots + \frac{\partial \left(f_k^{ij} \overline{f_N^{ij}} \right)}{\partial z_N} \right) = 0, \qquad k = 1, \ldots, N. \tag{18}$$

On the other hand, we can write equation (14) as

$$\begin{aligned}\delta\Omega &= -2^{2+2N}\sqrt{-1}\sum_{i<j}\mu_{ij}\left\{\sum_{k=1}^{N}\left(\frac{\partial\left(f_k^{ij}\overline{f_1^{ij}}\right)}{\partial z_1}+\cdots+\frac{\partial\left(f_k^{ij}\overline{f_N^{ij}}\right)}{\partial z_N}\right)dz_k+\right.\\ &\left.-\sum_{k=1}^{N}\left(\frac{\partial\left(\overline{f_1^{ij}}f_1^{ij}\right)}{\partial \overline{z_1}}+\cdots+\frac{\partial\left(\overline{f_1^{ij}}f_N^{ij}\right)}{\partial \overline{z_N}}\right)d\overline{z_k}\right\},\\ &= -2^{2+2N}\sqrt{-1}\sum_{i<j}\mu_{ij}\sum_{k=1}^{N}\left\{\left(\sum_{l=1}^{N}\frac{\partial}{\partial z_l}\left(f_k^{ij}\overline{f_l^{ij}}\right)\right)dz_k-\left(\sum_{l=1}^{N}\frac{\partial}{\partial \overline{z_l}}\left(\overline{f_k^{ij}}f_l^{ij}\right)d\overline{z_k}\right)\right\}.\end{aligned}$$

Like $z-\overline{z}=2\sqrt{-1}\,\mathrm{Im}z$, for every complex number z, then

$$\begin{aligned}\delta\Omega &= -2^{2+2N}\sqrt{-1}\sum_{i<j}\mu_{ij}\sum_{k=1}^{N}\left\{2\sqrt{-1}\ \mathrm{Im}\left(\sum_{l=1}^{N}\frac{\partial}{\partial z_l}\left(f_k^{ij}\overline{f_l^{ij}}\right)dz_k\right)\right\},\\ &= 2^{3+2N}\sum_{i<j}\mu_{ij}\left\{\sum_{k,l=1}^{N}\ \mathrm{Im}\left(\frac{\partial}{\partial z_l}\left(f_k^{ij}\overline{f_l^{ij}}\right)dz_k\right)\right\},\\ &= 2^{3+2N}\mathrm{Im}\left\{\sum_{i<j}\mu_{ij}\left(\sum_{k,l=1}^{N}\ \left(\frac{\partial}{\partial z_l}\left(f_k^{ij}\overline{f_l^{ij}}\right)dz_k\right)\right)\right\}.\end{aligned}$$

Then, we have the following proposition, equivalent to propositions (4.1) and (4.2).

Proposition 4.3 *A metric on $(\mathbb{F}(n),J)$ is cosymplectic if and only if the functions f_k^{ij} in the Kähler form satisfy the equation*

$$\mathrm{Im}\left\{\sum_{i<j}\mu_{ij}\left(\sum_{k,l=1}^{N}\ \left(\frac{\partial}{\partial z_l}\left(f_k^{ij}\overline{f_l^{ij}}\right)dz_k\right)\right)\right\}=0. \tag{19}$$

We can write this relation in real coordinates using the complex operators

$$\frac{\partial}{\partial z_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}-\sqrt{-1}\frac{\partial}{\partial y_i}\right),\qquad \frac{\partial}{\partial \overline{z}_i}=\frac{1}{2}\left(\frac{\partial}{\partial x_i}+\sqrt{-1}\frac{\partial}{\partial y_i}\right),$$

and the complex differential forms

$$dz_i=dx_i+\sqrt{-1}dy_i,\qquad d\overline{z}_i=dx_i-\sqrt{-1}dy_i.$$

So, we obtain that a metric on $(\mathbb{F}(n),J)$ is cosymplectic if and only if the functions f_k^{ij} in the Kähler form satisfy the following equations:

$$\sum_{i<j}\mu_{ij}\left\{\sum_{k,l}\left(\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial x_l}dx_k-\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial x_k}dx_l\right)+\left(\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial y_l}dy_k-\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial y_k}dy_l\right)\right\}=0 \tag{20}$$

$$\sum_{i<j}\mu_{ij}\left\{\sum_{k,l}\left(\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial x_l}dy_k-\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial y_k}dx_l\right)+\left(\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial x_k}dy_l-\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial y_l}dx_k\right)\right\}=0. \tag{21}$$

Here, the sums are calculated over all k and l, therefore the left side of the equation (20) is null. Then, we have the result.

Proposition 4.4 *A metric on $(\mathbb{F}(n), J)$ is cosymplectic if and only if the functions f_k^{ij} in the Kähler form satisfy the following equation*

$$\sum_{i<j}\mu_{ij}\left\{\sum_{k,l}\left(\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial x_l}dy_k-\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial y_k}dx_l\right)+\left(\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial x_k}dy_l-\frac{\partial\left(f_k^{ij}\overline{f_l^{ij}}\right)}{\partial y_l}dx_k\right)\right\}=0.$$

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Received 13.10.2008

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