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Some properties of C-fusion frames

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Abstract

In [10], we generalized the concept of fusion frames, namely, c-fusion frames, which is a continuous version of the fusion frames. In this article we give some important properties about the generalization, namely erasures of subspaces, the bound of c-erasure reconstruction error for Parseval c-fusion frames, perturbation of c-fusion frames and the frame operator for fusion pair.

Key Words: Operator, Hilbert space, Bessel, Frame, Fusion frame, c-fusion frame

1. Introduction and preliminaries

Throughout this paper H will be a Hilbert space and $\mathbb{H}$ will be the collection of all closed subspace of H. Also, (X, μ) will be a measure space, and $v : X \to [0, +\infty)$ a measurable mapping such that $v \neq 0$ almost everywhere (a.e.). We shall denote the unit closed ball of H by H_1.

Frames was first introduced in the context of non-harmonic Fourier series [9]. Outside of signal processing, frames did not seem to generate much interest until the ground breaking work in [8]. Since then the theory of frames began to be more widely studied. During the last 20 years the theory of frames has grown up rapidly, with the development of several new applications. For example, besides traditional application as signal processing, image processing, data compression, and sampling theory, frames are now used to mitigate the effect of losses in pocket-based communication systems and hence to improve the robustness of data transmission on [6], and to design high-rate constellation with full diversity in multiple-antenna code design [12]. In [2, 1, 3] some applications have been developed.

The fusion frames were considered by Casazza, Kutyniok and Li in connection with distributed processing and are related to the construction of global frames [4, 5]. The fusion frame theory is in fact more delicate due to complicated relations between the structure of the sequence of weighted subspaces and the local frames in the subspaces and due to the extreme sensitivity with respect to changes of the weights.

In [10] we extended fusion frames to their continuous versions in measure spaces and in this paper we shall investigate some properties about it.

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This article derives from the Ph.D thesis of which author, completed at the university of Tabriz.

Definition 1.1 *Let $\{f_i\}_{i\in I}$ be a sequence of members of H. We say that $\{f_i\}_{i\in I}$ is a frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$*

$$A\|h\|^2 \leq \sum_{i\in I} |< f_i, h >|^2 \leq B\|h\|^2. \tag{1.1}$$

The constants A and B are called the frame bounds. If A, B can be chosen so that $A = B$, we call this frame an A-tight frame and if $A = B = 1$, it is called a Parseval frame. If we only have the upper bound, we call $\{f_i\}_{i\in I}$ a Bessel sequence. If $\{f_i\}_{i\in I}$ is a Bessel sequence then the following operators are bounded:

$$T : l^2(I) \to H, \quad T(c_i) = \sum_{i\in I} c_i f_i \tag{1.2}$$

$$T^* : H \to l^2(I), \quad T^*(f) = \{< f, f_i >\}_{i\in I} \tag{1.3}$$

$$Sf = TT^*f = \sum_{i\in I} < f, f_i > f_i. \tag{1.4}$$

These operators are called synthesis operator; analysis operator and frame operator, respectively.

Definition 1.2 *For a countable index set I, let $\{W_i\}_{i\in I}$ be a family of closed subspace in H, and let $\{v_i\}_{i\in I}$ be a family of real numbers, called weights, i.e., $v_i > 0$ for all $i \in I$. Then $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H if there exist $0 < C \leq D < \infty$ such that for all $h \in H$*

$$C\|h\|^2 \leq \sum_{i\in I} {v_i}^2 \|\pi_{W_i}(f)\|^2 \leq D\|h\|^2, \tag{1.5}$$

where π_{W_i} is the orthogonal projection onto the subspace W_i.

We call C and D the fusion frame bounds. The family $\{(W_i, v_i)\}_{i\in I}$ is called a c-tight fusion frame, if in 1.5 the constants C and D can be chosen so that $C = D$, a Parseval fusion frame provided $C = D = 1$ and an orthonormal fusion basis if $H = \bigoplus_{i\in I} W_i$. If $\{(W_i, v_i)\}_{i\in I}$ possesses an upper fusion frame bound, but not necessarily a lower bound, we call it is a Bessel fusion sequence with Bessel fusion bound D. The representation space employed in this setting is

$$(\sum_{i\in I} \oplus W_i)_{l_2} = \{\{f_i\}_{i\in I} | f_i \in W_i \quad and \quad \{||f_i||\}_{i\in I} \in l^2(I)\}.$$

Let $\{(W_i, v_i)\}_{i\in I}$ be a fusion frame for H. The synthesis operator, analysis operator and frame operator are defined, respectively, by

$$T_W : (\sum_{i\in I} \oplus W_i)_{l_2} \to H \quad with \quad T_W(f) = \sum_{i\in I} v_i f_i,$$

$$T_W^* : H \to (\sum_{i\in I} \oplus W_i)_{l_2} \quad with \quad T_W^*(f) = \{v_i \pi_{W_i}(f)\}_{i\in I},$$

$$S_W(f) = T_W T_W^* = \sum_{i\in I} v_i^2 \pi_{W_i}(f).$$

By proposition 3.7 in [5], if $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H with fusion frame bounds C and D then S_W is a positive and invertible operator on H with $CId \leq S_W \leq DId$. The theory of frames has a continuous version as follows.

Definition 1.3 *Let (X, μ) be a measure space. Let $f : X \to H$ be weakly measurable (i.e., for all $h \in H$, the mapping $x \to < f(x), h >$ is measurable). Then f is called a continuous frame or c-frame for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$*

$$A\|h\|^2 \leq \int_X | < f(x), h > |^2 d\mu \leq B\|h\|^2. \tag{1.6}$$

The representation space employed in this setting is

$$L^2(X, \mu) = \{\varphi : X \to H| \quad \varphi \quad is \quad measurable \quad and \quad \|\varphi\|_2 < \infty\},$$

in which $\|\varphi\|_2 = (\int_X ||\varphi(x)||^2 d\mu)^{\frac{1}{2}}$. The synthesis operator, analysis operator and frame operator are defined, respectively, by

$$T_f : L^2(X, \mu) \to H$$

$$< T_f\varphi, h >= \int_X \varphi(x) < f(x), h > d\mu(x), \tag{1.7}$$

$$T_f^* : H \to L^2(X, \mu)$$

$$(T_f^* h)(x) =< h, f(x) >, \quad x \in X, \tag{1.8}$$

$$S_f = T_f T_f^*. \tag{1.9}$$

Also by Theorem 2.5. in [14], S_f is positive, self-adjoint and invertible.
We need the following theorems and the proofs can be found in [14].

Theorem 1.4 *Let f be a continuous frame for H with the frame operator S_f and let $V : H \to K$ be a bounded and invertible operator. Then $V \circ f$ is a continuous frame for K with the frame operator VS_fV^*.*

Theorem 1.5 *Let K be a closed subspace of H and let $P : H \to K$ be an orthogonal projection. Then the following holds:*
(i) If f is a continuous frame for H with bounds A and B, then Pf is a continuous frame for K with the bounds A and B.
(ii) If f is a continuous frame for K with the frame operator S_f, then for each $h, k \in H$

$$< Ph, k >= \int_X < h, S_f^{-1} f(x) >< f(x), k > d\mu(x).$$

The following lemmas and theorems can be found in operator theory text books [13, 16, 17, 18] which we shall use this work.

Lemma 1.6 *Let* $u : H \to K$ *be a bounded operator. Then*
(i) $\|u\| = \|u^*\|$ *and* $\|uu^*\| = \|u\|^2$.
(ii) R_u *is closed, if and only if,* R_{u^*} *is closed.*
(iii) u *is surjective, if and only if, there exists* $c > 0$ *such that for each* $h \in H$

$$c\|h\| \leq \|u^*(h)\|.$$

Lemma 1.7 *Let* u *be a self-adjoint bounded operator on* H. *Let*

$$m_u = \inf_{h \in H} < uh, h >, \quad M_u = \sup_{h \in H} < uh, h > .$$

Then, $m_u, M_u \in \sigma(u)$.

Theorem 1.8 *Let* $u : K \to H$ *be a bounded operator with closed range* R_u. *Then there exists a bounded operator* $u^\dagger : H \to K$ *for which* $uu^\dagger f = f$, $f \in R_u$.
Also, $u^* : H \to K$ *has closed range and* $(u^*)^\dagger = (u^\dagger)^*$. *The operator* $u^\dagger$ *is called the pseudo-inverse of* u.

Theorem 1.9 *Let* $u : K \to H$ *be a bounded surjective operator. Given* $y \in H$, *the equation* $ux = y$ *has a unique solution of minimal norm, namely,* $x = u^\dagger y$.

2. C-fusion frame

In this section we shall introduce the continuous version of fusion frames and we shall obtain some useful properties of it.

Definition 2.1 *Let* $F : X \to \mathbb{H}$ *be such that for each* $h \in H$, *the mapping* $x \mapsto \pi_{F(x)}(h)$ *is measurable (i.e. is weakly measurable), and let* $v : X \to [0, +\infty)$ *be a measurable mapping such that* $v \neq 0$ *a.e. We say that* (F, v) *is a c-fusion frame for* H *if there exist* $0 < A \leq B < \infty$ *such that for all* $h \in H$

$$A\|h\|^2 \leq \int_X v^2(x)\|\pi_{F(x)}\|^2 d\mu \leq B\|h\|^2. \tag{2.1}$$

(F, v) is called a tight c-fusion frame for H if A, B can be chosen so that $A = B$, and Parseval if $A = B = 1$. If we only have the upper bound, say call (F, v) is a Bessel c-fusion mapping for H.

Definition 2.2 *Let* $F : X \to \mathbb{H}$. *Let* $L^2(X, H, F)$ *be the class of all measurable mapping* $f : X \to H$ *such that for each* $x \in X$, $f(x) \in F(x)$ *and*

$$\int_X \|f(x)\|^2 d\mu < \infty.$$

It can be verified that $L^2(X,H,F)$ is a Hilbert space with inner product defined by

$$< f,g >= \int_X < f(x), g(x) > d\mu$$

for $f,g \in L^2(X,H,F)$.

Remark 2.3 *For brevity, we shall denote $L^2(X,H,F)$ by $L^2(X,F)$. Let (F,v) be a Bessel c-fusion mapping, $f \in L^2(X,F)$ and $h \in H$. Then:*

$$|\int_X v(x) < f(x), h > d\mu| = |\int_X v(x) < \pi_{F(x)}(f(x)), h > d\mu|$$

$$= |\int_X v(x) < f(x), \pi_{F(x)}(h) > d\mu| \leq \int_X v(x)\|f(x)\|.\|\pi_{F(x)}(h)\|d\mu$$

$$\leq (\int_X \|f(x)\|^2 d\mu)^{1/2} (\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu)^{1/2}$$

$$\leq B^{1/2}\|h\|(\int_X \|f(x)\|^2 d\mu)^{1/2}.$$

So we may define put forth the following definition.

Definition 2.4 *Let (F,v) be a Bessel c-fusion mapping for H. We define the c-fusion pre-frame operator (synthesis operator) $T_F : L^2(X,F) \to H$ by*

$$< T_F(f), h >= \int_X v(x) < f(x), h > d\mu, \tag{2.2}$$

where $f \in L^2(X,F)$ and $h \in H$.

By the Remark 2.3, $T_F : L^2(X,F) \to H$ is a bounded linear mapping. Its adjoint

$$T_F^* : H \to L^2(X,F)$$

will be called a c-fusion analysis operator, and $S_F = T_F \circ T_F^*$ will be called a c-fusion frame operator. The representation space in this setting is $L^2(X,F)$.

Remark 2.5 *Let (F,v) be a Bessel c-fusion mapping for H. Then $T_F : L^2(X,F) \to H$ is indeed a vector-valued integral, which for $f \in L^2(X,F)$ we shall put*

$$T_F(f) = \int_X vf d\mu \tag{2.3}$$

where

$$< \int_X vf d\mu, h >=< \int_X v(x) < f(x), h > d\mu, h \in H.$$

For each $h \in H$ *and* $f \in L^2(X,F)$, *we have*

$$< T_F^*(h), f >=< h, T_F(f) >= \int_X v(x) < h, f(x) > d\mu$$

$$= \int_X v(x) < \pi_{F(x)}(h), f(x) > d\mu =< v\pi_F(h), f > .$$

Hence for all $h \in H$

$$T_F^*(h) = v\pi_F(h). \tag{2.4}$$

So $T_F^* = v\pi_F$.

Remark 2.6 *A* c*-fusion frame is indeed a generalization of fusion frame. In definition 2.1, if we put* $X = I$ *and* μ *be the counting measure, then* F *is a fusion frame according to the definition 1.2. Also with this hypothesis,* $L^2(X,F)$ *changes to* $(\sum_{i\in I} \oplus W_i)_{l_2}$ *the representation space of fusion frame.*

Definition 2.7 *Let* (F,v) *and* (G,v) *be Bessel* c*-fusion mappings for* H. *We say* (F,v) *and* (G,v) *are weakly equal if* $T_F^* = T_G^*$, *which is equivalent to* $v\pi_F(h) = v\pi_G(h)$, *i.e. for all* $h \in H$. *Since,* $v \neq 0$ *i.e.,* (F,v) *and* (G,v) *are weakly equal if* $\pi_F(h) = \pi_G(h)$, *i.e. for all* $h \in H$.

Remark 2.8 *Let* $T_F = 0$. *Now, let* $O : X \to \mathbb{H}$ *be defined by* $O(x) = \{0\}$, *for almost all* $x \in X$. *Then* (O,v) *is a Bessel* c*-fusion mapping and* $T_O = 0$. *Let* $h \in H$. *Since* $v\pi_F(h) \in L^2(X,F)$, *so*

$$\int_X v^2(x) < \pi_{F(x)}(h), \pi_{F(x)}(h) > d\mu$$

$$= \int_X v(x) < v(x)\pi_{F(x)}(h), h > d\mu =< T_F(v\pi_F(h)), h >= 0.$$

Thus $\pi_{F(x)}(h) = 0$, *i.e. Therefore,* $\pi_F(h) = \pi_O(h)$, *a.e. Hence* (F,v) *and* (O,v) *are weakly equal.*

Definition 2.9 *For any Bessel* c*-fusion mapping* (F,v) *for* H, *we shall denote*

$$A_{F,v} = \inf_{h\in H_1} \|v\pi_F(h)\|^2, \tag{2.5}$$

$$B_{F,v} = \sup_{h\in H_1} \|v\pi_F(h)\|^2 = \|v\pi_F\|^2. \tag{2.6}$$

Remark 2.10 *Let* (F,v) *be a Bessel* c*-fusion mapping for* H. *Since, for each* $h \in H$

$$< T_F T_F^*(h), h >= \|v\pi_F(h)\|^2 = \int_X v^2(x)\|\pi_{F(x)}\|^2 d\mu,$$

$A_{F,v}$ *and* $B_{F,v}$ *are optimal scalars which satisfy*

$$A_{F,v} \leq T_F T_F^* \leq B_{F,v}.$$

In other words $A_{F,v}$ *is the spermum of all positive numbers* A, *and* $B_{F,v}$ *is the infimum of all positive numbers* B *which satisfies in 2.1. So* (F,v) *is a* c*-fusion frame for* H *if and only if* $A_{F,v} > 0$.

Lemma 2.11 *Let (F,v) be a Parseval c-fusion frame for H. Then $T_F^*T_F$ is the orthogonal projection of $L^2(X,F)$ onto $T_F^*(H)$.*

Proof. By remark 2.5 we have $T_F^*(h) = v\pi_F(h)$. Thus

$$||T_F^*(h)||^2 = ||v\pi_F(h)||^2 = \int_X ||v(x)\pi_{F(x)}(h)||^2 d\mu$$

$$= \int_X v^2(x)||\pi_{F(x)}(h)||^2 d\mu = ||h||^2.$$

Thus T_F^* is an isometry. So we can embed H into $L^2(X,F)$ by identifying H with $T_F^*(H)$. Let $P: L^2(X,F) \to T_F^*(H)$ be the orthogonal projection. For each $f \in L^2(X,F)$ and $h \in H$ we have:

$$< Pf, T_F^*(h) >=< f, PT_F^*(h) >=< f, T_F^*(h) >$$

$$=< T_F(f), h >=< T_F^*T_F(f), T_F^*(h) > .$$

Thus

$$Pf - T_F^*T_F(f) \perp T_F^*(H).$$

But $ran(P) = T_F^*(H)$, hence $P = T_F^*T_F$. □

Example 2.12 *Let $X = [-1,1]$ with the Lebesgue measure μ and let H_n be a n-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^n$. For each $x \in X$, let*

$$F(x) = \Big\{ \lambda \sum_{i=0}^{n-1} x^i e_{i+1} : \lambda \in \mathbb{C} \Big\} \quad and \quad v(x) = \sqrt{\sum_{i=1}^{n} x^{2i-2}}.$$

Then $F: X \to \mathbb{H}$ and (F,v) is a c-fusion frame for H. For the proof of the example we refer the reader to [15].

3. Erasures of subspaces and perturbation of C-fusion frames

Theorem 3.1 *Let (F,v) be a c-fusion frame for H with bounds C and D, and let $Y \subseteq X$ be measurable. Then the following assertions are satisfied:*
(i) If $\int_Y v^2(x)d\mu > D$ then $\cap_{x\in Y} F(x) = \{0\}$.
(ii) If $\int_Y v^2(x)d\mu = D$ then $\cap_{x\in Y} F(x) \perp span\{F(x)\}_{x\in X-Y}$ a.e.
(iii) If $c = \int_Y v^2(x)d\mu < C$, then $F: X - Y \to \mathbb{H}$ is a c-fusion frame with bounds $C - c$ and D.

Proof. (i) Suppose $h \in \cap_{x\in Y} F(x)$, then $\pi_{F(x)}(h) = h$ for all $x \in Y$. We have

$$D\|h\|^2 < \|h\|^2(\int_Y v^2(x)d\mu) = \int_Y \|h\|^2 v^2(x)d\mu$$

$$= \int_Y \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu$$

$$\leq \int_Y \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu + \int_{X-Y} \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu$$

$$= \int_X \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu \leq D\|h\|^2,$$

hence $h = 0$.
(ii) If $\int_Y v^2(x)d\mu = D$ and $h \in \cap_{x\in Y} F(x)$ then

$$D\|h\|^2 = \int_Y \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu$$

$$\leq \int_Y \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu + \int_{X-Y} \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu$$

$$\leq D\|h\|^2,$$

thus

$$D\|h\|^2 + \int_{X-Y} \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu \leq D\|h\|^2.$$

Hence

$$\int_{X-Y} \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu \leq 0,$$

therefore we have $\pi_{F(x)}(h) = 0$, for all $x \in X - Y$ $(a.e.)$ and we conclude that $h \perp F(x)$ for all $x \in X - Y$ $(a.e.)$. Thus $h \perp span\{F(x)\}_{x\in X-Y}$ $(a.e.)$ and we get

$$\cap_{x\in Y} F(x) \perp span\{F(x)\}_{x\in X-Y}.$$

(iii) For all $h \in H$ we have

$$\int_{X-Y} \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu$$

$$= \int_X \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu - \int_Y \|\pi_{F(x)}(h)\|^2 v^2(x) d\mu$$

$$\geq C\|h\|^2 - \|h\|^2 \int_Y v^2(x) d\mu = (C - c)\|h\|^2.$$

The upper bound is obvious. □

The following corollary immediately follows from the Theorem 3.1

Corollary 3.2 *Let (F, v) be a c-fusion frame for H with bounds C and D, and let $Y \subseteq X$ be measurable. Then the following statements are equivalent:*
(i) $c = \int_Y v^2(x)d\mu < C$.
(ii) $F : (X - Y) \to \mathbb{H}$ *is a c-fusion frame with bounds $C - c$ and D.*

Definition 3.3 *For any* $X_\circ \subseteq X$ *measurable, we define*

$$D_{X_\circ} : L^2(X, F) \to L^2(X, F)$$

$$D_{X_\circ}(f)(x) = \begin{cases} f(x) & \text{if } x \in X_\circ \\ 0 & \text{if } x \in X - X_\circ \end{cases}$$

for all $f \in L^2(X, F)$.

Definition 3.4 *Let* (F, v) *be a* c*-fusion frame with the pre-frame operator* T_F. *We define the* c*-erasure reconstruction error* $\xi_1(F)$ *to be*

$$\xi_1(F) = \sup\{\|T_F D_{X_\circ} T_F^*\| : X_\circ \subseteq X\}, \tag{3.1}$$

where $X_\circ$ *is measurable.*

Theorem 3.5 *Let* $v \in L^2(X)$ *and let* (F, v) *be a Parseval* c*-fusion frame with* c*-erasure reconstruction error* $\xi_1(F)$. *Then* $D_{X_\circ}(f) \in L^2(X, F)$ *and*

$$\xi_1(F) \leq \|v\|_2. \tag{3.2}$$

Proof. Since

$$\|D_{X_\circ}\| = \sup\{\|D_{X_\circ}(f)\| : \|f\| = 1 \quad \textit{and} \quad f \in L^2(X, F)\},$$

$$\|D_{X_\circ}(f)\|^2 = \int_X |D_{X_\circ}(f)(x)|^2 d\mu = \int_{X_\circ} |f(x)|^2 d\mu \leq \|f\|^2,$$

so $\|D_{X_\circ}\| \leq 1$.
Choose $X_\circ \subseteq X$ measurable, and fix it. By the Remark 2.5 we have

$$\|T_F D_{X_\circ} T_F^*\| = \sup_{h \in H_1} \|T_F D_{X_\circ} T_F^*(h)\|$$

$$= \sup_{h \in H_1} \|T_F D_{X_\circ}(v\pi_F(h))\| = \sup_{h \in H_1} \sup_{k \in H_1} | < T_F D_{X_\circ}(v\pi_F(h)), k > |$$

$$= \sup_{h \in H_1} \sup_{k \in H_1} | \int_{X_\circ} < v^2(x)\pi_{F(x)}(h)), k > d\mu|$$

$$\leq \sup_{h \in H_1} (\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu)^{1/2} (\int_X v^2(x) d\mu)^{1/2}$$

$$= \sup_{h \in H_1} \|h\| (\int_X v^2(x) d\mu)^{1/2} = \|v\|_2.$$

Since $X_\circ \subseteq X$ is arbitrary

$$\xi_1(F) \leq \|v\|_2.$$

□

Example 3.6 *Let* $X = [0,1]$ *and* μ *be the Lebesgue measure. Suppose that* $v(x) = \sqrt{2}$ *For all* $x \in X$, *then* v *is positive and measurable. For the Hilbert space* $H = \mathbb{C}$, *we put* $\mathbb{H} = \{W_1, W_2\}$ *which* $W_1 = span\{(1,0)\}$ *and* $W_2 = span\{(0,1)\}$. *If we define*

$$F : [0,1] \to \mathbb{H}$$

$$F(x) = \begin{cases} W_1 & \text{if} \quad 0 \leq x < \frac{1}{2}, \\ W_2 & \text{if} \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

then F *is weakly measurable and we have*

$$\int_0^1 v^2(x) \|\pi_{F(x)}(h)\|^2 d\mu$$

$$= \int_0^{1/2} 2a^2 dx + \int_{1/2}^1 2b^2 dx = a^2 + b^2 = \|h\|^2.$$

Hence (F, v) *is a Parseval* c*-fusion frame for* H. *Now by Theorem 3.5, we have* $\xi_1(F) \leq 2$.

According to the construction of fusion frame systems in [5], we introduce $\xi_1(F)$ for such c-fusion frame systems.

Definition 3.7 *Let* (X, μ) *and* (Y, λ) *be two measure spaces. Let* $f : X \times Y \to H$ *and* $F : X \to \mathbb{H}$. *Let for all* $x \in X$, $f(x,.) : Y \to F(x)$ *be a* c*-frame for* $F(x)$. *Then* (F, f, v) *is called a system of local* c*-frames. Also,* (F, f, v) *is called a* c*-fusion frame system if* (F, v) *is a* c*-fusion frame.*

Example 3.8 *Let* $X = Y = [0,1]$ *and* $\mu = \lambda$ *be the Lebesgue measure. Suppose that* $v(x) = e^{\frac{x}{2}}$ *For all* $x \in X$, *then* v *is positive and measurable. For Hilbert space* $H = \mathbb{C}$, *we put* $\mathbb{H} = \{W_1, W_2\}$ *which* $W_1 = span\{(1,1)\}$ *and* $W_2 = span\{(1,-1)\}$. *If we define*

$$F : [0,1] \to \mathbb{H}$$

$$F(x) = \begin{cases} W_1 & \text{if} \quad 0 \leq x < \frac{1}{2}, \\ W_2 & \text{if} \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

then F *is weakly measurable and* (F, v) *is a* c*-fusion frame for* H *with upper bound* $e - 1$ *and lower bound* $e^{\frac{1}{2}} - 1$. *If we define*

$$f : [0,1] \times [0,1] \to H$$

$$f(x,y) = x(1,1) + y(1,-1).$$

It is easy to show that f *is a* c*-frame for* H *with lower bound* $1/6$ *and upper bound* $\frac{7}{6}$. *to get a* c*-fusion frame system, for* $0 \leq x < \frac{1}{2}$ *we define*

$$f(x,.) : Y = [0,1] \to F(x) = span\{(1,1)\}$$

$$f(x,.)(y) = y(1,1),$$

and for $\frac{1}{2} \leq x \leq 1$ *we define*

$$f(x,.) : Y = [0,1] \to F(x) = span\{(1,-1)\}$$

$$f(x,.)(y) = y(1,-1).$$

Then for all $x \in [0,1]$, $f(x,.)$ *is a tight* c*-frame for* $F(x)$ *with* $A = B = \frac{2}{3}$. *Thus* (F,f,v) *is a* c*-fusion frame system for* $H = \mathbb{C}$.

Theorem 3.9 *Let* (X,μ) *and* (Y,λ) *be two* σ*-finite measure spaces. Let* $f : X \times Y \to H$, $F : X \to \mathbb{H}$ *be weakly measurable mappings. Let for all* $x \in X$, $f(x,.) : Y \to F(x)$ *be weakly measurable and for every* $x \in X$, $f(x,.)$ *be a continuous frame for* $F(x)$. *Let*

$$0 < A(x) = \inf_{h \in F(x)_1} \int_Y | < f(x,y), h > |^2 d\lambda$$

$$\leq \sup_{h \in F(x)_1} \int_Y | < f(x,y), h > |^2 d\lambda = B(x) < \infty,$$

and let

$$0 < A = \inf_x A(x) \leq \sup_x B(x) = B < \infty.$$

Then, (F,v) *is a* c*-fusion frame for* H *if and only if*

$$v.f : X \times Y \to H,$$

$$(x,y) \mapsto v(x)f(x,y)$$

is a continuous frame for H.

Proof. See [10]. $\square$

We denote the synthesis and analysis operator for $f(x,.)$ by T_x and T_x^*, respectively.

Definition 3.10 *Let* (F,f,v) *be a system of local* c*-frames. Let mappings* $x \mapsto T_x$, *of* X *into* $B(L^2(Y,\lambda),H)$ *and* $x \mapsto T_x^*$, *of* X *into* $B(H,L^2(Y,\lambda))$, *be measurable. We define the* c*-erasure reconstruction error* $\xi_1(F)$ *for the system of local* c*-frames* (F,f,v) *as follows:*

$$\xi_1(F) \tag{3.3}$$

$$= \sup\{\int_X v^2(x) \|T_x D_{Y_\circ} T_x^*\| d\mu : Y_\circ \subseteq Y, \quad measurable\}$$

Theorem 3.11 *With the hypothesis of the Theorem 3.9, let* $v.f : X \times Y \to H$, *be a* c*-frame for* H *with upper bound* B. *Let the mappings,* $x \mapsto T_x$, *of* X *into* $B(L^2(Y,\lambda),H)$ *and,* $x \mapsto T_x^*$, *of* X *into* $B(H,L^2(Y,\lambda))$ *be measurable. Then*

$$\xi_1(F) \leq B.$$

Proof. Choose $Y_\circ \subseteq Y$ and fix it. By the definition of synthesis and analysis operator for continuous frames, we have

$$\|T_x D_{Y_\circ} T_x^*\| \leq \|T_x\|\|T_x^*\| = \|T_x^*\|^2$$

$$= \sup_{h \in F(x)_1} \|T_x^*(h)\|^2 = \sup_{h \in F(x)_1} \int_Y |< h, f(x,y) >|^2 d\lambda$$

$$\leq \sup_{h \in H_1} \int_Y |< h, f(x,y) >|^2 d\lambda.$$

Hence

$$\int_X v^2(x)\|T_x D_{Y_\circ} T_x^*\| d\mu(x)$$

$$\leq \sup_{h \in H_1} \int_X \int_Y v^2(x)| < f(x,y), h > |^2 d\lambda(y) d\mu(x)$$

$$\leq \sup_{h \in H_1} \int_X \int_Y | < v(x)f(x,y), h > |^2 d\lambda(y) d\mu(x)$$

$$= \sup_{h \in H_1} \int_{X \times Y} | < v(x)f(x,y), h > |^2 d(\mu \times \lambda)$$

$$\leq \sup_{h \in H_1} B\|h\|^2 = B.$$

Since $Y_\circ \subseteq Y$ is arbitrary, thus

$$\xi_1(F) \leq B.$$

□

Example 3.12 *Let $X = Y = [0,1]$ and $\mu = \lambda$ be the Lebesgue measure. Let (F, v) be the c-fusion frame for $H = \mathbb{C}$ defined in the example 3.6 and we define*

$$v.f : [0,1] \times [0,1] \to H$$

$$v.f(x,y) = \sqrt{2}(x(1,0) + y(0,1)).$$

Then $v.f$ is a c-frame for H with lower bound $\frac{1}{6}$ and upper bound $\frac{7}{6}$. For $0 \leq x < \frac{1}{2}$ we define

$$f(x,.) : Y = [0,1] \to F(x) = span\{(1,0)\}$$

$$f(x,.)(y) = y(1,0),$$

and for $\frac{1}{2} \leq x \leq 1$ we define

$$f(x,.) : Y = [0,1] \to F(x) = span\{(0,1)\}$$

$$f(x,.)(y) = y(0,1).$$

Then $f(x,.)$ for any $x \in [0,1]$, is a tight c-frame for $F(x)$ with $A = B = \frac{1}{3}$. So by the Theorem 3.11 we have

$$\xi_1(F) \leq \frac{7}{6}.$$

Definition 3.13 *Let $F : X \to \mathbb{H}$, $\tilde{F} : X \to \mathbb{H}$. Let $0 \le \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$. We say that $(\tilde{F}, v)$ is a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of (F, v) if for all $h \in H$ and $x \in X$*

$$\|(\pi_{F(x)} - \pi_{\tilde{F}(x)})(h)\| \le \lambda_1 \|(\pi_{F(x)}(h)\| + \lambda_2 \|\pi_{\tilde{F}(x)})(h)\| + \varepsilon \|h\|.$$

Theorem 3.14 *Let (F, v) be a c-fusion frame for H with bounds C and D, and let $v \in L^2(X)$. Choose $0 \le \lambda_1 < 1$ and $\varepsilon > 0$ such that*

$$(1 - \lambda_1)\sqrt{C} - \varepsilon(\int_X v^2(x) d\mu)^{1/2} > 0.$$

Let $\tilde{F} : X \to \mathbb{H}$ be weakly measurable. Further, let $(\tilde{F}, v)$ be a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of (F, v) for some $0 \le \lambda_2 < 1$. Then $(\tilde{F}, v)$ is c-fusion frame for H with bounds

$$[\frac{(1 - \lambda_1)\sqrt{C} - \varepsilon(\int_X v^2(x) d\mu)^{1/2}}{1 + \lambda_2}]^2$$

and

$$[\frac{(1 + \lambda_2)\sqrt{D} - \varepsilon(\int_X v^2(x) d\mu)^{1/2}}{1 - \lambda_2}]^2.$$

Proof. We first prove the upper bound. For any $h \in H$, we get,

$$[\int_X v^2(x) \|\pi_{\tilde{F}(x)}(h)\|^2 d\mu]^{1/2}$$

$$\le [\int_X v^2(x) (\|\pi_{F(x)}(h) - \pi_{\tilde{F}(x)}(h)\| + \|\pi_{F(x)}(h)\|)^2 d\mu]^{1/2}$$

$$\le [\int_X v^2(x) (\|\pi_{F(x)}(h)\| + \lambda_1 \|\pi_{F(x)}(h)\| + \lambda_2 \|\pi_{\tilde{F}(x)}(h)\| + \varepsilon \|h\|)^2 d\mu]^{1/2}$$

$$= [\int_X ((1 + \lambda_1) v(x) \|\pi_{F(x)}(h)\| + \lambda_2 v(x) \|\pi_{\tilde{F}(x)}(h)\| + \varepsilon v(x) \|h\|)^2 d\mu]^{1/2}$$

$$\le [(1 + \lambda_1)^2 \int_X (v^2(x) \|\pi_{F(x)}(h)\|^2 d\mu]^{1/2}$$

$$+ [\lambda_2^2 \int_X v^2(x) \|\pi_{\tilde{F}(x)}(h)\|^2 d\mu]^{1/2} + [\varepsilon^2 \int_X v^2(x) \|h\|^2 d\mu]^{1/2}.$$

Thus

$$(1 - \lambda_2)[\int_X v^2(x) \|\pi_{\tilde{F}(x)}(h)\|^2 d\mu]^{1/2}$$

$$\le (1 + \lambda_1)[\int_X (v^2(x) \|\pi_{F(x)}(h)\|^2 d\mu]^{1/2} + \varepsilon \|h\| [\int_X v^2(x) d\mu]^{1/2}$$

$$\le [(1 + \lambda_1)\sqrt{D} + \varepsilon(\int_X v^2(x) d\mu)^{1/2}] \|h\|.$$

Hence

$$\int_X v^2(x)\|\pi_{\tilde{F}(x)}(h)\|^2 d\mu \leq [\frac{(1+\lambda_1)\sqrt{D}+\varepsilon(\int_X v^2(x)d\mu)^{1/2}}{1-\lambda_2}]^2\|h\|^2.$$

To prove the lower bound, for all $h \in H$ we have

$$[\int_X v^2(x)\|\pi_{\tilde{F}(x)}(h)\|^2 d\mu]^2$$

$$\geq [\int_X v^2(x)(\|\pi_{F(x)}(h)\| - \|\pi_{F(x)}(h) - \pi_{\tilde{F}(x)}(h)\|)^2 d\mu]^{1/2}$$

$$\geq [\int_X v^2(x)(\|\pi_{F(x)}(h)\| - \lambda_1\|\pi_{F(x)}(h)\| - \lambda_2\|\pi_{\tilde{F}(x)}(h)\| - \varepsilon\|h\|)^2 d\mu]^{1/2}$$

$$= [\int_X ((1-\lambda_1)v(x)\|\pi_{F(x)}(h)\| - \lambda_2 v(x)\|\pi_{\tilde{F}(x)}(h)\| - \varepsilon v(x)\|h\|)^2 d\mu]^{1/2}$$

$$\geq [\int_X (1-\lambda_1)^2 v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu]^{1/2}$$

$$-[\int_X \lambda_2^2 v^2(x)\|\pi_{\tilde{F}(x)}(h)\|^2 d\mu]^{1/2} - [\int_X \varepsilon^2 v^2(x)\|h\|^2 d\mu]^{1/2}.$$

Thus

$$(1+\lambda_2)[\int_X v^2(x)\|\pi_{\tilde{F}(x)}(h)\|^2 d\mu]^{1/2}$$

$$\geq (1-\lambda_1)[\int_X v^2(x)\|\pi_{F(x)}(h)\|^2 d\mu]^{1/2} - \varepsilon[\int_X v^2(x)d\mu]^{1/2}\|h\|$$

$$= [(1-\lambda_1)\sqrt{C} - \varepsilon[\int_X v^2(x)d\mu]^{1/2}]\|h\|.$$

Hence

$$\int_X v^2(x)\|\pi_{\tilde{F}(x)}(h)\|^2 d\mu \geq [\frac{(1-\lambda_1)\sqrt{C}+\varepsilon(\int_X v^2(x)d\mu)^{1/2}}{1+\lambda_2}]^2\|h\|^2.$$

This completes the proof. $\square$

Example 3.15 *Let* $X = [0,1]$ *and* μ *be the Lebesgue measure. Suppose that* $v(x) = \frac{1}{10}$ *For all* $x \in X$, *then* v *is positive and measurable. For Hilbert space* $H = \mathbb{C}$, *Let* $\mathbb{H} = \{W_1, W_2, \widetilde{W_1}, \widetilde{W_2}\}$ *which* $W_1 = span\{(1,0)\}$, $W_2 = span\{(0,1)\}$, $\widetilde{W_1} = span\{(1,0.1)\}$ *and* $\widetilde{W_2} = span\{(0.1,1)\}$. *If we define*

$$F : [0,1] \to \mathbb{H}$$

$$F(x) = \begin{cases} W_1 & \text{if} \quad 0 \leq x < \frac{1}{2}, \\ W_2 & \text{if} \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

then by Example 3.6, (F,v) *is a tight c-fusion frame for* $H=\mathbb{C}$ *with* $C=D=\frac{1}{200}$*. Now we define*

$$\widetilde{F}:[0,1]\to\mathbb{H}$$

$$\widetilde{F}(x)=\begin{cases}\widetilde{W_1} & \text{if} \quad 0\le x<\frac{1}{2},\\ \widetilde{W_2} & \text{if} \quad \frac{1}{2}\le x\le 1.\end{cases}$$

It is clear that $\widetilde{F}$ *is weakly measurable. For* $h=(a,b)\in H$ *we have*

$$h=\frac{a-0.1b}{0.99}(1,0.1)+\frac{b-0.1a}{0.99}(0.1,1).$$

Hence for $0\le x<\frac{1}{2}$,

$$\begin{aligned}\|(\pi_{F(x)}-\pi_{\tilde{F}(x)})(h)\| &= \|(\frac{-0.01a+0.1b}{0.99},\frac{0.01b-0.1a}{0.99})\|\\ &\le \frac{1}{99}|a|+\frac{1}{\sqrt{101}}|a-0.1b|\sqrt{1.01}+\frac{1}{9.9}\sqrt{a^2+b^2},\end{aligned}$$

thus

$$\|(\pi_{F(x)}-\pi_{\tilde{F}(x)})(h)\| \tag{3.4}$$

$$\le \frac{1}{99}\|(\pi_{F(x)}(h)\|+\frac{1}{\sqrt{101}}\|\pi_{\tilde{F}(x)})(h)\|+\frac{1}{9.9}\|h\|.$$

Similarly, 3.4 hold for $\frac{1}{2}\le x\le 1$*. It is easy to show that*

$$(1-\lambda_1)\sqrt{C}-\varepsilon(\int_X v^2(x)d\mu)^{\frac{1}{2}}>0.$$

Thus by Theorem 3.14, $(\widetilde{F},v)$ *is a c-fusion frame for* H *with respectively, lower and upper bounds*

$$\frac{(1-\frac{1}{99}).\frac{1}{\sqrt{200}}-\frac{1}{99}}{1+\frac{1}{\sqrt{101}}},\quad \frac{(1+\frac{1}{\sqrt{101}}).\frac{1}{\sqrt{200}}-\frac{1}{99}}{1-\frac{1}{\sqrt{101}}}.$$

Definition 3.16 *Let* f *be a continuous frame for* H*, where* μ *is* $\sigma-finite$*. Let* $\tilde{f}:X\to H$ *be a weakly-measurable vector-valued function and assume that there exist constants* $\lambda_1,\lambda_2,\gamma\ge 0$ *such that* $\max(\lambda_1+\frac{\gamma}{\sqrt{A}},\lambda_2)<1$ *and*

$$|\int_X \varphi(x)<f(x)-\tilde{f}(x),h>d\mu(x)|\le\lambda_1|\int_X\varphi(x)<f(x),h>d\mu(x)|$$

$$+\lambda_2|\int_X\varphi(x)<\tilde{f}(x),h>d\mu(x)|+\gamma\|\varphi\|_2 \tag{3.5}$$

for all $\varphi\in L^2(X,\mu)$ *and for all* $h\in H_1$*. Then* $\tilde{f}$ *is called the* $(\lambda_1,\lambda_2,\gamma)$ *perturbation of* f*.*

Theorem 3.17 *Let $\tilde{f}$ be a $(\lambda_1, \lambda_2, \gamma)$ perturbation of f. Then $\tilde{f} : X \to H$ is a continuous frame for H with the bounds*

$$A[\frac{1-(\lambda_1 + \frac{\gamma}{\sqrt{A}})}{1+\lambda_2}]^2 \quad and \quad B[\frac{1+\lambda_1 + \frac{\gamma}{\sqrt{B}}}{1-\lambda_2}]^2,$$

where A, B are frame bounds for f.

Proof. See [14]. □

Proposition 3.18 *Let $f : X \to H$ and let $\tilde{f} : X \to H$ be two weakly-measurable mappings. Let f be a continuous frame for $\overline{span}\{f(x)\}_{x\in X}$ and let $\tilde{f}$ be a $(\lambda_1, \lambda_2, \gamma)$ perturbation of f.*

(i) $\tilde{f}$ is a continuous frame for $\overline{span}\{\tilde{f}(x)\}_{x\in X}$ and for all $\varphi \in L^2(X, \mu)$ and for all h in the unit sphere in H we have

$$|\int_X \varphi(x) < f(x), h > d\mu(x)|$$

$$\leq \frac{1+\lambda_2}{1-\lambda_1}|\int_X \varphi(x) < \tilde{f}(x), h > d\mu(x)| + \frac{\gamma}{1-\lambda_1}\|\varphi\|_2, \tag{3.6}$$

$$|\int_X \varphi(x) < f(x), h > d\mu(x)|$$

$$\geq \frac{1-\lambda_1}{1+\lambda_2}|\int_X \varphi(x) < \tilde{f}(x), h > d\mu(x)| - \frac{\gamma}{1+\lambda_2}\|\varphi\|_2. \tag{3.7}$$

(ii) Let π_f be the orthogonal projection of H onto $\overline{span}\{f(x)\}_{x\in X}$ and $\pi_{\tilde{f}}$ be the orthogonal projection of H onto $\overline{span}\{\tilde{f}(x)\}_{x\in X}$. For all k in H we have

$$\|\pi_f(\pi_{\tilde{f}}(k))\|^2 \geq \frac{1-\lambda_1}{N(1+\lambda_2)}\|\pi_{\tilde{f}}(k)\|^2 - \frac{M}{N}\|\varphi_{k\tilde{f}}\|_2,$$

which $M = \gamma(1 + \frac{1}{1-\lambda_1} + \frac{1}{1+\lambda_2})$, $N = 1 + \lambda_1\frac{1+\lambda_2}{1-\lambda_1} + \lambda_2$ and $\varphi_{k\tilde{f}} = T^*_{S^{-1}_{\tilde{f}}\circ\tilde{f}}(k)$.

Proof. (i) By Theorem 3.17, $\tilde{f}$ is a continuous frame for $\overline{span}\{\tilde{f}(x)\}_{x\in X}$. For all $\varphi \in L^2(X, \mu)$ and for all $h \in H_1$ we have

$$|\int_X \varphi(x) < f(x), h > d\mu(x)| = |\int_X \varphi(x) < f(x) - \tilde{f}(x) + \tilde{f}(x), h > d\mu(x)|$$

$$\leq |\int_X \varphi(x) < f(x) - \tilde{f}(x), h > d\mu(x)| + |\int_X \varphi(x) < \tilde{f}(x), h > d\mu(x)|$$

$$\leq |\int_X \varphi(x) < \tilde{f}(x), h > d\mu(x)|$$

$$+\lambda_1|\int_X \varphi(x) < f(x),h > d\mu(x)| + \lambda_2|\int_X \varphi(x) < \tilde{f}(x),h > d\mu(x)| + \gamma\|\varphi\|_2.$$

Hence

$$|\int_X \varphi(x) < f(x),h > d\mu(x)|$$

$$\leq \frac{1+\lambda_2}{1-\lambda_1}|\int_X \varphi(x) < \tilde{f}(x),h > d\mu(x)| + \frac{\gamma}{1-\lambda_1}\|\varphi\|_2.$$

By replacing f with $\tilde{f}$ in the above argument we have

$$|\int_X \varphi(x) < f(x),h > d\mu(x)|$$

$$\geq \frac{1-\lambda_1}{1+\lambda_2}|\int_X \varphi(x) < \tilde{f}(x),h > d\mu(x)| - \frac{\gamma}{1+\lambda_2}\|\varphi\|_2.$$

(ii) Since $\tilde{f}$ is a c-frame for $\overline{span}\{\tilde{f}(x)\}_{x\in X}$, thus $S_{\tilde{f}}$ is bounded and invertible. Hence by Theorem 1.4, $S_{\tilde{f}}^{-1}\circ\tilde{f}$ is a c-frame for $S_{\tilde{f}}^{-1}(\overline{span}\{\tilde{f}(x)\}_{x\in X})$ and for all $k\in S_{\tilde{f}}^{-1}(\overline{span}\{\tilde{f}(x)\}_{x\in X})$, $T^*_{S_{\tilde{f}}^{-1}\circ\tilde{f}}(k)\in L^2(X,\mu)$. Fix $k\in S_{\tilde{f}}^{-1}(\overline{span}\{\tilde{f}(x)\}_{x\in X})$ and let

$$\varphi_{k\tilde{f}}(x) = (T^*_{S_{\tilde{f}}^{-1}\circ\tilde{f}}(k))(x) =< k, S_{\tilde{f}}^{-1}(\tilde{f}(x)) > .$$

By the Theorem 1.5, for all $h\in H$ we have

$$|\int_X < k, S_{\tilde{f}}^{-1}(\tilde{f}(x)) >< f(x)-\tilde{f}(x),h > d\mu|$$

$$= |\int_X \varphi_{k\tilde{f}}(x) < f(x)-\tilde{f}(x),h > d\mu|$$

$$\leq \lambda_1|\int_X \varphi_{k\tilde{f}}(x) < f(x),h > d\mu| + \lambda_2|\int_X \varphi_{k\tilde{f}}(x) < \tilde{f}(x),h > d\mu| + \gamma\|\varphi_{k\tilde{f}}\|_2$$

$$\leq \lambda_1\frac{1+\lambda_2}{1-\lambda_1}|\int_X \varphi_{k\tilde{f}}(x) < \tilde{f}(x),h > d\mu|$$

$$+\lambda_2|\int_X \varphi_{k\tilde{f}}(x) < \tilde{f}(x),h > d\mu| + \frac{\gamma}{1-\lambda_1}\|\varphi_{k\tilde{f}}\|_2 + \gamma\|\varphi_{k\tilde{f}}\|_2$$

$$= (\lambda_1\frac{1+\lambda_2}{1-\lambda_1}+\lambda_2)|\int_X \varphi_{k\tilde{f}}(x) < \tilde{f}(x),h > d\mu| + (\frac{\gamma}{1-\lambda_1}+\gamma)\|\varphi_{k\tilde{f}}\|_2$$

$$= (\lambda_1\frac{1+\lambda_2}{1-\lambda_1}+\lambda_2)|\int_X < k, S_{\tilde{f}}^{-1}(\tilde{f}(x)) >< \tilde{f}(x),h > d\mu| + (\frac{\gamma}{1-\lambda_1}+\gamma)\|\varphi_{k\tilde{f}}\|_2$$

$$= (\lambda_1\frac{1+\lambda_2}{1-\lambda_1}+\lambda_2)| < \pi_{\tilde{f}}k,h > | + (\frac{\gamma}{1-\lambda_1}+\gamma)\|\varphi_{k\tilde{f}}\|_2.$$

Using the above inequality, for all $h \in H$ we obtain

$$|<\pi_f\pi_{\tilde{f}}k,h>| = |<\pi_{\tilde{f}}k,\pi_f h>|$$

$$=|\int_X <k,S_{\tilde{f}}^{-1}(\tilde{f}(x))><\tilde{f}(x),\pi_f h>d\mu|$$

$$\geq |\int_X <k,S_{\tilde{f}}^{-1}(\tilde{f}(x))><f(x),\pi_f h>d\mu|$$

$$-|\int_X <k,S_{\tilde{f}}^{-1}(\tilde{f}(x))><f(x)-\tilde{f}(x),\pi_f h>d\mu|$$

$$\geq |\int_X <k,S_{\tilde{f}}^{-1}(\tilde{f}(x))><f(x),h>d\mu|$$

$$-|\int_X <k,S_{\tilde{f}}^{-1}(\tilde{f}(x))><f(x)-\tilde{f}(x),\pi_f h>d\mu|$$

$$\geq \frac{1-\lambda_1}{1+\lambda_2}|<\pi_{\tilde{f}}k,h>|-(\lambda_1\frac{1+\lambda_2}{1-\lambda_1}+\lambda_2)|<\pi_{\tilde{f}}k,\pi_f h>|-M\|\varphi_{k\tilde{f}}\|_2.$$

Thus

$$|<\pi_f\pi_{\tilde{f}}k,h>| \geq \frac{1-\lambda_1}{N(1+\lambda_2)}|<\pi_{\tilde{f}}k,h>|-\frac{M}{N}\|\varphi_{k\tilde{f}}\|_2.$$

Since $h \in H$ was arbitrary we get

$$\|\pi_f(\pi_{\tilde{f}}k)\|^2 \geq \frac{1-\lambda_1}{N(1+\lambda_2)}\|\pi_{\tilde{f}}k\|^2-\frac{M}{N}\|\varphi_{k\tilde{f}}\|_2.$$

□

Remark 3.19 *Our argument in Proposition 3.18 is symmetric in π_f and $\pi_{\tilde{f}}$. Thus we have*

$$\|\pi_{\tilde{f}}(\pi_f k)\|^2 \geq \frac{1-\lambda_1}{N(1+\lambda_2)}\|\pi_f k\|^2-\frac{M}{N}\|\varphi_{kf}\|_2.$$

Theorem 3.20 *Let (F,f,v) be a c-fusion frame system for H with with bounds C and D and $v \in L^2(X)$. Choose $0 \leq \lambda_1,\lambda_2 < 1$ and $\varepsilon > 0$ such that*

$$1-\frac{\varepsilon^2}{2}=\frac{1-\lambda_1}{N(1+\lambda_2)} \quad \textit{and} \quad \sqrt{C}-\varepsilon(\int_X v^2(x)d\mu)^{1/2}>0.$$

Let $\tilde{F}: X \to H$. For every $x \in X$, let $\tilde{f}_x=\tilde{f}(x,.): Y \to \tilde{F}(x)$ be a $(\lambda_1,\lambda_2,\gamma)$ perturbation of $f_x=f(x,.)$. Let $R=\sup_{x\in X}\|S_{f_x}^{-1}\|<\infty$ and $\tilde{R}=\sup_{x\in X}\|S_{\tilde{f}_x}^{-1}\|<\infty$. Then $(\tilde{F},\tilde{f},v)$ is a c-fusion frame system for H with fusion frame bounds

$$[\sqrt{C}-\varepsilon(\int_X v^2(x)d\mu)^{1/2}]^2 \quad \textit{and} \quad [\sqrt{D}+\varepsilon(\int_X v^2(x)d\mu)^{1/2}]^2.$$

Proof. Fix $x \in X$ and choose $k \in H$. Recalling Proposition 3.18 we have

$$\|\pi_{F(x)}(k)\|^2 = \|\pi_{\tilde{F}(x)}(\pi_{F(x)})(k)\|^2 + \|(I - \pi_{\tilde{F}(x)})\pi_{F(x)}(k)\|^2$$

$$\geq (1 - \frac{\varepsilon^2}{2})\|\pi_{F(x)}(k)\|^2 + \|(I - \pi_{\tilde{F}(x)})\pi_{F(x)}(k)\|^2 - \frac{M}{N}\|\varphi_{kf_x}\|_2.$$

Hence

$$\|(I - \pi_{\tilde{F}(x)})\pi_{F(x)}(k)\|^2 \leq \frac{\varepsilon^2}{2}\|\pi_{F(x)}(k)\|^2 + \frac{M}{N}\|\varphi_{kf_x}\|_2.$$

By a similar argument we have

$$\|(I - \pi_{F(x)})\pi_{\tilde{F}(x)}(k)\|^2 \leq \frac{\varepsilon^2}{2}\|\pi_{\tilde{F}(x)}(k)\|^2 + \frac{M}{N}\|\varphi_{k\tilde{f}_x}\|_2.$$

Thus

$$\|(\pi_{F(x)} - \pi_{\tilde{F}(x)})(k)\|^2 = | < (\pi_{F(x)} - \pi_{\tilde{F}(x)})(k), (\pi_{F(x)} - \pi_{\tilde{F}(x)})(k) > |$$

$$= | < (\pi_{F(x)} - \pi_{\tilde{F}(x)})^2(k), k > |$$

$$= | < (\pi_{F(x)} - \pi_{\tilde{F}(x)}\pi_{F(x)} + \pi_{\tilde{F}(x)} - \pi_{F(x)}\pi_{\tilde{F}(x)})(k), k > |$$

$$\leq \|(I - \pi_{\tilde{F}(x)})(\pi_{F(x)}(k)) + (I - \pi_{F(x)})(\pi_{\tilde{F}(x)})(k)\|\|k\|$$

$$\leq [\|(I - \pi_{\tilde{F}(x)})(\pi_{F(x)}(k))\| + \|(I - \pi_{F(x)})(\pi_{\tilde{F}(x)})(k)\|]\|k\|$$

$$\leq [(\frac{\varepsilon^2}{2}\|\pi_{F(x)}(k)\|^2 + \frac{M}{N}\|\varphi_{kf_x}\|_2)^{1/2} + (\frac{\varepsilon^2}{2}\|\pi_{\tilde{F}(x)}(k)\|^2$$

$$+\frac{M}{N}\|\varphi_{k\tilde{f}_x}\|_2)^{1/2}]\|k\|. \tag{3.8}$$

Now by the Proposition 3.18 (ii), $\varphi_{k\tilde{f}_x}, \varphi_{kf_x} \in L^2(Y, \lambda)$ and by the Theorem 3.9 we have

$$\|\varphi_{kf_x}\|_2 = \int_X | < k, S_{f_x}^{-1}(f(x,y)) > |^2 d\lambda \leq B(x)\|S_{f_x}^{-1}(k)\|^2$$

$$\leq B\|S_{f_x}^{-1}\|\|k\|^2 \leq BR\|k\|^2.$$

Since for all $x \in X$, $\tilde{f}_x$ is c-frame thus, $\|\varphi_{k\tilde{f}_x}\|_2 \leq \tilde{B}\tilde{R}\|k\|^2$. Hence by inequality 3.8 we have

$$\|(\pi_{F(x)} - \pi_{\tilde{F}(x)})(k)\|^2$$

$$\leq [(\frac{\varepsilon^2}{2}\|k\|^2 + \frac{MBR}{N}\|k\|^2)^{1/2} + (\frac{\varepsilon^2}{2}\|k\|^2 + \frac{M\tilde{B}\tilde{R}}{N}\|k\|^2)^{1/2}]\|k\| = t\|k\|^2,$$

with $t > 0$. By Theorem 3.14 this complete the proof. □

Now we consider the duality topics in continuous version of fusion frame. The next result shows when two Bessel c-fusion mappings are a fusion pair.

Theorem 3.21 *Let (F,v) and (G,v) be two Bessel c-fusion mapping for H, Then the following assertions are equivalent:*
(i) For all $h\in H$, $h=\int_X v^2\pi_G\pi_F(h)d\mu$.
(ii) For all $h\in H$, $h=\int_X v^2\pi_F\pi_G(h)d\mu$.
(iii) For all $h,k\in H$, $<h,k>=\int_X v^2<\pi_G(h),\pi_F(k)>d\mu$.
(iv) For all $h\in H$, $||h||^2=\int_X v^2<\pi_G(h),\pi_F(h)>d\mu$.

Proof. $(i)\to(ii)$ Let $h,k\in H$. We have

$$<h,k>=<T_F(v\pi_F\pi_G(h)),k>=\int_X v<v\pi_F\pi_G(h)),k>d\mu$$

$$=\int_X v<h,v\pi_G\pi_F(k)>d\mu=<h,T_G(v\pi_F\pi_G(k))>.$$

Hence $k=T_G(v\pi_F\pi_G(k))$.
$(ii)\to(iii)$ It is evident by the proof of $(i)\to(ii)$.
$(iii)\to(i)$ For all $h,k\in H$, we have

$$<h,k>=\int_X v^2<\pi_G(h)),\pi_F(k)>d\mu=<T_F(v\pi_F\pi_G(h)),k>.$$

Thus $h=T_F(v\pi_F\pi_G(h))$.
$(iv)\to(i)$ Let $L:H\to H$ be defined by

$$L(h)=T_F(v\pi_F\pi_G(h)).$$

It is clear that L is linear. Since

$$\|L(h)\|=\sup_{k\in H_1}|<L(h),k>|=\sup_{k\in H_1}|\int_X v^2<\pi_F\pi_G(h)),k>d\mu|$$

$$\leq(\int_X v^2\|\pi_G(h)\|^2d\mu)^{1/2}\times(\sup_{k\in H_1}(\int_X v^2\|\pi_F(k)\|^2d\mu)^{1/2})$$

$$\leq B_{F,v}^{1/2}B_{G,v}^{1/2}\|h\|.$$

Hence $L\in B(H)$. For all $g\in H$, we have

$$<h,h>=\|h\|^2=\int_X v^2<\pi_G(h)),\pi_F(h)>d\mu$$

$$=<T_F(v\pi_F\pi_G(h)),h>.$$

Hence for all $h\in H$

$$h=T_F(v\pi_F\pi_G(h)).$$

$(iii)\to(iv)$ is evident. □

Definition 3.22 *Let (F, v) and (G, v) be Bessel c-fusion mapping for H. We say that F, G is a fusion pair if one of the assertions of the Theorem 3.21 satisfies.*

Lemma 3.23 *Let F, G be a fusion pair. Then $A_{F,v} > 0$.*

Proof. By Theorem 3.21 (iv), for all $h \in H$, we have

$$||h||^2 = \int_X v^2 < \pi_G(h), \pi_F(h) > d\mu \leq B_{G,v}^{1/2}(||v\pi_F(h)||)||h||.$$

Since, (G, v) is a Bessel c-fusion mapping, we have:

$$A_{F,v} = \inf_{h \in H_1} ||v\pi_F(h)||^2 \geq B_{G,v}^{-1} > 0.$$

□

Definition 3.24 *Let F, G be two Bessel c-fusion mappings for H. We define frame operator for fusion pair by*

$$S_{F,G} : H \to H$$

$$S_{F,G}(h) = T_F T_G^* \pi_F(h) = \int_X v^2 \pi_G \pi_F(h) d\mu,$$

$$< S_{F,G}(h), k >= \int_X v^2(x) < \pi_{G(x)} \pi_{F(x)}, k > d\mu.$$

Theorem 3.25 *We assume (F, v) and (G, v) be two Bessel c-fusion mappings for H.*
(i) $S_{F,G}$ is bounded and $S_{F,G}^* = S_{G,F}$.
(ii) Let there exists $\lambda_1 < 1$, $\lambda_2 > -1$ such that

$$\|h - S_{F,G}(h)\| = ||h - \int_X v^2 \pi_G \pi_F(h) d\mu|| \leq \lambda_1 ||h|| + \lambda_2 || \int_X v^2 \pi_G \pi_F(h) d\mu||,$$

for all $h \in H$. Then (G, v) is a c-fusion frame for H and for all $h \in H$ we have

$$(\frac{1 - \lambda_1}{1 + \lambda_2})^2 \frac{1}{B_{F,v}} \|h\|^2 \leq \int_X v^2 ||\pi_G(h)||^2 d\mu.$$

Proof. (i) We have

$$< S_{F,G}(h), g >=< \int_X v^2 \pi_G \pi_F(h) d\mu, g >= \int_X v^2 < \pi_G \pi_F(h), g > d\mu$$

$$= \int_X v^2 < \pi_F(h), \pi_G(g) > d\mu.$$

Thus

$$| < S_{F,G}(h), g > |^2 \tag{3.9}$$

$$\leq (\int_X v^2||\pi_F(h)||^2 d\mu)(\int_X v^2||\pi_G(g)||^2 d\mu)$$

$$\leq B_{F,v}B_{G,v}||h||^2||g||^2.$$

Hence $S_{F,G}$ is bounded a operator with

$$||S_{F,G}|| \leq B_{F,v}^{1/2}B_{G,v}^{1/2}.$$

Also $S^*_{F,G}$ is bounded and we have:

$$< h, S^*_{F,G}(g) >=< S_{F,G}(h), g >= \int_X v^2 < \pi_G\pi_F(h), g > d\mu$$

$$= \int_X v^2 < h, \pi_F\pi_G(g) > d\mu =< h, S_{G,F}(g) > .$$

Thus $S^*_{F,G} = S_{G,F}$.
(ii) Since

$$S_{F,G}(h) = \int_X v^2\pi_G\pi_F(h)d\mu,$$

$$||h - S_{F,G}(h)|| \leq \lambda_1||h|| + \lambda_2||S_{F,G}(h)||.$$

Thus

$$\lambda_1||h|| + \lambda_2||S_{F,G}|| \geq ||h|| - ||S_{F,G}(h)||,$$

hence

$$||S_{F,G}(h)|| \geq \frac{1-\lambda_1}{1+\lambda_2}||h||.$$

By inequality 3.9 for all $h \in H$ we have

$$| < S_{F,G}(h), h > |^2 \leq (B_{F,v}\|h\|^2)(\int_X v^2||\pi_G(h)||^2 d\mu).$$

Thus

$$\int_X v^2||\pi_G(h)||^2 d\mu \geq \frac{1}{B_{F,v}}||S_{F,G}(h)||^2 \geq \frac{1}{B_{F,v}}(\frac{1-\lambda_1}{1+\lambda_2})^2||h||^2.$$

□

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