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Notes on null curves in Minkowski spaces

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Abstract

We show a correspondence between the evolute of a null curve and the involute of a certain spacelike curve in the 4-dimensional Minkowski space. Also we characterize pseudo-spherical null curves in the n-dimensional Minkowski space in terms of the curvature functions.

Key Words: Null curve, Minkowski space, Frenet frame, Cartan curvature.

1. Introduction

In a semi-Riemannian manifold, there exist three families of curves, that is, spacelike, timelike, and null or lightlike curves, according to their causal characters. In the case of null curves, many different situations appear compared with the cases of spacelike and timelike curves. The theory of Frenet frames for a null curve has been studied and developed by several researchers in this field (cf. [2], [4], [1] and [3]). In [4] Ferrandez, Gimenez and Lucas introduced a Frenet frame with curvature functions for a null curve in a Lorentzian manifold, and studied null helices in Lorentzian space forms. In [1] Cöken and Ciftci studied null curves in the 4-dimensional Minkowski space R_1^4, and characterized pseudo-spherical null curves and Bertrand null curves.

In this paper we discuss null curves in the n-dimensional Minkowski space R_1^n. We define the evolute of a null curve in R_1^4 and the involute of a spacelike curve in R_1^4, and show a correspondence between them which is similar to that between the plane evolute and involute. Also, we characterize pseudo-spherical null curves in R_1^n in terms of the curvature functions, which is a generalization of [1, Theorem 3.2] for R_1^4.

2. Preliminaries

In this section, following [4] and [1], we recall the Frenet frame and curvature functions for a null curve in R_1^n.

Let $\langle\ ,\ \rangle$ denote the metric on R_1^n. A curve $\gamma(t)$ in R_1^n is called a null curve if $\langle \gamma'(t), \gamma'(t)\rangle = 0$ and $\gamma'(t) \neq 0$ for all t. We note that a null curve $\gamma(t)$ in R_1^n satisfies $\langle \gamma''(t), \gamma''(t)\rangle \geq 0$ (cf. [2, Chap. 3]). We say that a null curve $\gamma(t)$ in R_1^n is parametrized by the pseudo-arc if $\langle \gamma''(t), \gamma''(t)\rangle = 1$. If a null curve $\gamma(t)$ in R_1^n

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satisfies $\langle\gamma''(t), \gamma''(t)\rangle \neq 0$, then $\langle\gamma''(t), \gamma''(t)\rangle > 0$, and

$$u(t) = \int_{t_0}^{t} \langle\gamma''(t), \gamma''(t)\rangle^{1/4} dt$$

becomes the pseudo-arc parameter.

Let us say that a null curve $\gamma(t)$ in R_1^n with $\langle\gamma''(t), \gamma''(t)\rangle \neq 0$ is a Cartan curve if $\{\gamma'(t), \gamma''(t), \cdots, \gamma^{(n-1)}(t)\}$ is linearly independent for any t.

For a Cartan curve $\gamma(t)$ in R_1^n, $n = m + 2$ with pseudo-arc parameter t, there exists a unique Frenet frame $\{L, N, W_1, \cdots, W_m\}$ such that

$$\gamma' = L, \quad L' = W_1, \quad N' = k_1 W_1 + k_2 W_2,$$

$$W_1' = -k_1 L - N, \quad W_2' = -k_2 L + k_3 W_3,$$

$$W_i' = -k_i W_{i-1} + k_{i+1} W_{i+1}, \quad 3 \leq i \leq m-1,$$

$$W_m' = -k_m W_{m-1},$$

where N is null, $\langle L, N\rangle = 1$, $\{L, N\}$ and $\{W_1, \cdots, W_m\}$ are orthogonal, $\{W_1, \cdots, W_m\}$ is orthonormal, $\{\gamma', \gamma'', \cdots, \gamma^{(i+2)}\}$ and $\{L, N, W_1, \cdots, W_i\}$ have the same orientation for $2 \leq i \leq m-1$, and $\{L, N, W_1, \cdots, W_m\}$ is positively oriented. The functions $\{k_1, k_2, \cdots, k_m\}$ are called the Cartan curvatures of γ, which satisfy $k_i \neq 0$ for $2 \leq i \leq m-1$.

Remark. In [4] it is assumed that $\{\gamma'(t), \gamma''(t), \cdots, \gamma^{(n)}(t)\}$ is linearly independent. But, for the existence of the Frenet frame, the linear independence of $\{\gamma'(t), \gamma''(t), \cdots, \gamma^{(n-1)}(t)\}$ is sufficient.

3. Evolutes and involutes in R_1^4

Let $\gamma(t)$ be a Cartan curve in R_1^4 parametrized by the pseudo-arc. Then the Frenet equations are given as

$$\gamma' = L, \quad L' = W_1, \quad N' = k_1 W_1 + k_2 W_2,$$

$$W_1' = -k_1 L - N, \quad W_2' = -k_2 L.$$

When $k_2(t) \neq 0$, we shall define the evolute of $\gamma(t)$ by

$$E(t) = \gamma(t) + \frac{1}{k_2(t)} W_2(t),$$

which is the center of the osculating pseudo-sphere at $\gamma(t)$ (cf. [1]).

On the other hand, for a spacelike curve $c(t)$ in R_1^4, we define the involute of $c(t)$ from a point $c(t_0)$ by

$$I(t) = c(t) - s(t)T(t),$$

where $s(t)$ is the arc length of $c(t)$ from $c(t_0)$ and $T(t) = c'(t)/|c'(t)|$ is the unit tangent vector to $c(t)$.

In this section, we show a correspondence between the evolute of a null curve in R_1^4 and the involute of a certain spacelike curve in R_1^4, which is similar to that between the plane evolute and involute (cf. [6]).

Theorem 1. (i) *Let $\gamma(t)$ be a Cartan curve in R_1^4 with pseudo-arc parameter t such that $k_2(t) \neq 0$ and $(1/k_2(t))' \neq 0$. Then the evolute $E(t)$ of $\gamma(t)$ is a spacelike curve in R_1^4, and the involute $I_E(t)$ of $E(t)$ from some point coincides with $\gamma(t)$.*

(ii) *Let $c(s)$ be a spacelike curve in R_1^4 with arc parameter s such that $c''(s)$ is null, $\langle c^{(3)}(s), c^{(3)}(s)\rangle \neq 0$, and $\{c''(s), c^{(3)}(s), c^{(4)}(s)\}$ is linearly independent. Then, for $s > 0$, the involute $I(s)$ of $c(s)$ is a Cartan curve in R_1^4, and the evolute $E_I(s)$ of $I(s)$ coincides with $c(s)$.*

Proof. (i) By the Frenet equation for the Cartan curve γ, the evolute $E(t)$ of $\gamma(t)$ satisfies

$$E'(t) = L + \left(\frac{1}{k_2(t)}\right)' W_2 + \frac{1}{k_2(t)}(-k_2(t)L) = \left(\frac{1}{k_2(t)}\right)' W_2.$$

So

$$\langle E', E'\rangle = \left(\left(\frac{1}{k_2(t)}\right)'\right)^2 > 0,$$

and $E(t)$ is a spacelike curve.

We only consider the case where $(1/k_2(t))' > 0$, because the case where $(1/k_2(t))' < 0$ is similar. Then $E(t)$ has unit tangent vector $T_E = W_2$, and the arc length $s_E(t)$ of $E(t)$ from $E(t_0)$ is given by

$$s_E(t) = \int_{t_0}^{t} |E'|dt = \int_{t_0}^{t} \left(\frac{1}{k_2(t)}\right)' dt = \frac{1}{k_2(t)} - \frac{1}{k_2(t_0)}.$$

So we have

$$\frac{1}{k_2(t)} = s_E(t) + \frac{1}{k_2(t_0)},$$

which is the arc length of $E(t)$ from another point $E(t_1)$.

The involute $I_E(t)$ of $E(t)$ from $E(t_1)$ satisfies

$$I_E(t) = E(t) - \left(s_E(t) + \frac{1}{k_2(t_0)}\right) T_E$$

$$= \gamma(t) + \frac{1}{k_2(t)} W_2 - \frac{1}{k_2(t)} W_2 = \gamma(t).$$

Thus we get the conclusion of (i).

(ii) As $T' = c''$ is null, we may view $T(s)$ as a null curve in R_1^4, and we have $\langle T'', T''\rangle \geq 0$ (cf. [2, Chap. 3]). So the assumption $\langle T'', T''\rangle = \langle c^{(3)}, c^{(3)}\rangle \neq 0$ implies that $\langle T'', T''\rangle > 0$.

The involute $I(s) = c(s) - sT(s)$ of the spacelike curve $c(s)$ satisfies

$$I'(s) = T - T - sT' = -sT', \qquad I''(s) = -T' - sT'',$$

$$I^{(3)}(s) = -2T'' - sT^{(3)}.$$

For $s > 0$, $I(s)$ is a null curve, and

$$\langle I'', I'' \rangle = s^2 \langle T'', T'' \rangle > 0.$$

Set $b = \langle T'', T'' \rangle^{1/2}$. The pseudo-arc length $u(s)$ of $I(s)$ is given by

$$u(s) = \int_{s_0}^{s} \langle I'', I'' \rangle^{1/4} ds = \int_{s_0}^{s} s^{1/2} b^{1/2} ds,$$

and

$$\frac{du}{ds} = s^{1/2} b^{1/2}.$$

Since $\{T', T'', T^{(3)}\} = \{c'', c^{(3)}, c^{(4)}\}$ is linearly independent, $\{I', I'', I^{(3)}\}$ is also linearly independent, and the null curve $I(s)$ is a Cartan curve with pseudo-arc length $u(s)$. Let $\{L, N, W_1, W_2\}$ be the Frenet frame for the Cartan curve $I(s)$ with Cartan curvatures $\{k_1, k_2\}$. Then we have

$$L = \frac{dI}{du} = \frac{dI}{ds}\frac{ds}{du} = s^{-1/2} b^{-1/2}(-sT') = -s^{1/2} b^{-1/2} T',$$

$$W_1 = \frac{dL}{du} = \frac{1}{2}(b^{-2}b' - s^{-1}b^{-1})T' - b^{-1}T'',$$

and

$$N = -k_1 L - \frac{dW_1}{du}.$$

Noting that

$$\mathrm{span}\{L, N, W_1\} = \mathrm{span}\{T', T'', T^{(3)}\},$$

and

$$\langle T, T' \rangle = \langle T, T'' \rangle = \langle T, T^{(3)} \rangle = 0,$$

we can find that $W_2 = \pm T$. We consider the case where $W_2 = T$, because the case where $W_2 = -T$ is similar. Then

$$\begin{aligned} k_2 &= -\left\langle \frac{dW_2}{du}, N \right\rangle = -\left\langle \frac{dT}{du}, N \right\rangle = -\langle s^{-1/2} b^{-1/2} T', N \rangle \\ &= s^{-1} \langle L, N \rangle = s^{-1}. \end{aligned}$$

Thus the evolute $E_I(s)$ of $I(s)$ satisfies

$$E_I(s) = I(s) + \frac{1}{k_2} W_2 = c(s) - sT + sT = c(s),$$

which is the conclusion of (ii). □

Remark. In (ii), by the conditions, $T(s) = c'(s)$ may be viewed as a Cartan curve in the pseudo-sphere $S_1^3(1)$, where

$$S_1^3(1) = \{x \in R_1^4 | \langle x, x \rangle = 1\}.$$

4. Pseudo-spherical null curves in R_1^{m+2}

The pseudo-sphere of radius r and center p_0 in R_1^{m+2} is given by

$$S_1^{m+1}(r) = \{x \in R_1^{m+2} | \langle x - p_0,\ x - p_0 \rangle = r^2\}$$

(cf. [5]). A curve $\gamma(t)$ in R_1^{m+2} is called pseudo-spherical if it lies on a pseudo-sphere. In this section, we characterize pseudo-spherical null curves in R_1^{m+2} in terms of the curvature functions, which is a generalization of [1, Th.3.2] for R_1^4.

For a Cartan curve $\gamma(t)$ in R_1^{m+2} parametrized by the pseudo-arc with Cartan curvatures $\{k_1, k_2, \cdots, k_m\}$ and $k_m \neq 0$, let us define a sequence of functions $\{a_1, a_2, \cdots, a_m\}$ inductively by

$$a_1 = 0, \quad a_2 = -\frac{1}{k_2}, \quad a_{i+1} = \frac{1}{k_{i+1}}(a_i' + k_i a_{i-1}), \quad 2 \leq i \leq m-1.$$

Theorem 2. *Let $\gamma(t)$ be a Cartan curve in R_1^{m+2} parametrized by the pseudo-arc such that $k_m \neq 0$.*

(i) *If $\gamma(t)$ lies on a pseudo-sphere of radius r, then*

$$\sum_{i=2}^{m} a_i^2 = r^2.$$

(ii) *If $a_m \neq 0$ and*

$$\sum_{i=2}^{m} a_i^2 = r^2$$

for some positive constant r, then $\gamma(t)$ lies on a pseudo-sphere of radius r.

Proof. We will use the Frenet equations for $\gamma(t)$ given in Section 2.

(i) Suppose that $\gamma(t)$ lies on a pseudo-sphere of radius r. That is, there exists a fixed point $p_0 \in R_1^{m+2}$ such that

$$\langle \gamma(t) - p_0,\ \gamma(t) - p_0 \rangle = r^2. \tag{1}$$

Set

$$\gamma(t) - p_0 = b_1 L + b_2 N + c_1 W_1 + c_2 W_2 + \cdots + c_m W_m.$$

Differentiating (1), we have

$$\langle \gamma(t) - p_0,\ L \rangle = 0, \tag{2}$$

and $b_2 = 0$. Differentiating (2), we have

$$\langle L,\ L \rangle + \langle \gamma(t) - p_0,\ W_1 \rangle = 0,$$

$$\langle \gamma(t) - p_0,\ W_1 \rangle = 0, \tag{3}$$

and $c_1 = 0 = a_1$. Differentiating (3), we have

$$\langle L,\ W_1\rangle + \langle \gamma(t) - p_0,\ -k_1 L - N\rangle = 0,$$

$$\langle \gamma(t) - p_0,\ N\rangle = 0, \tag{4}$$

and $b_1 = 0$. Differentiating (4), we have

$$\langle L,\ N\rangle + \langle \gamma(t) - p_0,\ k_1 W_1 + k_2 W_2\rangle = 0,$$

$$\langle \gamma(t) - p_0,\ W_2\rangle = -\frac{1}{k_2}, \tag{5}$$

and $c_2 = -1/k_2 = a_2$. Differentiating (5), we have

$$\langle L,\ W_2\rangle + \langle \gamma(t) - p_0,\ -k_2 L + k_3 W_3\rangle = -\left(\frac{1}{k_2}\right)',$$

and

$$c_3 = -\frac{1}{k_3}\left(\frac{1}{k_2}\right)' = a_3.$$

For $3 \le i \le m-1$, differentiating

$$\langle \gamma(t) - p_0,\ W_i\rangle = c_i,$$

we have

$$\langle L, W_i\rangle + \langle \gamma(t) - p_0,\ -k_i W_{i-1} + k_{i+1} W_{i+1}\rangle = c_i',$$

and

$$-k_i c_{i-1} + k_{i+1} c_{i+1} = c_i'.$$

So we get

$$c_{i+1} = \frac{1}{k_{i+1}}(c_i' + k_i c_{i-1}).$$

By $c_2 = a_2,\ c_3 = a_3$ and the definition of $\{a_i\}$, we can see that $c_i = a_i$ for $1 \le i \le m$. Hence we have

$$\gamma(t) - p_0 = a_2 W_2 + a_3 W_3 + \cdots + a_m W_m,$$

and by (1),

$$\sum_{i=2}^{m} a_i^2 = r^2.$$

(ii) Suppose that $a_m \neq 0$ and

$$\sum_{i=2}^{m} a_i^2 = r^2 \tag{6}$$

for some positive constant r. Set

$$A(t) = \gamma(t) - a_2W_2 - a_3W_3 - \cdots - a_mW_m.$$

Then, using the Frenet equations and the definition of $\{a_i\}$, we may obtain

$$\begin{aligned} A'(t) = (1 + k_2a_2)L + (k_3a_3 - a_2')W_2 + (k_4a_4 - a_3' - k_3a_2)W_3 + (k_5a_5 - a_4' - k_4a_3)W_4 \\ + \cdots + (k_ma_m - a_{m-1}' - k_{m-1}a_{m-2})W_{m-1} - (a_m' + k_ma_{m-1})W_m \\ = -(a_m' + k_ma_{m-1})W_m. \end{aligned}$$

Differentiating (6) we have

$$a_2a_2' + a_3a_3' + \cdots + a_{m-1}a_{m-1}' + a_ma_m' = 0.$$

Using it together with the definition of $\{a_i\}$, we can get

$$\begin{aligned} a_m(a_m' + k_ma_{m-1}) &= k_ma_{m-1}a_m - a_{m-1}a_{m-1}' - a_{m-2}a_{m-2}' - \cdots - a_2a_2' \\ &= k_{m-1}a_{m-2}a_{m-1} - a_{m-2}a_{m-2}' - a_{m-3}a_{m-3}' - \cdots - a_2a_2' \\ &= k_{m-2}a_{m-3}a_{m-2} - a_{m-3}a_{m-3}' - \cdots - a_2a_2' \\ &= \cdots = k_3a_2a_3 - a_2a_2' = 0. \end{aligned}$$

So $A'(t) = 0$, and $A(t) = p_0$ for some fixed point $p_0 \in R_1^{m+2}$. Thus we have

$$\gamma(t) - p_0 = \sum_{i=2}^{m} a_iW_i,$$

and by (6),

$$\langle \gamma(t) - p_0,\ \gamma(t) - p_0 \rangle = r^2.$$

Hence $\gamma(t)$ lies on a pseudo-sphere of radius r. $\square$

5. A remark

In [1, Theorem 3.2], it is shown that a Cartan curve in R_1^4 is pseudo-spherical if and only if k_2 is a nonzero constant. This seems similar to the classical fact that a plane curve is a part of a circle if and only if the curvature is a nonzero constant.

In the case of R_1^5, Theorem 2 says the following:

(i) If a Cartan curve $\gamma(t)$ in R_1^5 parametrized by the pseudo-arc with $k_3 \neq 0$ lies on a pseudo-sphere of radius r, then

$$\left(\frac{1}{k_2}\right)^2 + \left(\frac{1}{k_3}\left(\frac{1}{k_2}\right)'\right)^2 = r^2.$$

(ii) If a Cartan curve $\gamma(t)$ in R_1^5 parametrized by the pseudo-arc satisfies $k_3 \neq 0$, $(1/k_2)' \neq 0$ and

$$\left(\frac{1}{k_2}\right)^2 + \left(\frac{1}{k_3}\left(\frac{1}{k_2}\right)'\right)^2 = r^2$$

for some positive constant r, then $\gamma(t)$ lies on a pseudo-sphere of radius r.

This is similar to the condition for a curve in the 3-dimensional Euclidean space R^3 to lie on a sphere (cf. [6]). Also the correspondence in Theorem 1 for R_1^4 is similar to that between the plane evolute and involute.

So, the results in [1] and this paper suggest that null curves in R_1^{m+2} may have various properties similar to those of curves in the m-dimensional Euclidean space R^m.

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