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Some sufficient conditions for starlikeness and convexity

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Abstract

There are many results for sufficient conditions of functions $f(z)$ which are analytic in the open unit disc $\mathbb{U}$ to be starlike and convex in $\mathbb{U}$. In view of the results due to S. Ozaki, I. Ono and T. Umezawa (1956), P.T. Mocanu (1988), and M. Nunokawa (1993), some sufficient conditions for starlikeness and convexity of $f(z)$ are discussed.

Key word and phrases: Univalent, starlike, convex.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1,1)$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} | |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are univalent in $\mathbb{U}$. Also, let $\mathcal{S}^*$ and $\mathcal{K}$ be the subclasses of $\mathcal{S}$ consisting of all starlike functions $f(z)$ in $\mathbb{U}$ and of all convex functions $f(z)$ in $\mathbb{U}$, respectively.

To discuss our problems, we have to recall here the following results.

Theorem A ([3]) *If* $f(z) \in \mathcal{A}$ *satisfies* $|f''(z)| < 1$ $(z \in \mathbb{U})$, *then* $f(z) \in \mathcal{S}$.

Theorem B ([1]) *If* $f(z) \in \mathcal{A}$ *satisfies*

$$|f'(z) - 1| < \frac{\sqrt{20}}{5} \quad (z \in \mathbb{U}), \qquad (1.2)$$

then $f(z) \in \mathcal{S}^*$.

Theorem C ([1]) *If* $f(z) \in \mathcal{A}$ *satisfies*

$$|\arg f'(z)| < \frac{\pi}{2}\alpha_0 \quad (z \in \mathbb{U}), \qquad (1.3)$$

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then $f(z) \in \mathcal{S}^*$, *where* $\alpha_0 = 0.6165...$ *is the unique root of the equation*

$$2\tan^{-1}(1-\alpha) + (1-2\alpha)\pi = 0. \tag{1.4}$$

2. Starlikeness and convexity

We begin with the statement and the proof of the following result for sufficient condition of $f(z)$ to be in the class $\mathcal{S}^*$.

Theorem 1 *If* $f(z) \in \mathcal{A}$ *satisfies*

$$|f''(z)| \leqq \frac{\sqrt{20}}{5} = 0.8944... \quad (z \in \mathbb{U}), \tag{2.1}$$

then $f(z) \in \mathcal{S}^*$.

Proof. Noting that

$$\begin{aligned} |f'(z)-1| &= \left|\int_0^z f''(t)dt\right| \\ &\leqq \int_0^{|z|} |f''(\varphi e^{i\theta})|d\varphi \\ &\leqq \frac{\sqrt{20}}{5}\int_0^{|z|} d\varphi = \frac{\sqrt{20}}{5}|z| < \frac{\sqrt{20}}{5}, \end{aligned}$$

Theorem B gives us that $f(z) \in \mathcal{S}^*$. □

Remark 1 In view of the result by Obradović [4], we see that the sharp bound in (2.1) is 1.

Next we derive the following theorem.

Theorem 2 *If* $f(z) \in \mathcal{A}$ *satisfies*

$$|f''(z)| \leqq \frac{\sqrt{5}}{5} = 0.4472... \quad (z \in \mathbb{U}), \tag{2.2}$$

then $f(z) \in \mathcal{K}$.

Proof. It follows that

$$\begin{aligned} |(zf'(z))'-1| &= |f'(z)+zf''(z)-1| \\ &\leqq |f'(z)-1| + |zf''(z)| \\ &\leqq \left|\int_0^z f''(t)dt\right| + |zf''(z)| \\ &\leqq \int_0^{|z|} |f''(t)dt| + \frac{\sqrt{5}}{5}|z| \\ &\leqq \frac{2\sqrt{5}}{5}|z| < \frac{\sqrt{20}}{5}. \end{aligned}$$

Therefore, using Theorem B, we see that $zf'(z) \in \mathcal{S}^*$, or $f(z) \in \mathcal{K}$. □

Remark 2 By virtue of the result due to Mocanu [2], we know that the sharp bound in (2.2) is $\frac{1}{2}$.

Finally, applying the result due to Nunokawa [2], we can prove

Theorem 3 *If $f(z) \in \mathcal{A}$ satisfies*

$$|\arg(f'(z) + zf''(z))| < \frac{\pi}{2}\left(\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right) \quad (z \in \mathbb{U}), \tag{2.3}$$

then $f(z) \in \mathcal{K}$, where $\alpha_1 = 0.3834...$ is the root of the equation

$$2\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1 = 1.$$

Proof. Note that

$$\arg(f'(z) + zf''(z)) = \arg f'(z) + \arg\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

If there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg f'(z)| < \frac{\pi}{2}\alpha_1 \quad (|z| < |z_0|),$$

and

$$|\arg f'(z_0)| = \frac{\pi}{2}\alpha_1,$$

then Nunokawa's theorem in [2] gives us that

$$\frac{z_0 f''(z_0)}{f'(z_0)} = i\alpha_1 k,$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad (\text{when } \arg f'(z_0) = \frac{\pi}{2}\alpha_1),$$

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad (\text{when } \arg f'(z_0) = -\frac{\pi}{2}\alpha_1),$$

and

$$f'(z_0)^{1/\alpha_1} = \pm ia \quad (a > 0).$$

Therefore, if $\arg f'(z_0) = \frac{\pi}{2}\alpha_1$, then we have

$$\begin{aligned}
\arg f'(z_0) + \arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= \frac{\pi}{2}\alpha_1 + \arg(1 + i\alpha_1 k) \\
&\geqq \frac{\pi}{2}\alpha_1 + \tan^{-1}\alpha_1 \\
&= \frac{\pi}{2}\left(\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right),
\end{aligned}$$

which contradicts (2.3). If $\arg f'(z_0) = -\frac{\pi}{2}\alpha_1$, then, applying the same method for the previous case, we also have

$$\arg f'(z_0) + \arg\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) \leqq -\frac{\pi}{2}\left(\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right),$$

which contradicts (2.3). Therefore, there exists no $z_0 \in \mathbb{U}$ such that $|\arg f'(z_0)| = \frac{\pi}{2}\alpha_1$. This implies that

$$|\arg f'(z)| < \frac{\pi}{2}\alpha_1 \quad (z \in \mathbb{U}).$$

Furthermore, since

$$\begin{aligned} \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| - |\arg f'(z)| \quad &\leqq \quad |\arg(f'(z) + zf''(z))| \\ &< \quad \frac{\pi}{2}\left(\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right) \quad (z \in \mathbb{U}), \end{aligned}$$

we conclude that

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\left(2\alpha_1 + \frac{2}{\pi}\tan^{-1}\alpha_1\right) = \frac{\pi}{2} \quad (z \in \mathbb{U}),$$

or

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{U}).$$

This completes the proof of Theorem 3. □

3. Appendix

Let us consider a function $f(z) \in \mathcal{A}$ which satisfies

$$f'(z) = \frac{n}{n+z}, \; f'(0) = 1,$$

where n is a sufficiently large positive integer. Then we see that, for $z_0 = \frac{\sqrt{2}}{2}(1+i)$,

$$\begin{aligned} \lim_{\substack{z\to z_0 \\ n\to n_0}} \arg\left(\frac{zf'(z)}{f(z)}\right) &= \lim_{\substack{z\to z_0 \\ n\to n_0}} \arg\left(\frac{z}{(n+z)\log(n+z)}\right) \\ &= \lim_{\substack{z\to z_0 \\ n\to n_0}} (\arg z - \arg(n+z) - \arg(\log(n+z))) \\ &= \frac{3}{4}\pi > \frac{\pi}{2}. \end{aligned}$$

This shows that, for arbitrary real α which is sufficiently close to 1 but less than 1, if $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re} f'(z) > \alpha \quad (z \in \mathbb{U}),$$

then $f(z)$ is not necessarily starlike in $\mathbb{U}$.

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