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An expansion result for a Sturm-Liouville eigenvalue problem with impulse

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Abstract

The paper is concerned with an eigenvalue problem for second order differential equations with impulse. Such a problem arises when the method of separation of variables applies to the heat conduction equation for two-layered composite. The existence of a countably infinite set of eigenvalues and eigenfunctions is proved and a uniformly convergent expansion formula in the eigenfunctions is established.

Key Words: Green's function; Completely continuous operator; Impulse conditions; Eigenvalue; Eigenvector.

1. Introduction

An equation for temperatures in a solid $0 \le x \le b$ composed of a layer $0 \le x < a$ of material in contact with a layer $a < x \le b$ of another material is given by (see, for example, [6, 13])

$$\rho(x)\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u(x,t)}{\partial x}\right] - q(x)u(x,t), \tag{1}$$

$$x \in [0,a) \cup (a,b], \ t > 0.$$

We shall assume that $\rho(x)$, $p(x)$, and $q(x)$ are real-valued, $p(x)$ is differentiable on $[0,a)\cup(a,b]$, $\rho(x)$, $p'(x)$, and $q(x)$ are piecewise continuous on $[0,a)\cup(a,b]$ and $\rho(x)>0$, $p(x)>0$, $q(x)\ge 0$. In addition, it is assumed that there exist finite left-sided and right-sided limits $\rho(a\pm 0)$, $p(a\pm 0)$, and $q(a\pm 0)$, and that $\rho(a\pm 0)>0$, $p(a\pm 0)>0$.

For solution $u(x,t)$ of equation (1) we take at $x=a$ interface conditions of the form

$$u(a-0,t) = \alpha u(a+0,t), \quad u_x(a-0,t) = \beta u_x(a+0,t), \tag{2}$$

in which α and β are given positive real numbers, and at the end faces $x=0$ and $x=b$ we take the zero temperature conditions

$$u(0,t) = u(b,t) = 0. \tag{3}$$

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The initial temperature of the composite is given by

$$u(x,0) = f(x), \quad x \in [0,a) \cup (a,b]. \tag{4}$$

Note that the conditions in (2) represent an impulse phenomenon at $x = a$ (see [2, 3, 10, 14]).

Let us look for a nontrivial solution of (1)–(3), ignoring the initial condition (4), which has the form

$$u(x,t) = e^{-\lambda t} y(x), \quad x \in [0,a) \cup (a,b], \tag{5}$$

where λ is a complex constant and $y(x)$ is a function independent of t (but, in general, dependent on λ) that is not identically zero. Substituting (5) into (1)–(3), we obtain

$$-[p(x)y']' + q(x)y = \lambda \rho(x) y, \quad x \in [0,a) \cup (a,b], \tag{6}$$

$$y(a-0) = \alpha y(a+0), \quad y'(a-0) = \beta y'(a+0), \tag{7}$$

$$y(0) = y(b) = 0. \tag{8}$$

So, the function (5) is a nontrivial solution of problem (1)–(3) if and only if λ is an eigenvalue and $y(x)$ is a corresponding eigenfunction of problem (6)–(8).

In Section 2 of the present paper it is shown that the eigenvalue problem (6)–(8) has a countably infinite set of eigenvalues $\lambda_1, \lambda_2, \ldots$ tending to $+\infty$, with the corresponding eigenfunctions $\upsilon_1(x), \upsilon_2(x), \ldots$. Primary tools in our proof are a suitable Green's function and the Hilbert-Schmidt theorem on symmetric completely continuous operators.

By linearity of problem (1)–(3) the function

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} \upsilon_k(x) \tag{9}$$

is a formal solution of (1)–(3), where $c_1, c_2, \ldots$ are arbitrary constants. Now we try to choose the constants c_k so that (9) will also satisfy the initial condition (4). This leads to

$$f(x) = \sum_{k=1}^{\infty} c_k \upsilon_k(x), \quad x \in [0,a) \cup (a,b]. \tag{10}$$

Thus the problem of possibility to expand a given function $f(x)$ in eigenfunctions $\upsilon_1(x), \upsilon_2(x), \ldots$ arises.

In Section 3 it is proved that if f is a continuous function on $[0,a) \cup (a,b]$ having a piecewise continuous derivative on $[0,a) \cup (a,b]$ and finite limits $f(a \pm 0)$, $f'(a \pm 0)$ and satisfying the impulse condition $f(a-0) = \alpha f(a+0)$ and boundary conditions in (8), then for f a uniformly convergent on $[0,a) \cup (a,b]$ expansion formula (10) holds. The coefficients c_k in (10) and hence in (9), are found by the formula

$$c_k = \int_0^a \rho(x) f(x) \upsilon_k(x) dx + \omega \int_a^b \rho(x) f(x) \upsilon_k(x) dx, \tag{11}$$

where

$$\omega = \alpha\beta \frac{p(a-0)}{p(a+0)} \tag{12}$$

is a positive real number.

Note that in [7] the eigenvalue problem $(6)-(8)$ was investigated by the authors in the case

$$p(x) \equiv 1, \quad \rho(x) = \begin{cases} \rho_1^2, & x \in [0,a), \\ \rho_2^2, & x \in (a,b], \end{cases}$$

where ρ_1 and ρ_2 are positive constants. There it was proved by examining the asymptotic behavior of the resolvent function of problem $(6)-(8)$ and taking then a contour integral of the resolvent function that there is a positive constant c not depending on x and n such that

$$\left| f(x) - \sum_{k=1}^{n} c_k \upsilon_k(x) \right| \le \frac{c}{n}, \quad x \in [0,a) \cup (a,b], \tag{13}$$

provided that f is a twice differentiable function on $[0,a)\cup(a,b]$ satisfying impulse conditions (7) and boundary conditions (8) and being such that $f''(x)$ is integrable over $[0,b]$. In (13), the coefficients c_k are defined by (11). Result (13), which gives an asymptotic estimate when $n \to \infty$ of the amount by which the nth partial sum of the series (10) differs from $f(x)$, of course implies the uniform convergence of expansion (10). Thus the main result of the present paper is that in the theorem on uniformly convergent expansion the condition of existence of a second derivative of f is reduced to the condition of existence of a first derivative of f. This is achieved by making use of a method employed by Steklov in the case of absence of the impulse (see [15, Section 182]).

Conditions in (7), which we call the impulse conditions, are called in the literature also as the discontinuity conditions or the transmission conditions; see [1, 4, 5, 11, 12, 16]. In these papers the uniformity of convergence of eigenfunction expansions is not investigated. In our presentation in the present paper we follow the main outlines of [8].

2. Mean square convergent expansions

Denote by $L^2_\rho[a,b]$ the Hilbert space of all real-valued measurable functions y on $[0,b]$ such that

$$\int_0^b \rho(x) y^2(x) dx < \infty,$$

with the inner product (scalar product)

$$\langle y, z \rangle = \int_0^a \rho(x)y(x)z(x)dx + \omega \int_a^b \rho(x)y(x)z(x)dx \tag{14}$$

and the norm $\|y\|$ defined by

$$\|y\|^2 = \langle y, y \rangle = \int_0^a \rho(x)y^2(x)dx + \omega \int_a^b \rho(x)y^2(x)dx,$$

where ω is given by (12).

Next denote by D the set of all functions $y \in L^2_\rho[a,b]$ satisfying the following three conditions:

1. y is continuous on $[0,a) \cup (a,b]$ and $y(0) = y(b) = 0$.

2. y is continuously differentiable on $[0,a) \cup (a,b]$, there exist finite limits $y(a \pm 0)$, $y'(a \pm 0)$, and

$$y(a-0) = \alpha y(a+0), \quad y'(a-0) = \beta y'(a+0).$$

3. y' is differentiable almost everywhere on $[0,a) \cup (a,b]$ and

$$\int_0^b |y''|^2 \, dx < \infty.$$

Obviously D is a linear subset dense in $L^2_\rho[0,b]$. Now we define the operator $A : D \subset L^2_\rho[0,b] \longrightarrow L^2_\rho[0,b]$ as follows. The domain of definition of A is D and we put

$$(Ay)(x) = \frac{1}{\rho(x)}\{-[p(x)y'(x)]' + q(x)y(x)\}, \quad x \in [0,a) \cup (a,b],$$

for $y \in D$.

We see that the eigenvalue problem (6)–(8) is equivalent to the equation

$$Ay = \lambda y, \quad y \in D, \quad y \neq 0. \tag{15}$$

Theorem 1 *We have for all $y, z \in D$,*

$$\langle Ay, z \rangle = \langle y, Az \rangle, \tag{16}$$

$$\begin{aligned} \langle Ay, y \rangle &= \int_0^a [p(x)y'^2(x) + q(x)y^2(x)]dx \\ &\quad + \omega \int_a^b [p(x)y'^2(x) + q(x)y^2(x)]dx. \end{aligned} \tag{17}$$

Proof. Integrating by parts we have for all $y, z \in D$

$$\begin{aligned}\langle Ay, z\rangle &= \int_0^a [-(py')' + qy]z dx + \omega \int_a^b [-(py')' + qy]z dx \\ &= -p(x)y'(x)z(x)|_0^{a-0} + \int_0^a (py'z' + qyz)dx \\ &\quad -\omega p(x)y'(x)z(x)|_{a+0}^b + \omega \int_a^b (py'z' + qyz)dx \\ &= \int_0^a (py'z' + qyz)dx + \omega \int_a^b (py'z' + qyz)dx \\ &= p(x)y(x)z'(x)|_0^{a-0} + \int_0^a y[-(pz')' + qz]dx \\ &\quad +\omega p(x)y(x)z'(x)|_{a+0}^b + \omega \int_a^b y[-(pz')' + qz]dx \\ &= \int_0^a y[-(pz')' + qz]dx + \omega \int_a^b y[-(pz')' + qz]dx = \langle y, Az\rangle,\end{aligned}$$

where we have used the boundary conditions $u(a) = u(b) = 0$ and the impulse conditions $u(a-0) = \alpha u(a+0)$, $u'(a-0) = \beta u'(a+0)$ for functions $u \in D$. Simultaneously, we have also got (17). The theorem is proved. □

Relation (16) shows that the operator A is symmetric (self-adjoint), while (17) shows that it is positive:

$$\langle Ay, y\rangle > 0 \quad \text{for all} \quad y \in D,\ y \neq 0.$$

Therefore all eigenvalues of the operator A are real and positive and two eigenfunctions corresponding to the distinct eigenvalues are orthogonal in the sense of inner product (14). Besides, it can easily be seen that eigenvalues of problem (6)–(8) are simple, that is, to each eigenvalue there corresponds a single eigenfunction up to a constant factor (equation (6) with the impulse conditions (7) can not have two linearly independent solutions satisfying $y(0) = 0$).

Note that the kernel of A,

$$\ker A = \{y \in D : Ay = 0\}$$

consists only of the zero element. Indeed, if $y \in D$ and $Ay = 0$, then from (17) we have $y'(x) = 0$ for $x \in [0, a) \cup (a, b]$ and hence

$$y(x) = c_1 \quad \text{for} \quad x \in [0, a) \quad \text{and} \quad y(x) = c_2 \quad \text{for} \quad x \in (a, b],$$

where c_1 and c_2 are constants. Then from the impulse condition $y(a-0) = \alpha y(a+0)$ we have $c_1 = \alpha c_2$ and hence by the condition $y(0) = 0$ (or $y(b) = 0$) we find $c_1 = c_2 = 0$ so that $y(x) \equiv 0$.

It follows that the inverse operator A^{-1} exists. To present its explicit form we introduce the Green function (see [7])

$$G(x, \xi) = -\frac{1}{W_\xi(\varphi, \psi)} \begin{cases} \varphi(x)\psi(\xi) & \text{if} \quad 0 \leq x \leq \xi \leq b, \\ \varphi(\xi)\psi(x) & \text{if} \quad 0 \leq \xi \leq x \leq b, \end{cases} \tag{18}$$

for $x,\xi \in [0,a)\cup(a,b]$, where $\varphi(x)$ and $\psi(x)$ are solutions of the problem

$$-[p(x)y']' + q(x)y = 0, \quad x \in [0,a)\cup(a,b],$$

$$y(a-0) = \alpha y(a+0), \quad y'(a-0) = \beta y'(a+0),$$

satisfying the initial conditions

$$\varphi(0)=0,\ \varphi'(0)=1; \quad \psi(b)=0,\ \psi'(b)=1$$

(for the existence and uniqueness of such solutions see [7]), and

$$W_x(\varphi,\psi) = p(x)[\varphi(x)\psi'(x) - \varphi'(x)\psi(x)], \quad x \in [0,a)\cup(a,b],$$

the Wronskian of solutions φ, ψ, is constant on each of the intervals $[0,a)$ and $(a,b]$ and hence (see [7])

$$W_x(\varphi,\psi) = \begin{cases} -p(0)\psi(0) & \text{if} \quad x \in [0,a), \\ -\frac{1}{\omega}p(0)\psi(0) & \text{if} \quad x \in (a,b]. \end{cases} \tag{19}$$

Note that $\psi(0) \neq 0$. Otherwise we would have $\psi \in D$ and $A\psi = 0$, so that $\psi \in \ker A$. But this is a contradiction, since we showed above that $\ker A = \{0\}$, but ψ is not equal to the zero element (we have $\psi'(b) = 1$).

Then it is not difficult to see that

$$(A^{-1}u)(x) = \int_0^b G(x,\xi)\rho(\xi)u(\xi)d\xi, \quad u \in L^2_\rho[0,b]. \tag{20}$$

The equations (18), (19), and (20) imply that A^{-1} is a completely continuous (or compact) operator. It is also symmetric with respect to the inner product (14).

The eigenvalue problem (15) is equivalent (note that $\lambda = 0$ is not an eigenvalue of A) to the eigenvalue problem

$$Bu = \mu u, \quad u \in L^2_\rho[0,b], \quad u \neq 0,$$

where

$$B = A^{-1} \quad \text{and} \quad \mu = \frac{1}{\lambda}.$$

In other words, if λ is an eigenvalue and $y \in D$ is a corresponding eigenfunction for A, then $\mu = \lambda^{-1}$ is an eigenvalue for B with the same eigenfunction y; conversely, if $\mu \neq 0$ is an eigenvalue and $u \in L^2_\rho[0,b]$ is a corresponding eigenfunction for B, then $u \in D$ and $\lambda = \mu^{-1}$ is an eigenvalue for A with the same eigenfunction u.

Note that $\mu = 0$ can not be an eigenvalue for B. In fact, if $Bu = 0$, then applying to both sides A we get that $u = 0$.

Next we use the following well-known Hilbert-Schmidt theorem (see, for example, [9, Sec. 24.3]): *For every completely continuous symmetric linear operator B in a Hilbert space H there is an orthonormal system*

$\{\upsilon_k\}$ of eigenvectors corresponding to eigenvalues $\{\mu_k\}$ $(\mu_k \neq 0)$ such that each $f \in H$ can be written uniquely in the form

$$f = \sum_k c_k \upsilon_k + h,$$

where $h \in \ker B$, that is, $Bh = 0$. Moreover,

$$Bf = \sum_k \mu_k c_k \upsilon_k$$

and if the system $\{\upsilon_k\}$ is infinite, then $\lim \mu_k = 0$ $(k \to \infty)$.

As a corollary of the Hilbert-Schmidt theorem we have: *If B is a completely continuous symmetric linear operator in a Hilbert space H and if $\ker B = \{0\}$, then the eigenvectors of B form an orthogonal basis of H.*

Applying the corollary of the Hilbert-Schmidt theorem to the operator $B = A^{-1}$ and using the above described connection between the eigenvalues and eigenfunctions of A and the eigenvalues and eigenfunctions of B we obtain the following result.

Theorem 2 *For the eigenvalue problem (6)–(8), there exists an orthonormal system $\{\upsilon_k\}$ of eigenfunctions corresponding to eigenvalues $\{\lambda_k\}$. Each eigenvalue λ_k is positive and simple and $\lambda_k \to \infty$ as $k \to \infty$. The system $\{\upsilon_k\}$ forms an orthonormal basis for the Hilbert space $L^2_\rho[0,b]$. Therefore any function $f \in L^2_\rho[0,b]$ can be expanded in eigenfunctions υ_k in the form*

$$f(x) = \sum_{k=1}^{\infty} c_k \upsilon_k(x), \tag{21}$$

where c_k are the Fourier coefficients of f defined by

$$c_k = \int_0^a \rho(x) f(x) \upsilon_k(x) dx + \omega \int_a^b \rho(x) f(x) \upsilon_k(x) dx. \tag{22}$$

Note that series (21) converges to the function f in metric of the space $L^2_\rho[0,b]$, which means that

$$\begin{aligned} & \int_0^a \rho(x) \left[f(x) - \sum_{k=1}^{n} c_k \upsilon_k(x) \right]^2 dx + \omega \int_a^b \rho(x) \left[f(x) - \sum_{k=1}^{n} c_k \upsilon_k(x) \right]^2 dx \\ \to \quad & 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{23}$$

Since

$$\begin{aligned} & \int_0^a \rho(x) \left[f(x) - \sum_{k=1}^{n} c_k \upsilon_k(x) \right]^2 dx + \omega \int_a^b \rho(x) \left[f(x) - \sum_{k=1}^{n} c_k \upsilon_k(x) \right]^2 dx \\ = \quad & \int_0^a \rho(x) f^2(x) dx + \omega \int_a^b \rho(x) f^2(x) dx - \sum_{k=1}^{n} c_k^2, \end{aligned}$$

we get from (23) the Parseval equality

$$\int_0^a \rho(x)f^2(x)dx + \omega\int_a^b \rho(x)f^2(x)dx = \sum_{k=1}^{\infty} c_k^2. \tag{24}$$

Remark 3 *Since* $\alpha,\ \beta,\ p(a-0)$, *and* $p(a+0)$ *are positive numbers and* $c_1 \le \rho(x) \le c_2$ *for some positive constants* c_1 *and* c_2, *we see that the condition* $f \in L^2_\rho[0,b]$ *is equivalent to the condition* $f \in L^2[0,b]$ *and (23) is equivalent to*

$$\lim_{n\to\infty}\int_0^b \left[f(x) - \sum_{k=1}^{n} c_k \upsilon_k(x)\right]^2 dx = 0$$

so that the series in (21) converges to the function f *in usual metric of the space* $L^2[0,b]$, *that is, in mean square metric.*

3. Uniformly convergent expansions

In this section we prove the following result.

Theorem 4 *Let* f *be a continuous function on* $[0,a)\cup(a,b]$ *satisfying the boundary conditions* $f(0)=f(b)=0$ *and such that it has a derivative* $f'(x)$ *everywhere on* $[0,a)\cup(a,b]$, *except at a finite number of points* $x_1, x_2, \ldots, x_m$, *the derivative being continuous everywhere except at these points, at which* f' *has finite limits from the left and right. Further, suppose that there exist finite limits* $f(a\pm 0)$, $f'(a\pm 0)$ *and* f *satisfies the impulse condition* $f(a-0)=\alpha f(a+0)$. *Then the series*

$$\sum_{k=1}^{\infty} c_k \upsilon_k(x), \tag{25}$$

where c_k *is defined by (22), converges uniformly on* $[0,a)\cup(a,b]$ *to the function* f.

Proof. First we assume for simplicity that the function f is differentiable everywhere on $[0,a]\cup(a,b]$. Consider the functional

$$J(y) = \int_0^a \left[p(x)y'^2(x) + q(x)y^2(x)\right]dx + \omega\int_a^b \left[p(x)y'^2(x) + q(x)y^2(x)\right]dx$$

so that we have $J(y)\ge 0$. Substituting in the functional $J(y)$

$$y = f(x) - \sum_{k=1}^{n} c_k \upsilon_k(x),$$

where c_k are defined by (22), we obtain

$$\begin{aligned}
& J\left(f-\sum_{k=1}^{n} c_k \upsilon_k\right) \\
= & \int_0^a p\left(f'-\sum_{k=1}^{n} c_k \upsilon_k'\right)^2 dx+\int_0^a q\left(f-\sum_{k=1}^{n} c_k \upsilon_k\right)^2 dx \\
& +\omega \int_a^b p\left(f'-\sum_{k=1}^{n} c_k \upsilon_k'\right)^2 dx+\omega \int_a^b q\left(f-\sum_{k=1}^{n} c_k \upsilon_k\right)^2 dx \\
= & \int_0^a\left(p f'^2+q f^2\right) dx+\omega \int_a^b\left(p f'^2+q f^2\right) dx \\
& -2 \sum_{k=1}^{n} c_k\left\{\int_0^a\left(p f' \upsilon_k'+q f \upsilon_k\right) dx+\omega \int_a^b(p f' \upsilon_k'+q f \upsilon_k) dx\right\} dx \\
& +\sum_{k, l=1}^{n} c_k c_l\left\{\int_0^a(p \upsilon_k' \upsilon_l'+q \upsilon_k \upsilon_l) dx+\omega \int_a^b(p \upsilon_k' \upsilon_l'+q \upsilon_k \upsilon_l) dx\right\}. \qquad (26)
\end{aligned}$$

Next, integrating by parts, we have

$$\begin{aligned}
& \int_0^a(p f' \upsilon_k'+q f \upsilon_k) dx+\omega \int_a^b(p f' \upsilon_k'+q f \upsilon_k) dx \\
= & p(x) f(x) \upsilon_k'(x)|_0^{a-0}+\int_0^a f[-(p \upsilon_k')'+q \upsilon_k] dx \\
& +\omega p(x) f(x) \upsilon_k'(x)|_{a+0}^b+\omega \int_a^b f[-(p \upsilon_k')'+q \upsilon_k] dx \\
= & \lambda_k\left(\int_0^a \rho f \upsilon_k dx+\omega \int_a^b \rho f \upsilon_k dx\right) \\
= & \lambda_k c_k,
\end{aligned}$$

$$\begin{aligned}
& \int_0^a(p \upsilon_k' \upsilon_l'+q \upsilon_k \upsilon_l) dx+\omega \int_a^b(p \upsilon_k' \upsilon_l'+q \upsilon_k \upsilon_l) dx \\
= & p(x) \upsilon_k(x) \upsilon_l'(x)|_0^{a-0}+\int_0^a \upsilon_k[-(p \upsilon_l')'+q \upsilon_l] dx \\
& +\omega p(x) \upsilon_k(x) \upsilon_l'(x)|_{a+0}^b+\omega \int_a^b \upsilon_k[-(p \upsilon_l')'+q \upsilon_l] dx \\
= & \lambda_l\left(\int_0^a \rho \upsilon_k \upsilon_l dx+\omega \int_a^b \rho \upsilon_k \upsilon_l dx\right) \\
= & \lambda_l \delta_{kl},
\end{aligned}$$

where δ_{kl} is the Kronecker symbol and where we have used the boundary conditions

$$f(a) = f(b) = 0, \quad \upsilon_k(a) = \upsilon_k(b) = 0,$$

and the impulse conditions

$$f(a-0) = \alpha f(a+0), \quad \upsilon_k(a-0) = \alpha \upsilon_k(a+0), \quad \upsilon_k'(a-0) = \beta \upsilon_k'(a+0).$$

Therefore we get from (26)

$$J\left(f - \sum_{k=1}^{n} c_k \upsilon_k\right)$$
$$= \int_0^a \left(pf'^2 + qf^2\right) dx + \omega \int_a^b \left(pf'^2 + qf^2\right) dx - \sum_{k=1}^{n} \lambda_k c_k^2.$$

Since the left-hand side is nonnegative for all n, we get the inequality

$$\sum_{k=1}^{\infty} \lambda_k c_k^2 \le \int_0^a \left(pf'^2 + qf^2\right) dx + \omega \int_a^b \left(pf'^2 + qf^2\right) dx \tag{27}$$

analogous to Bessel's inequality, and the convergence of the series on the left follows. All the terms of this series are nonnegative, since $\lambda_k > 0$. Note that the proof of (27) is entirely unchanged if we assume that the function $f(x)$ satisfies only the conditions stated in the theorem. Indeed, when integrate by parts, it is sufficient to integrate over the intervals on which f' is continuous and then add all these integrals (the integrated terms are vanished by $f(a) = f(b) = 0$ and the fact that f, φ_k, and φ_k' are continuous on $[0,a) \cup (a,b]$). We now show that the series

$$\sum_{k=1}^{\infty} |c_k \upsilon_k(x)| \tag{28}$$

is uniformly convergent on $[0,a) \cup (a,b]$. Obviously from this the uniformly convergence of series (25) will follow. Using the integral equation

$$\upsilon_k(x) = \lambda_k \int_0^b G(x,\xi)\rho(\xi)\upsilon_k(\xi)d\xi,$$

which follows from $\upsilon_k = \lambda_k A^{-1} \upsilon_k$ by (20), we can rewrite (28) as

$$\sum_{k=1}^{\infty} \lambda_k |c_k g_k(x)|, \tag{29}$$

where

$$g_k(x) = \int_0^b G(x,\xi)\rho(\xi)\upsilon_k(\xi)d\xi. \tag{30}$$

Setting

$$K(x,\xi) = \frac{1}{p(0)\psi(0)} \begin{cases} \varphi(x)\psi(\xi) & \text{if} \quad 0 \le x \le \xi \le b, \\ \varphi(\xi)\psi(x) & \text{if} \quad 0 \le \xi \le x \le b, \end{cases}$$

we have from (30), by (18) and (19),

$$g_k(x) = \int_0^a K(x,\xi)\rho(\xi)\upsilon_k(\xi)d\xi + \omega\int_a^b K(x,\xi)\rho(\xi)\upsilon_k(\xi)d\xi.$$

Therefore $g_k(x)$ can be regarded as the Fourier coefficient of $K(x,\xi)$ as a function of ξ. By using inequality (27), we can write

$$\begin{aligned}\sum_{k=1}^{\infty}\lambda_k g_k^2(x) \quad &\le \quad \int_0^a \left[p(\xi)K_\xi^2(x,\xi) + q(\xi)K^2(x,\xi)\right]d\xi \\ &+\omega\int_a^b \left[p(\xi)K_\xi^2(x,\xi) + q(\xi)K^2(x,\xi)\right]d\xi, \end{aligned} \tag{31}$$

where $K_\xi(x,\xi)$ is the derivative of $K(x,\xi)$ with respect to ξ. All the functions appearing under the integral sign are bounded and it follows from (31) that

$$\sum_{k=1}^{\infty}\lambda_k g_k^2(x) \le M,$$

where M is a positive constant. Now replacing λ_k by $\sqrt{\lambda_k}\sqrt{\lambda_k}$ we apply the Cauchy-Schwarz inequality to the segment of series (29):

$$\sum_{k=m}^{m+n}\lambda_k\left|c_k g_k(x)\right| \le \sqrt{\sum_{k=m}^{m+n}\lambda_k c_k^2}\sqrt{\sum_{k=m}^{m+n}\lambda_k g_k^2(x)} \le \sqrt{\sum_{k=m}^{m+n}\lambda_k c_k^2}\sqrt{M}$$

and this inequality, together with the convergence of the series with terms $\lambda_k c_k^2$ (see (27)), at once implies that series (29), and hence series (28) is uniformly convergent on $[0,a)\cup(a,b]$. Denote the sum of series (25) by $f_1(x)$:

$$f_1(x) = \sum_{k=1}^{\infty} c_k\upsilon_k(x). \tag{32}$$

Since the series in (32) is uniformly convergent on $[0,a)\cup(a,b]$, we get that

$$\int_0^a \rho(x)f_1(x)\upsilon_k(x)dx + \omega\int_a^b \rho(x)f_1(x)\upsilon_k(x)dx = c_k.$$

Therefore the Fourier coefficients of f_1 and f are the same. Then the Fourier coefficients of the difference $f_1 - f$ are zero and applying the Parseval equality (24) to the function $f_1 - f$ we get that $f_1 - f = 0$, so that the sum of series (25) is equal to $f(x)$. $\square$

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