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Traveling wavefronts in a single species model with nonlocal diffusion and age-structure*

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Abstract

This paper is concerned with the existence of monotone traveling wavefronts in a single species model with nonlocal diffusion and age-structure. We first apply upper and lower solution technique to prove the result if the wave speed is larger than a threshold depending only on the basic parameters. When the wave speed equals to the threshold, we show the conclusion by passing to a limit function.

Key Words: Age-structure, nonlocal diffusion, traveling wavefront, upper and lower solutions.

1. Introduction

Due to the different behavior of individuals with different ages in population dynamics, Aiello and Freedmann [1] first introduced the following single species model with time delay and age-structure

$$\begin{cases} u_i'(t) = \alpha u_m(t) - r u_i(t) - \alpha e^{-r\tau} u_m(t-\tau), \\ u_m'(t) = \alpha e^{-r\tau} u_m(t-\tau) - \beta u_m^2(t), \end{cases} \tag{1.1}$$

in which all the parameters are positive, u_i and u_m denote the number of immature and mature individuals of a single species, and time delay $\tau > 0$ describes the time taken from birth to maturity. Based on the model (1.1), Gourley and Kuang [5] further considered the spatial inhomogeneity of the individuals distribution and proposed the following reaction-diffusion system with non-local delays:

$$\begin{cases} \frac{\partial u_i(x,t)}{\partial t} = d_i \Delta u_i(x,t) + \alpha u_m(x,t) - r u_i(x,t) - \alpha e^{-r\tau} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i \tau}} e^{-\frac{y^2}{4 d_i \tau}} u_m(x-y, t-\tau) dy, \\ \frac{\partial u_m(x,t)}{\partial x} = d_m \Delta u_m(x,t) + \alpha e^{-r\tau} \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi d_i \tau}} e^{-\frac{y^2}{4 d_i \tau}} u_m(x-y, t-\tau) dy - \beta u_m^2(x,t), \end{cases} \tag{1.2}$$

where d_i, d_m are positive constants accounting for the diffusivity. In view of the background of the Gaussian kernel, the random migration of the individuals of model (1.2) is obvious, see also [10, 12, 13, 14, 15].

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Recently, Al-Omari and Gourley [2] further derived a model that describes the immobility of the immature members in the model proposed by [5], namely, let $d_i = 0$ in (1.2). In particular, they got the following reaction-diffusion equation with time delay to model the dynamical behavior of $u_m(x,t)$

$$\frac{\partial u_m(x,t)}{\partial t} = d\frac{\partial^2 u_m(x,t)}{\partial x^2} + \alpha e^{-\gamma\tau} u_m(x,t-\tau) - \beta u_m^2(x,t), x \in \mathbb{R}. \tag{1.3}$$

By the abstract results in Wu and Zou [16], the authors proved that (1.3) has a monotone traveling wavefront connecting 0 with $K_\tau = \frac{\alpha}{\beta}e^{-\gamma\tau}$. Yang and Fang [17] further considered a spatially discrete equation as

$$\frac{du_j(t)}{dt} = d[u_{j+1}(t) - 2u_j(t) + u_{j-1}(t)] + \alpha e^{-\gamma\tau} u_j(t-\tau) - \beta u_j^2(t), j \in \mathbb{Z} \tag{1.4}$$

and proved the existence of traveling wavefronts connecting 0 with K_τ. Model (1.4) can model the evolution of individuals that live in a discrete patch environment, and the migration of individuals is formulated by a discrete Laplacian operator [4].

In this paper, we shall consider the non-local diffusion version of models (1.3) and (1.4) as follows

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(y-x)[u(y,t) - u(x,t)]dy + \alpha e^{-\gamma\tau} u(x,t-\tau) - \beta u^2(x,t), x \in \mathbb{R} \tag{1.5}$$

where the parameters are same to those of (1.3) and (1.4), J is a probability function describing the migration of the individual. The reason why we formulate the migration by such an integral operator is that the (discrete) Laplacian operator is not sufficiently accurate in describing the spatial diffusion for some evolutionary process, such as the embryological development case [8]; we also refer to Bates [3] for more on the nonlocal diffusion model.

Similar to these of [2, 5, 17], we shall consider the existence of traveling wavefronts of (1.5) in this paper, and the main result is listed as follows.

Theorem 1.1 *For any given* $\tau \geq 0$*, there exists* $c^* = c^*(\tau) > 0$ *such that* (1.5) *has a traveling wavefront connecting* 0 *with* K_τ *if the wave speed is not smaller than* c^**.*

We first establish the existence of traveling wavefronts if the wave speed is larger than some threshold of the wave speed, which is based on an abstract result established in Pan et al. [9]; see also Li and Lin [7]. The main technique is to construct proper upper and lower solutions. When the wave speed equals to the threshold of the wave speed, we also prove the existence of traveling wavefronts by passing to a limit function, which is motivated by the idea of Thieme and Zhao [11]; see also Pan et al. [9] for a non-local diffusion model with time delay. Note that our result remains true if $\tau = 0$, then our result implies that the longer the maturation delay τ, the lower is the threshold of the wave speed, which is similar to the results in [2].

2. Preliminaries

We first consider the following nonlocal reaction-diffusion equation with time delay

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} J(y-x)[u(y,t) - u(x,t)]dy + f(u_t(x),\tau), \tag{2.6}$$

where $t \geq 0, x \in \mathbb{R}, u \in \mathbb{R}, f : C([-\tau, 0], \mathbb{R}) \to \mathbb{R}$ is a continuous functional (which may involve the parameter τ), $u_t(x) \in C([-\tau, 0], \mathbb{R})$ is parametered by $x \in \mathbb{R}$ and given by $u_t(x)(s) = u(x, t+s), s \in [-\tau, 0], t \geq 0, x \in \mathbb{R}$.

A *traveling wave solution* of (2.6) is a spacial translation invariant solution of the form $u(x,t) = U(x+ct)$, in which $c > 0$ describes the wave speed and $U \in C(\mathbb{R}, \mathbb{R})$ means the wave profile function. Substituting it into (2.6) and setting $z = x + ct$, then (2.6) becomes

$$\int_{\mathbb{R}} J(y-z)[U(y) - U(z)]dy - cU'(z) + f_c(U_z; \tau) = 0, \tag{2.7}$$

where $f_c : C([-c\tau, 0], \mathbb{R}) \to \mathbb{R}$ is defined by $f_c(U; \tau) = f(U^c; \tau)$ with $U^c(s) = U(cs), s \in [-\tau, 0]$. If such a traveling wave solution is monotone in $t \in \mathbb{R}$, then it is also called a *traveling wavefront*. In particular, from the background of traveling wavefronts in the population dynamics, we also require the traveling wavefronts satisfy the asymptotic boundary conditions

$$\lim_{z \to +\infty} U(z) = U_\tau^+, \lim_{z \to -\infty} U(z) = U_\tau^-. \tag{2.8}$$

Without loss of generality, we assume that $U_\tau^+ = U_\tau, U_\tau^- = 0$ such that (2.8) becomes

$$\lim_{z \to +\infty} U(z) = U_\tau, \lim_{z \to -\infty} U(z) = 0.$$

For convenience, we list the following conditions on (2.6)

(N1) $f(\hat{0}; \tau) = f(\widehat{U}_\tau; \tau) = 0$, and $f(\widehat{n}; \tau) \neq 0$ for $n \in (0, U_\tau)$. Herein $\widehat{\cdot}$ denotes the constant value function in $C([-c\tau, 0], \mathbb{R})$;

(N2) There exists a constant $\alpha_\tau \geq 0$ such that

$$f_c(U_1; \tau) - f_c(U_2; \tau) + \alpha_\tau[U_1(0) - U_2(0)] \geq \int_{\mathbb{R}} J(x)dx[U_1(0) - U_2(0)]$$

for all $U_1, U_2 \in C([-c\tau, 0], \mathbb{R})$ with $0 \leq U_2(z) \leq U_1(z) \leq U_\tau, z \in [-c\tau, 0]$;

(N3) $\int_{\mathbb{R}} J(x)u(x)dx \geq 0$ for any $u(x) \in C(\mathbb{R}, \mathbb{R})$ with $u(x) \geq 0, x \in \mathbb{R}$;

(N4) For any given $\mu \geq 0$, $\int_{\mathbb{R}} J(x)e^{\mu|x|}dx < \infty$;

(N5) $\int_{\mathbb{R}} J(x)u(x)dx = \int_{\mathbb{R}} J(x)u(-x)dx$ for any $u(x) \in C(\mathbb{R}, \mathbb{R})$;

(N6) $\int_{\mathbb{R}\setminus 0} J(x)dx > 0$.

In order to prove the existence of traveling wavefronts, we need to construct proper profile set by upper and lower solutions, which are defined as follows.

Definition 2.1 Assume that (N2) holds. A continuous function $U(z) : \mathbb{R} \to [0, U_\tau]$ is called an **upper** (**lower**) **solution** of (2.7), if $U(z)$ is differentiable on $\mathbb{R}$ except finite points $\mathbb{T}$, and $U'(z)$ are bounded for $z \in \mathbb{R} \setminus \mathbb{T}$ such that

$$cU'(z) \geq (\leq) \int_{\mathbb{R}} J(y-z)[U(y) - U(z)]dy + f_c(U_z; \tau), z \in \mathbb{R} \setminus \mathbb{T}. \tag{2.9}$$

Similar to the proof of Pan et al. [9, Theorem 3.2], we can obtain the following conclusion although our positive equilibrium depending on the time delay.

Theorem 2.2 *Assume that* (N1)–(N4) *hold. Suppose that* (2.7) *has an upper solution* $\overline{U}(z)$ *and a lower solution* $\underline{U}(z)$ *such that*

(i) $\sup_{s\leq z}\underline{U}(s)\leq \overline{U}(z), z\in\mathbb{R}$;

(ii) $\sup_{z\in\mathbb{R}}\underline{U}(z)>0, \inf_{z\in\mathbb{R}}\overline{U}(z)<U_\tau$.

Then (2.6) *has a traveling wavefront* $U(z)$ *connecting* 0 *with* U_τ.

3. Proof of theorem 1.1

Let $\phi(z)$ be a traveling wavefront of (1.5), then $\phi(z)$ satisfies

$$c\phi'(z)=\int_{\mathbb{R}}J(y-z)[\phi(y)-\phi(z)]dy+\alpha e^{-\gamma\tau}\phi(z-c\tau)-\beta\phi^2(z), \tag{3.10}$$

and we are interested in the asymptotic boundary conditions

$$\phi(-\infty)=0, \phi(+\infty)=K_\tau. \tag{3.11}$$

Denote

$$f_c(\phi;\tau)=-\beta\phi^2(0)+\alpha e^{-\gamma\tau}\phi(-c\tau).$$

We first need to to verify that assumption (N1) and (N2) in the previous section are satisfied. Note that (N1) is clear, we formulate the proof of (N2) as follows.

Lemma 3.1 *For any* $c>0$, $f_c(\phi)$ *satisfies* (N2).

Proof. Let $\phi_1,\phi_2\in C([-c\tau,0];\mathbb{R})$ such that $0\leq\phi_1(z)\leq\phi_2(z)\leq K_\tau, z\in[-c\tau,0]$. Then

$$\begin{aligned}
& f_c(\phi_1;\tau)-f_c(\phi_2;\tau)\\
=\ & -\beta\phi_1^2(0)+\alpha e^{-\gamma\tau}\phi_1(-c\tau)+\beta\phi_2^2(0)-\alpha e^{-\gamma\tau}\phi_2(-c\tau))\\
=\ & -\beta\left[\phi_1^2(0)-\phi_2^2(0)\right]+\alpha e^{-\gamma\tau}\left[\phi_1(-c\tau)-\phi_2(-c\tau)\right]\\
\geq\ & -\beta\left[\phi_1^2(0)-\phi_2^2(0)\right]\\
\geq\ & -2\beta K_\tau\left[\phi_1(0)-\phi_2(0)\right].
\end{aligned}$$

We choose $\delta_\tau=2\beta K_\tau+\int_{\mathbb{R}}J(x)dx$, then

$$f_c(\phi_1;\tau)-f_c(\phi_2;\tau)+\delta_\tau[\phi_1(0)-\phi_2(0)]\geq\int_{\mathbb{R}}J(x)dx[\phi_1(0)-\phi_2(0)].$$

Therefore, $f_c(\phi)$ satisfies (N2). The proof is complete. □

In order to construct such a pair of upper and lower solutions, we define

$$\Delta(\lambda, c) = \int_{\mathbb{R}} J(x)[e^{\lambda x} - 1]dx - c\lambda + \alpha e^{-\gamma\tau} e^{-\lambda c\tau} \tag{3.12}$$

for positive constants c, λ. Then $\Delta(\lambda, c)$ is well-defined if (N4) holds and satisfies the following result.

Lemma 3.2 *Assume that* (N3)–(N6) *hold. For any given* $\tau \geq 0$ *, there exists* $c^* = c^*(\tau) > 0$ *such that* (3.12) *has two distinct positive roots* $\lambda_1(c)$ *and* $\lambda_2(c)$ *with* $\lambda_1(c) < \lambda_2(c)$ *for any* $c > c^*$*, and* (3.12) *has no positive root if* $c < c^*$*. In particular,*

$$\Delta(\lambda, c) = \begin{cases} > 0 & \textit{for } \lambda > \lambda_2; \\ < 0 & \textit{for } \lambda \in (\lambda_1, \lambda_2); \\ > 0 & \textit{for } \lambda < \lambda_1. \end{cases}$$

Remark 3.3 It is also clear that the longer the maturation delay τ, the lower is the threshold of the wave speed $c^*(\tau)$.

Now, we construct upper and lower solutions of equation (2.7).

Lemma 3.4 *Assume that* (N3)–(N6) *and* $c > c^*$ *hold. Then*

$$\overline{\phi}(z) = \min\left\{K_\tau, K_\tau e^{\lambda_1(c)z}\right\}$$

is an upper solution of (2.7).

Proof. It is sufficient to prove that

$$\int_{\mathbb{R}} J(y-z)[\overline{\phi}(y) - \overline{\phi}(z)]dy - c\overline{\phi}'(z) + \alpha e^{-\gamma\tau}\overline{\phi}(z - c\tau) - \beta\overline{\phi}^2(z) \leq 0, z \neq 0. \tag{3.13}$$

If $z > 0$, then $\overline{\phi}(z) = K_\tau$, $\overline{\phi}(z - c\tau) \leq K_\tau$ such that (3.13) is clear.

If $z < 0$, then $\overline{\phi}(z) = K_\tau e^{\lambda_1 z}, \overline{\phi}'(z) = K_\tau \lambda_1 e^{\lambda_1 z}$. Then

$$\begin{aligned} & \int_{\mathbb{R}} J(y-z)[\overline{\phi}(y) - \overline{\phi}(z)]dy - c\overline{\phi}'(z) + \alpha e^{-\gamma\tau}\overline{\phi}(z - c\tau) - \beta\overline{\phi}^2(z) \\ < & \int_{\mathbb{R}} J(y-z)[\overline{\phi}(y) - \overline{\phi}(z)]dy - c\overline{\phi}'(z) + \alpha e^{-\gamma\tau}\overline{\phi}(z - c\tau) \\ \leq & \int_{\mathbb{R}} J(y-z)[K_\tau e^{\lambda_1 y} - K_\tau e^{\lambda_1 z}]dy - cK_\tau\lambda_1 e^{\lambda_1 z} + \alpha e^{-\gamma\tau} K_\tau \lambda_1 e^{\lambda_1 z}. \\ = & \ 0. \end{aligned}$$

by the definition of $\lambda_1(c)$, namely (3.13) holds. The proof is complete. □

Lemma 3.5 *Assume that* (N3)–(N6) *and* $c > c^*$ *hold. Then*

$$\underline{\phi}(z) = \max\left\{0, K_\tau(1 - Me^{\varepsilon z})e^{\lambda_1(c)z}\right\}$$

is a lower solution of (2.7) *if*

$$\varepsilon \in (0, \min\{\lambda_1(c), \lambda_2(c) - \lambda_1(c)\}),\ M \geq \frac{\beta K_\tau}{-\Delta(\lambda_1(c) + \varepsilon, c)} + 1.$$

Proof. Let $z_1 = -\frac{1}{\varepsilon}\ln M$, then

$$\underline{\phi}(z) = \begin{cases} 0, & z \geq z_1, \\ K_\tau(1 - Me^{\varepsilon z})e^{\lambda_1 z}, & z \leq z_1, \end{cases}$$

and it is sufficient to prove that

$$\int_{\mathbb{R}} J(y-z)\left[\underline{\phi}(y) - \underline{\phi}(z)\right]dy - c\underline{\phi}'(z) + \alpha e^{-\gamma\tau}\underline{\phi}(z - c\tau) - \beta\underline{\phi}^2(z) \geq 0, z \neq z_1. \tag{3.14}$$

If $z > z_1$, then $\underline{\phi}(z) = 0$ and $\underline{\phi}(z - c\tau) \geq 0$ such that (3.14) is clear.

If $z < z_1$, then $\underline{\phi}(z) = K_\tau(1 - Me^{\varepsilon z})e^{\lambda_1 z}$ such that

$$\begin{aligned} & \int_{\mathbb{R}} J(y-z)\left[\underline{\phi}(y) - \underline{\phi}(z)\right]dy - c\underline{\phi}'(z) + \alpha e^{-\gamma\tau}\underline{\phi}(z - c\tau) - \beta\underline{\phi}^2(z) \\ \geq & \int_{\mathbb{R}} J(y-z)\left[K_\tau(1 - Me^{\varepsilon y})e^{\lambda_1 y} - K_\tau(1 - Me^{\varepsilon z})e^{\lambda_1 z}\right]dy \\ & -cK_\tau\lambda_1 e^{\lambda_1 z} + M(\lambda_1 + \varepsilon)e^{(\lambda_1+\varepsilon)z} + \alpha e^{-\gamma\tau}K_\tau(1 - Me^{\varepsilon(z-c\tau)})e^{\lambda_1(z-c\tau)} - \beta K_\tau^2 e^{2\lambda_1 z} \\ = & -MK_\tau\Delta(\lambda_1(c) + \varepsilon, c)\, e^{(\lambda_1+\varepsilon)z} - \beta K_\tau^2 e^{2\lambda_1 z}. \end{aligned}$$

By the definition of $\Delta(\lambda, c)$ and the choice of ε, we see that $\Delta(\lambda_1(c) + \varepsilon, c) < 0$. Thus,

$$-M\Delta(\lambda_1(c) + \varepsilon, c)\, e^{(\lambda_1+\varepsilon)z} - \beta K_\tau e^{2\lambda_1 z} \geq 0$$

if $M \geq \frac{\beta K_\tau}{-\Delta(\lambda_1(c)+\varepsilon,c)} + 1$ holds, which implies that (3.14) holds. The proof is complete. □

By Theorem 2.2 and Lemmas 3.2–3.5, the following result is obvious.

Lemma 3.6 *For any* $c > c^*(\tau)$, (3.10) *and* (3.11) *have a monotone solution* $\phi(z)$ *such that* $\lim_{z\to-\infty}\phi(z)e^{-\lambda_1(c)z} = K_\tau$.

For the case of $c = c^*$, we also have the following result.

Lemma 3.7 *If* $c = c^*(\tau)$, *then* (3.10) *and* (3.11) *have a monotone solution* $\phi(z)$.

Proof. Let $\{c_n\}_{n=1}^{\infty}$ be a sequence satisfying

$$2c^* = c_1 < c_2 < \cdots, \lim_{n\to+\infty} c_n = c^*.$$

Let $\phi_n(z)$ denote the monotone solution of (3.10) and (3.11) with $c = c_n$, then

$$c_n\phi_n'(z) = \int_{\mathbb{R}} J(y-z)[\phi_n(y) - \phi_n(z)]dy + \alpha e^{-\gamma\tau}\phi_n(z - c_n\tau) - \beta\phi_n^2(z). \tag{3.15}$$

Since the traveling wave solution is invariant in the sense of phase shift, then we can assume that $\phi_n(0) = \frac{K_\tau}{2}$ for all $n \in \mathbb{N}$.

Note that $\phi_n(z)$ is bounded, it is clear that $\phi_n(z), \phi_n'(z)$ are equicontinuous for $n \in \mathbb{N}$ and $z \in \mathbb{R}$. By Ascoli-Arzela lemma and a nested subsequence argument, there exists a subsequence of ϕ_n, still denote it by $\phi_n(t)$, and a continuous function $\phi_*(z)$ such that

$$\lim_{n\to\infty} \phi_n(z) = \phi_*(z), \lim_{n\to\infty} \phi_n'(z) = \phi_*'(z),$$

and the convergence is uniform on any bounded interval of $\mathbb{R}$ and is pointwise on $\mathbb{R}$. Then the Lebesgue's dominant theorem implies that

$$c^*\phi_*'(z) = \int_{\mathbb{R}} J(y-z)[\phi_*(y) - \phi_*(z)]dy + \alpha e^{-\gamma\tau}\phi_*(z - c^*\tau) - \beta\phi_*^2(z).$$

Since $\phi_n(0) = \frac{K_\tau}{2}$ and $\phi_n(z)$ are monotone in $z \in \mathbb{R}$, then $\lim_{z\to\pm\infty}\phi_*(z)$ exists and $\lim_{z\to-\infty}\phi_*(z) \le \frac{K_\tau}{2} \le \lim_{z\to+\infty}\phi_*(z)$, which further implies that

$$\lim_{z\to-\infty} \phi_*(z) = 0, \lim_{z\to+\infty} \phi_*(z) = K_\tau.$$

Therefore, $\phi_*(z)$ satisfies (3.10)–(3.11) with $c = c^*$. The proof is complete. □

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