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The linear functionals on fundamental locally multiplicative topological algebras

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Abstract

In this paper we study the dual space of fundamental locally multiplicative topological algebras and prove some results for linear and multiplicative linear functionals on these algebras. An investigation on locally compactness of the carrier space of these algebras is the last part of this note.

Key word and phrases: Multiplicative linear functionals, carrier space, fundamental locally multiplicative topological algebras.

1. Introduction

In [2] we have extended and proved the famous Cohen factorization theorem for complete metrizable fundamental topological algebras, where the meaning of fundamental topological algebras generalizing both local boundedness and local convexity is initially introduced in [1] and [2]. Yet, some of the basic theorems are proved on fundamental topological vector spaces in [3]. To answer the wide question, which properties of the well-known topological algebras can be extended to fundamental topological algebras, we have introduced in [5] the notion of fundamental locally multiplicative topological algebra (abbreviated by FLM) with a property very similar to the normed algebras.

In this note, we give an example of an FLM algebra which is neither locally bounded nor locally convex and in [6], by discussing on a necessary condition for a fundamental topological algebra to be FLM, we gave an example of a complete metrizable separable locally multiplicative convex topological algebra which is not FLM.

In section 2, we have gathered a collection of definitions and related results, and in section 3 we present a discussion on the relation between FLM and locally bounded algebras. In section 4, we define a norm on a subspace of the algebraic dual space of an FLM algebra where we show that the multiplicative linear functionals have a norm no greater than one.

In section 5, we prove that the exponential map can be defined on complete metrizable FLM algebras, and then we extend the Gleason, Kahane- Zelazko theorem [7; Theorem 16.7] to these algebras.

We close the paper with a theorem on locally compactness of the carrier space of FLM algebras, which is an extension of theorem (17.2) in [7].

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2. Basic definitions and related theorems

In this section we recall some related definitions and theorems of fundamental topological algebras.

Definition 2.1 *Let A be a topological vector space. We say A is a fundamental topological vector space if there exists $b > 1$ such that for every sequence (a_n) of A, the convergence of $b^n(a_n - a_{n-1})$ to zero in A implies that (a_n) is a Cauchy sequence.*

Proposition 2.2 *([2; Proposition 2.4]) Let A be a fundamental topological vector space. Then, for every $c > 1$ and every sequence (a_n) of A, the convergence of $c^n(a_n - a_{n-1})$ to zero in A implies that (a_n) is a Cauchy sequence.*

Theorem 2.3 *([3;Theorem 3.5]) Suppose $b > 1$ and A is a fundamental metrizable topological vector space. Then A is complete if and only if for every sequence (x_n) with the condition $b^n x_n \to 0$ in A, the series $\sum(x_n)$ is summable.*

Definition 2.4 *A fundamental topological algebra is a topological algebra whose underlying topological vector space is fundamental.*

Theorem 2.5 *([5;Theorem 4.1]) Let A be a complete metrizable fundamental topological algebra with unit, and $x \in A$. If for some $b > 1, b^n x^n \to 0$ in A, then $1 - x$ is invertible and*

$$(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n.$$

Definition 2.6 *A fundamental topological algebra is said to be locally multiplicative, if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero, the "sufficiently large powers" of U_0 lie in V. We call such an algebra an FLM algebra.*

Theorem 2.7 *([5;Theorem 4.5]) Let A be a complete metrizable FLM algebra. Then, every multiplicative linear functional is automatically continuous.*

3. The relation between FLM and locally bounded topological algebras

In this section we study the relation between FLM and locally bounded topological algebras.

It is easy to see that every locally bounded algebra is an FLM. Is every metrizable FLM necessarily a locally bounded one? In theorem 3.1 we show that if the algebra has a unit element the answer is yes, but the following example shows that, when the algebra has no unit element, the answer is not positive.

Example 3.1 *Let A be a locally bounded topological algebra which is not locally convex, and B be a metrizable locally convex topological vector space which is not locally bounded. Define the product on B trivially by $xy = 0$ for all $x, y \in B$. Then B is a locally convex algebra which is not locally bounded.*

Now, $X = A \oplus B$, with the usual point-wise definitions for addition, scalar multiplication and product, is a fundamental topological algebra which is not locally bounded and not locally convex. This X is obviously an FLM algebra.

Theorem 3.2 *Let A be a topological algebra with unit element. Then A is FLM if and only if it is locally bounded.*

Proof. Suppose U_0 satisfies in definition (2.6) for A and $1 \in A$. For some $\lambda > 0, \lambda.1 \in U_0$, and for the neighborhood V of zero $\exists n_0 \in \mathbb{N}$ such that for $n \geq n_0, U_0^n \subseteq V$. Now $\lambda^n U_0 = \lambda^n.1^n U_0 \subseteq {U_0}^n U_0 = {U_0}^{n+1} \subseteq V$ and therefore U_0 is bounded. □

4. A norm on a subspace of the algebraic dual space of an FLM algebra

Let A be a metrizable FLM algebra, $\hat{A}$ be the space of all linear functionals on A, and $f \in \hat{A}$. Define $S(A) = \{x \in A : \exists\ b > 1 \quad \text{such that } b^n x^n \to 0\}$ and $v(f) = \sup\{|f(x)| : x \in S(A)\}$. Then we have the following theorem.

Theorem 4.1 *Taking the above notations, one has:*

i) $S(A)$ is a balanced absorbing set;

ii) $0 \leq v(f) \leq \infty$ and $v(f) = 0 \Leftrightarrow f \equiv 0$;

iii) if $A^* = \{f \in \hat{A} : v(f) < \infty\}$ then, $v(.)$ is a a norm on A^* and $(A^*; v(.))$ is a Banach space;

iv) $A^* \subseteq A'$ where A' is the set of all continuous linear functionals on A,

v) if A is complete, for a multiplicative linear functional $f, v(f) \leq 1$; moreover, if A has a unit element 1 such that $1 \in \overline{S(A)}$, then $v(f) = 1$.

Proof. (i) is clear; and for (ii), suppose $v(f) = 0$. Since for $b > 1$ and $x \in b^{-1}U_0$, where U_0 is the neighborhood satisfying in the Definition (2.6) of FLM, $b^n x^n \to 0$, we have $|f(x)| \leq v(f) = 0$ and therefore $f \equiv 0$ on A. Since obviously $v(f + g) \leq v(f) + v(g)$ and $v(\alpha f) = |\alpha| v(f)$, to prove (iii) it suffices to show that $(A^*; v(.))$ is complete. Let (f_n) be a Cauchy sequence in $(A^*; v(.))$, $b > 1$, and $x \in V = b^{-1}U_0$. The sequence $(f_n(x))$ is a Cauchy sequence of $\mathbb{C}$ and so there exists a function $f : A \to \mathbb{C}$ such that $f_n(x) \to f(x)$ for all $x \in X$. Suppose $b > 1$. For $\epsilon > 0$ and $x \in S(A)$ there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $|f_n(x) - f(x)| < \epsilon$ and therefore $v(f) < \infty$ which says $f \in A^*$ and $v(f_n - f) < \epsilon$.

Let $f \in A^*$ then, for all $x \in V = 2^{-1}U_0$, $|f(x)| \leq v(f)$ *i.e.* f is bounded on V and hence $f \in A'$ which proves (iv).

To prove (v) let f be a multiplicative linear functional on A. Since f is continuous on A (2.7), for $x \in S(A)$, if $b > 1$ be such that $b^n x^n \to 0$, then $f(b^n x^n) \to 0$ and thus $f(x)^n = f(x^n) \to 0$. Now $|f(x)| < 1$ and so $v(f) \leq 1$.

Let moreover $1 \in \overline{S(A)}$ and (x_n) be a sequence in $S(A)$ such that $x_n \to 1$. Then $|f(x_n)| \leq v(f) \leq 1$. Since f is continuous, $1 = f(1) = |f(1)| = v(f)$. □

5. The Gleason, Kahane- Zelazko theorem in FLM algebras

On a Banach algebra A, the definition of the exponential map $E(x)=\sum_{n=1}^{\infty}\frac{x^n}{n!}$ for $x\in A$ is based on the well-known theorem that the absolutely convergent series in Banach spaces are summable.

Theorem 2.3 extends this fact and proves it for complete metrizable fundamental topological vector spaces.

Notation. For a complete metrizable topological algebra A, by $E(A)$ we mean the set of all elements $x\in A$ for which $E(x)$ can be defined, and if $1\in A$, by $exp(a)$ we mean $1+E(a)$ for $a\in E(A)$.

Proposition 5.1 *Let A be a complete metrizable topological algebra with $1\in A$. If $\{a,-a\}\subseteq E(A)$ then, $\exp(a)$ is invertible in A.*

Proof. If temporary, we put $x^0=1$ for $x\in A$, then for $a\in A$ we have:

$$1=\exp(0)=\sum_{n=0}^{\infty}\frac{(a+(-a))^n}{n!}=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}a^k(-a)^{n-k}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{n!}\binom{n}{k}(-1)^{n-k}a^n=\sum_{k=0}^{n}\sum_{n=0}^{\infty}\frac{1}{n!}\binom{n}{k}(-1)^{n-k}a^n$$

$$=\sum_{k=0}^{n}\sum_{n=k}^{\infty}\frac{1}{n!}\binom{n}{k}(-1)^{n-k}a^n=\sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{1}{k!}\frac{1}{(n-k)!}a^k(-a)^{n-k}$$

$$=\sum_{k=0}^{\infty}\frac{a^k}{k!}\sum_{n=k}^{\infty}\frac{(-a)^{n-k}}{(n-k)!}=\sum_{k=0}^{\infty}\frac{a^k}{k!}\sum_{n=0}^{\infty}\frac{(-a)^n}{n!}=\exp(a)\exp(-a).$$

□

Proposition 5.2 *Suppose A is a complete metrizable fundamental topological algebra. If for $x\in A$ the set $\{x^n:n\in\mathbb{N}\}$ is bounded, then for all $\lambda\in\mathbb{C},\lambda x\in E(A)$.*

Proof. Suppose (x^n) is bounded and $b>1$. Since $\frac{b^n\lambda^n}{n!}\to 0$, we see that $b^n\,\frac{(\lambda x)^n}{n!}\to 0$ and so by Theorem (2.3) the result holds. □

Theorem 5.3 *Let A be a commutative complete metrizable locally convex topological algebra, $G=\{x\in A:(x^n)$ is bounded$\}$; and $L(G)$ be the subalgebra generated by G. Then*

i) $L(G)\subseteq E(A)$,

ii) if A has a unit element, then for all $x,y\in L(G)$,

$$exp(x+y)=exp(x)\ exp(y).$$

Proof. Let $x, y \in G$ and U *and* W be convex neighborhoods of zero such that $W \subseteq U$, $W^2 \subseteq U$, and let for all $j \in \mathbb{N}$, $x^j \in lW$ *and* $y^j \in lW$ with some $l > 1$. Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \in \sum_{k=0}^{n} \binom{n}{k} l^2 W^2 \subseteq 2^n l^2 U,$$

and so $\frac{(x+y)}{2} \in G$. Now by (5.2), $(x+y) \in E(A)$. If $1 \in A$, Since A^* separates the points of A, (ii) holds by [4; 3.6]. □

When A is an FLM algebra we can get a better result viz. The following theorem.

Theorem 5.4 *Let A be a complete metrizable FLM algebra, then $E(A) = A$.*

Proof. Let A be a complete metrizable FLM algebra and $x \in A$. Let also U_0 satisfies in the Definition of FLM , and $b > 1$. There exists $\lambda > 0$ such that $\lambda x \in U_0$ and so $\lambda^n x^n \to 0$. Since $\frac{b^n \lambda^{-n}}{n!} \to 0$, so $b^n(\frac{x^n}{n!}) \to 0$ in A. Now, by theorem 2.3 we have $x \in E(A)$. □

Now we can prove the Gleason, Kahane-Zelazko theorem in FLM algebras without using the notion of locally bounded and p-norms [8].

Theorem 5.5 *(The Gleason, Kahane - Zelazko theorem in FLM algebras). Let A be a complete metrizable FLM algebra with unit 1 and φ be a non-zero linear functional on A. Then the following conditions are equivalent:*

(i) $\varphi(1) = 1$ *and* $\ker(\varphi) \subseteq SingA$,

(ii) $\varphi(a) \in Sp(a)$ *for all* $a \in A$,

(iii) $\varphi : A \to \mathbb{C}$ *is multiplicative.*

Proof. By a simple algebraic proof, $(i) \Leftrightarrow (ii)$ and $(iii) \Rightarrow (i)$ can be proved and is omitted.

Now, let $\varphi(1) = 1$ and $\ker(\varphi) \subseteq SingA$. For $x \in A$ and $b > 1$ if $b^n x^n \to 0$ then $\forall \lambda \in \mathbb{C}$ with $|\lambda| > 1, \lambda^{-n} b^n x^n \to 0$ *i.e.* $b^n(\lambda^{-n} x^n) \to 0$ and by 2.5 $1 - \lambda^{-1}x \in InvA$. Therefore $\varphi(x) \neq \lambda$ and so, $b^n x^n \to 0 \Rightarrow |\varphi(x)| \leq 1$. Thus, φ is bounded on $V = b^{-1}U_0$ and so is continuous where U_0 is the neighborhood satisfying in the definition of FLM (2.6).

Suppose $b > 1$ and $a \in V = b^{-1}U_0$. Put $x = a^k$ for a natural $k \in \mathbb{N}$. Since $(b^k)^n x^n = b^{kn} a^{kn} \to 0$, therefore $|\varphi(x)| \leq 1$ *i.e.* $|\varphi(a^k)| \leq 1$.

Now, for $a \in V$, define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \varphi(\exp(za)) = \varphi(1 + \lim_{N\to\infty} \sum_{1}^{N} \frac{z^n a^n}{n!})$$

$$= \varphi(1) + \lim_{N\to\infty} \sum_{1}^{N} \frac{z^n \varphi(a^n)}{n!}$$

$$= 1 + \sum_{1}^{\infty} \frac{\varphi(a^n)z^n}{n!}. \qquad (*)$$

Since $\frac{|\varphi(a^n)z^n|}{n!} \leq \frac{|z^n|}{n!}$, the series in $(*)$ converges and therefore F is an entire function and

$$|F(z)| = |1 + \lim_{N\to\infty} \sum_{1}^{N} \frac{\varphi(a^n)z^n}{n!}|$$

$$\leq 1 + \lim_{N\to\infty} |\sum_{1}^{N} \frac{\varphi(a^n)z^n}{n!}|$$

$$\leq 1 + \sum_{1}^{\infty} \frac{|z^n|}{n!} = \exp(|z|) \quad \forall z \in \mathbb{C},$$

so that F has order at most 1. Since $\exp(za) \in InvA,$ for all $z \in \mathbb{C}$, we have $F(z) \neq 0$. Therefore there exists $\alpha \in \mathbb{C}$ such that:

$$F(z) = \exp(\alpha z) = 1 + \sum_{1}^{\infty} \frac{\alpha^n z^n}{n!} \quad (\forall z \in \mathbb{C}).$$

So, by comparing the two formulas for $F(z)$, we get

$$\varphi(a) = \alpha, \varphi(a^2) = \alpha^2.$$

Now let $x \in A$. There exists $\gamma > 0$ such that $a = \gamma x \in V$ and $\varphi(\gamma^2 x^2) = \varphi(\gamma x)^2$ and so $\varphi(x^2) = (\varphi(x))^2$, *i.e.* φ is a Jordan functional and hence by a famous algebraic proof [7; Proposition 16.6], it is multiplicative. □

Theorem 5.6 *([7; Theorem 17.2]) Let A be a complete metrizable FLM algebra.The carrier space Φ_A is a locally compact Hausdorff space with one point compactification Φ_A^∞. If A has a unit element, Φ_A is compact.*

Proof. Suppose $g \notin \{f \in A' : v(f) \leq 1\}$, then $v(g) > 1$ and so $\exists x \in S(A)$ such that $v(g) \geq |g(x)| > 1$. Choose $0 < \epsilon < |g(x)| - 1$, then the *weak** neighborhood of g, *i.e.* $g + x^{-1}B(0;\epsilon)$ does not intersect $\{f \in A' : v(f) \leq 1\}$ and hence it is *weak**-closed. Let $b > 1$ and $V = b^{-1}U_0$. Since we have $\{f \in A' : v(f) \leq 1\} \subseteq \{f \in A' : |f(x)| \leq 1 \text{ on } V\}$, so by Banach- Alaoglu theorem [9;Theorem 4.3], the set $\{f \in A' : v(f) \leq 1\}$ is *weak**-compact. Now $\Phi_A^\infty \subseteq \{f \in A' : v(f) \leq 1\}$. Hence Φ_A^∞ is *weak**-compact and therefore the carrier space is locally compact. □

Corollary 5.7 *Suppose A is a complex Banach algebra, then for every continuous linear functional f, $\|f\| \leq v(f)$ and the set $\{f \in A' : v(f) \leq 1\}$ is weak*-compact.*

Proof. Suppose f is a linear functional on complex Banach algebra A, and let $x \in A$ with $\|x\| \leq 1$. Choose $x_k \in A$ with $\|x_k\| < 1$ such that $x_k \to x$, Then $\exists b > 1$ *such that* $b^n {x_k}^n \to 0$ and therefore $|f(x_k)| \leq v(f)$ and so $|f(x)| \leq v(f)$, *i.e* $\|f\| \leq v(f)$. $\square$

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