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Transversal lightlike submanifolds of indefinite sasakian manifolds

Cumali Yıldırım and Bayram Şahin

Abstract

We study both radical transversal and transversal lightlike submanifolds of indefinite Sasakian manifolds. We give examples, investigate the geometry of distributions and obtain necessary and sufficient conditions for the induced connection on these submanifolds to be metric connection. We also study totally contact umbilical radical transversal and transversal lightlike submanifolds of indefinite Sasakian manifolds and obtain a classification theorem for totally contact umbilical transversal lightlike submanifolds.

Key Words: Indefinite Sasakian Manifold, Lightlike Submanifold, Radical Transversal Lightlike Submanifold, Transversal Lightlike Submanifold.

1. Introduction

A submanifold M of a semi-Riemannian manifold $\bar{M}$ is called lightlike (degenerate) submanifold if the induced metric on M is degenerate. Lightlike submanifolds have been studied widely in mathematical physics. Indeed, lightlike submanifolds appear in general relativity as some smooth parts of event horizons of the Kruskal and Kerr black holes [10]. Lightlike submanifolds of semi-Riemannian manifold have been studied by Duggal-Bejancu and Kupeli in [4] and [12], respectively. Kupeli's approach is intrinsic while Duggal-Bejancu's approach is extrinsic.

Lightlike submanifolds of indefinite Sasakian manifolds are defined according to the behaviour of the almost contact structure of indefinite Sasakian manifolds and such submanifolds were studied by Duggal-Şahin in [8]. They defined and studied invariant, screen real, contact CR-lightlike and screen CR-lightlike submanifolds of indefinite Sasakian manifolds. Later on, Duggal and Şahin studied contact generalized CR-lightlike submanifolds of indefinite Sasakian manifolds [9]. It is known that contact geometry has been used differential equations, optics, and phase spaces of a dynamical system (see Arnold; [1]). We note that invariant screen real, screen CR and generalized CR-lightlike submanifolds of indefinite Kaehler manifolds were defined and studied in [6], [7]. On the other hand, CR-lightlike submanifolds of a Kaehler manifolds were studied in [4].

All these submanifolds of indefinite Sasakian manifolds mentioned above have invariant radical distribution on their tangent bundles i.e $\phi(Rad\ TM) \subset TM$, where ϕ is the almost contact structure of indefinite Sasakian manifold, $Rad\ TM$ is the radical distribution and TM is the tangent bundle. The above property is

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also valid for lightlike hypersurfaces [11] which are examples of contact CR-lightlike submanifolds of indefinite Sasakian manifolds.

In this paper, we define and study transversal and radical transversal lightlike submanifolds of indefinite Sasakian manifolds such that radical distributions of such submanifolds do not belong to the tangent bundle of the submanifold under the action of the almost contact structure of indefinite Sasakian manifold. More precisely, $\phi(Rad\ TM) = ltr(TM)$, where $ltr(TM)$ is the lightlike transversal bundle of lightlike submanifold. We note that transversal lightlike submanifolds were defined by Ṣahin in [13] for indefinite Kaehler manifolds and he showed that such submanifolds can be considered a lightlike version of anti-invariant (or totally real) submanifolds of Kaehler manifolds. We also note that anti-invariant submanifolds of contact manifolds (also, Hermitian manifolds) have been studied widely in differential geometry [15]. Therefore, the aim of this paper is to study radical transversal and transversal lightlike submanifolds as lightlike versions of anti-invariant submanifolds of Sasakian manifolds.

The paper is organized as follows. In section 2, we give basic information needed for this paper. In section 3, we introduce radical transversal lightlike submanifolds, give examples of such submanifolds and study the integrability of distributions. We also obtain a necessary and sufficient conditions for the induced connection to be metric connection. In section 4, we investigate the geometry of totally contact umbilical transversal lightlike submanifolds. We also investigate the existence(or non-existence) of transversal lightlike submanifolds in an indefinite Sasakian space form. In section 5, we give examples of transversal lightlike submanifolds, obtain integrability conditions of distributions and give geometric conditions for the induced connection to be metric connection.

2. Preliminaries

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+k}, \bar{g})$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution $Rad(TM)$ is of rank r, where $1 \leq r \leq m$. $Rad(TM) = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \cup_{x\in M}\{u \in T_x\bar{M} / \bar{g}(u, v) = 0,\ \forall v \in T_x M\}.$$

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, i.e., $TM = Rad\,(TM) \perp S(TM)$.

We consider a screen transversal vector bundle $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$ [4, page 144]. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$. Then

$$\begin{aligned} tr(TM) &= ltr(TM) \perp S(TM^{\perp}), \\ T\bar{M}|_M &= S(TM) \perp [Rad(TM) \oplus ltr(TM)] \perp S(TM^{\perp}). \end{aligned}$$

Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/Rad\,TM$ [12]. The following result is important to this paper.

Proposition 2.1. [4, page 157]. *The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM), S(TM^{\perp})$ and $ltr(TM)$.*

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

Case 1: r-lightlike if $r < min\{m, k\}$;

Case 2: Co-isotropic if $r = k < m$; $S(TM^{\perp}) = \{0\}$;

Case 3: Isotropic if $r = m < k$; $S(TM) = \{0\}$;

Case 4: Totally lightlike if $r = m = k$; $S(TM) = \{0\} = S(TM^{\perp})$.

The Gauss and Weingarten equations are:

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.1}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \quad \forall X \in \Gamma(TM), V \in \Gamma(tr(TM)), \tag{2.2}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X, N), \quad N \in \Gamma(ltr(TM)), \tag{2.4}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X, W), \quad W \in \Gamma(S(TM^{\perp})). \tag{2.5}$$

Denote the projection of TM on $S(TM)$ by $\bar{P}$. Then, by using (2.1), (2.3)–(2.5) and a metric connection $\bar{\nabla}$, we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.6}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(N, A_W X). \tag{2.7}$$

From the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{2.8}$$

$$\nabla_X \xi = -A^*_\xi X + {\nabla^*}^t_X \xi, \tag{2.9}$$

for $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\ TM)$. By using above equations we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \tag{2.10}$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y), \tag{2.11}$$

$$\bar{g}(h^l(X, \xi), \xi) = 0 \quad , \quad A^*_\xi \xi = 0. \tag{2.12}$$

In general, the induced connection ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.3) we get

$$(\nabla_X g)(Y, Z) = \bar{g}(h^l(X, Y), Z) + \bar{g}(h^l(X, Z), Y). \tag{2.13}$$

However, it is important to note that $\nabla^\star$ is a metric connection on $S(TM)$. Finally, we recall that the Gauss equation of lightlike submanifolds is given by

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + A_{h^l(X,Z)}Y - A_{h^l(Y,Z)}X + A_{h^s(X,Z)}Y \\ &\quad - A_{h^s(Y,Z)}X + (\nabla_X h^l)(Y,Z) - (\nabla_Y h^l)(X,Z) \\ &\quad + D^l(X, h^s(Y,Z)) - D^l(Y, h^s(X,Z)) + (\nabla_X h^s)(Y,Z) \\ &\quad - (\nabla_Y h^s)(X,Z) + D^s(X, h^l(Y,Z)) - D^s(Y, h^l(X,Z))\end{aligned} \tag{2.14}$$

for $\forall X, Y, Z \in \Gamma(TM)$.

Finally, we recall some basic definitions and results of indefinite Sasakian manifolds.

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold [3] if there is a $(1,1)$ tensor field ϕ, a vector field V, called the characteristic vector field and its 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \epsilon\, \eta(X)\eta(Y),\ \bar{g}(V,V) = \epsilon \tag{2.15}$$

$$\phi^2(X) = -X + \eta(X)V, \qquad \bar{g}(X, V) = \epsilon\eta(X) \tag{2.16}$$

$$d\eta(X, Y) = \bar{g}(X, \phi Y), \qquad \forall X, Y \in \Gamma(TM), \quad \epsilon = \pm 1. \tag{2.17}$$

It follows that

$$\phi V = 0 \tag{2.18}$$

$$\eta \circ \phi = 0, \quad \eta(V) = 1. \tag{2.19}$$

Then $(\phi, V, \eta, \bar{g})$ is called contact metric structure of $\bar{M}$. We say that $\bar{M}$ has a normal contact structure if $N_\phi + d\eta \otimes V = 0$, where N_ϕ is the Nijenhuis tensor field of ϕ [15]. A normal contact metric manifold is called a Sasakian manifold [15, 14] for which we have

$$\bar{\nabla}_X V = -\phi X, \tag{2.20}$$

$$(\bar{\nabla}_X \phi)Y = \bar{g}(X,Y)V - \epsilon\, \eta(Y)X. \tag{2.21}$$

Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of $(\bar{M}, \bar{g})$. For any vector field X tangent to M, we put

$$\phi X = PX + FX \tag{2.22}$$

where PX and FX are the tangential and the transversal parts of ϕX, respectively. Moreover, P is skew symmetric on $S(TM)$.

3. Radical transversal lightlike submanifolds

In this section, we define radical transversal lightlike submanifolds, give examples and study the geometry of such lightlike submanifolds. First we note that it is known that if M is tangent to the structure vector field V, then V belongs to $S(TM)$ [2].

Definition 3.1 *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. We say that M is a radical transversal lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

$$\phi(Rad\,TM) = ltr\,(TM), \tag{3.1}$$

$$\phi(S(TM)) = S\,(TM). \tag{3.2}$$

In this paper, we assume that the characteristic vector field is a spacelike vector field. If V is a timelike vector field then one can obtain similar results. But it is known that V can not be lightlike [3].

From now on, $(\mathbf{R}^{2m+1}_q, \phi_o, V, \eta, \bar{g})$ will denote the manifold $\mathbf{R}^{2m+1}_q$ with its usual Sasakian structure given by

$$\begin{aligned}
\eta &= \frac{1}{2}(dz - \sum_{i=1}^{m} y^i dx^i), V = 2\partial\, z \\
\bar{g} = \eta \otimes \eta &+ \frac{1}{4}(-\sum_{i=1}^{\frac{q}{2}} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i) \\
\phi_o(\sum_{i=1}^{m}(X_i \partial\, x^i &+ Y_i\, \partial\, y^i) + Z\, \partial\, z) = \sum_{i=1}^{m}(Y_i \partial\, x^i - X_i\, \partial\, y^i) + \sum_{i=1}^{m} Y_i y^i\, \partial\, z
\end{aligned}$$

where $(x^i; y^i; z)$ are the Cartesian coordinates. The above construction will help in understanding how the contact structure is recovered in next four examples.

Example 1. Let $\bar{M} = (\mathbf{R}^9_2, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Suppose M is a submanifold of $\mathbf{R}^9_2$ defined by

$$x^1 = y^2, x^2 = y^1, x^3 = -y^4, x^4 = y^3$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned}
Z_1 &= 2(\partial\, x_1 + \partial\, y_2 + y^1 \partial\, z) \\
Z_2 &= 2(\partial\, x_2 + \partial\, y_1 + y^2 \partial\, z) \\
Z_3 &= 2(\partial\, x_3 - \partial\, y_4 + y^3 \partial\, z) \\
Z_4 &= 2(\partial\, x_4 + \partial\, y_3 + y^4 \partial\, z) \\
Z_5 &= V = 2\partial\, z.
\end{aligned}$$

Hence, $Rad\, TM = span\{Z_1, Z_2\}$ and lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = (-\partial\, x_1 + \partial\, y_2 - y^1 \partial\, z),\ N_2 = (\partial\, x_2 - \partial\, y_1 + y^2 \partial\, z).$$

It follows that $\phi Z_1 = \frac{1}{2}N_2$, $\phi Z_2 = -\frac{1}{2}N_1$. Thus $\phi Rad\ TM = ltr(TM)$. Also, $\phi Z_3 = -Z_4$ implies that $\phi S(TM) = S(TM)$. Hence M is a radical transversal 2-lightlike submanifold.

Example 2. Let $\bar{M} = (\mathbf{R}^9_4, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,+,+,-,-,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Suppose M is a submanifold of $\mathbf{R}^9_4$ defined by

$$x^1 = u^1, x^2 = u^2 \sin\theta, x^3 = u^3, x^4 = -u^4\cos\theta$$

$$y^1 = u^3, y^2 = -u^4\sin\theta, y^3 = u^1, y^4 = -u^2\cos\theta.$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned} Z_1 &= 2(\partial\, x_1 + \partial\, y_3 + y^1\partial\, z), \\ Z_2 &= 2(\sin\theta\partial\, x_2 - \cos\theta\partial\, y_4 + \sin\theta y^2\partial\, z) \\ Z_3 &= 2(\partial\, x_3 + \partial\, y_1 + y^3\partial\, z) \\ Z_4 &= 2(-\cos\theta\partial\, x_4 - \sin\theta\partial\, y_2 - \cos\theta y^4\partial\, z) \\ Z_5 &= V = 2\partial\, z. \end{aligned}$$

Hence $Rad\ TM = span\{Z_1, Z_3\}$ and lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = (-\partial\, x_1 + \partial\, y_3 - y^1\partial\, z),\ N_3 = (\partial\, x_3 - \partial\, y_1 + y^3\partial\, z).$$

It follows that $\phi Z_1 = \frac{1}{2}N_3$, $\phi Z_3 = -\frac{1}{2}N_1$. Thus $\phi Rad\ TM = ltr(TM)$. Also, $\phi Z_2 = Z_4$ implies that $\phi S(TM) = S(TM)$.Hence M is a radical transversal 2-lightlike submanifold.

In the sequel we show that there is a restriction on the nullity degree, i.e., $dim(Rad\ TM)$.

Proposition 3.1. *There do not exist 1-lightlike radical transversal lightlike submanifolds of an indefinite Sasakian manifold $\bar{M}$.*

Proof. Let us suppose that M is an 1-lightlike radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. In this case $Rad\ TM = span\{\xi\}$. This implies that $ltr(TM) = span\{N\}$. Using (2.15) we have

$$\bar{g}(\phi\xi, \xi) = \bar{g}(\phi^2\xi, \phi\xi) + \eta(\phi\xi)\eta(\xi).$$

Then (2.16) and (2.18) imply that

$$\bar{g}(\phi\xi, \xi) = \bar{g}(-\xi + \eta(\xi)V, \phi\xi).$$

Since V belongs to $S(TM)$, we get

$$\bar{g}(\phi\xi, \xi) = 0.$$

On the other hand, (3.1) implies that $\phi\xi = N \in \Gamma(ltr(TM))$. Thus, we obtain $g(\phi\xi, \xi) = g(N, \xi) = 1$, which is a contradiction. Thus, we conclude that M can not be 1-lightlike radical transversal lightlike submanifold. □

From Definition 3.1, we have the following result.

Proposition 3.2. *There exist no isotropic or totally lightlike radical transversal lightlike submanifolds of an indefinite Sasakian manifold* $\bar{M}$.

Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. We say that M is a proper radical transversal lightlike if $S(TM) \neq 0$. From Definition 3.1 and Proposition 3.1, we can note the following special features:(i): $dim(Rad\ TM) \geq 2$, (ii): $dim(S(TM)) = 2s, s > 1$, (iii): Any proper 5-dimensional radical transversal lightlike submanifold must be 2-lightlike.

Theorem 3.1. *Let* M *be a radical transversal lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$. *Then the distribution* $S(TM^{\perp})$ *is invariant with respect to* ϕ.

Proof. For $W \in \Gamma(S(TM^{\perp}))$ and $\xi \in \Gamma(Rad\ TM)$, from (2.15) we have

$$g(\phi W, \xi) = -g(W, \phi\xi) = 0 \tag{3.3}$$

$$g(\phi W, N) = -g(W, \phi N) = 0 \tag{3.4}$$

which imply that $\phi(S(TM^{\perp})) \cap Rad\ TM = \{0\}$ and $\phi(S(TM^{\perp})) \cap ltr(TM) = \{0\}$. For $X \in \Gamma(S(TM))$, (2.15), (2.16) and (2.20) imply

$$g(\phi W, X) = -g(W, \phi X) = 0 \tag{3.5}$$

which shows that $\phi(S(TM^{\perp})) \cap S(TM) = \{0\}$. Thus, (3.3), (3.4) and (3.5) complete the proof. □

Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Let Q and T be the projection morphisms on $Rad\ TM$ and *S(TM)*, respectively. Then, for $X \in \Gamma(TM)$, we have

$$X = TX + QX \tag{3.6}$$

where $TX \in \Gamma(S(TM)), QX \in \Gamma(Rad\ TM)$. Applying ϕ to (3.6) we obtain

$$\phi X = \phi TX + \phi QX. \tag{3.7}$$

If we put $\phi TX = SX$ and $\phi QX = LX$, we rewrite (3.7) as

$$\phi X = SX + LX \tag{3.8}$$

where $SX \in \Gamma(S(TM))$ and $LX \in \Gamma(ltrTM)$.

Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, from (2.21), we have

$$\bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y = g(X, Y)V - \eta(Y)X.$$

Then, using (3.8), (2.3) and (2.4) we have

$$\begin{aligned} g(X,Y)V - \eta(Y)X &= \nabla_X SY + h^l(X,SY) + h^s(X,SY) - A_{LY}X \\ &\quad + \nabla^l_X LY + D^s(X,LY) - S\nabla_X Y - L\nabla_X Y \\ &\quad - \phi h^l(X,Y) - \phi h^s(X,Y). \end{aligned}$$

Then, taking the tangential, screen transversal and lightlike transversal parts of the above equation, respectively, we obtain

$$(\nabla_X S)Y = A_{LY}X + \phi h^l(X,Y) + g(X,Y)V - \eta(Y)X \tag{3.9}$$

$$h^l(X,SY) + \nabla^l_X LY - L\nabla_X Y = 0 \tag{3.10}$$

$$h^s(X,SY) + D^s(X,LY) - \phi h^s(X,Y) = 0. \tag{3.11}$$

It is known that the induced connection of a lightlike submanifold is not a metric connection. In the sequel, we obtain a necessary and sufficient condition for the induced connection to be a metric connection.

Theorem 3.2. *Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the induced connection ∇ on M is a metric connection if and only if $A_{\phi Y}X$ has no components in $S(TM)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad\ TM)$.*

Proof. We know from [4, page 161] that the induced connection is a metric connection if and only if $\nabla_X Y \in \Gamma(Rad\ TM)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad\ TM)$. Suppose that ∇ is a metric connection. Then, for $Z \in \Gamma(S(TM))$ from (2.3), we have

$$0 = \bar{g}(\bar{\nabla}_X Y, Z).$$

Using (2.15), we get

$$g(\phi\bar{\nabla}_X Y, \phi Z) + \eta(\bar{\nabla}_X Y)\eta(Z) = 0.$$

Hence

$$g(-(\bar{\nabla}_X \phi)Y + \bar{\nabla}_X \phi Y, \phi Z) = 0.$$

Thus, using (2.21) and (2.4), we obtain

$$g(A_{\phi Y}X, \phi Z) = 0.$$

Let us prove the converse. Suppose that $A_{\phi Y}X$ has no components in $S(TM)$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad\ TM)$. Then from (2.4) we get

$$g(\bar{\nabla}_X \phi Y, Z) = 0.$$

Thus, we have

$$g((\bar{\nabla}_X \phi)Y + \phi\bar{\nabla}_X Y, Z) = 0.$$

Using (2.21) and (2.3), we derive

$$g(\phi\nabla_X Y, Z) = 0.$$

Then from (2.15) and (2.16) we arrive at

$$g(\nabla_X Y, \phi Z) = 0,$$

which proves the assertion. $\square$

We now investigate the integrability of the distributions which are involved in the definition of a radical transversal lightlike submanifold.

Theorem 3.3. *Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $S(TM)$ is integrable if and only if*

$$h^l(X, SY) = h^l(Y, SX)$$

for $\forall X, Y \in \Gamma(S(TM))$.

Proof. If we interchange the role of X and Y in (3.10), then, we obtain

$$h^l(Y, SX) + \nabla^l_Y LX - L\nabla_Y X = 0. \tag{3.12}$$

Thus, from (3.10) and (3.12) we get

$$h^l(X, SY) - h^l(Y, SX) = L[X, Y]. \tag{3.13}$$

Then proof follows from (3.13). $\square$

Theorem 3.4. *Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $(Rad\ TM)$ is integrable if and only if*

$$A_{LX}Y = A_{LY}X$$

for $\forall X, Y \in \Gamma(Rad\ TM)$.

Proof. From (3.9), we get

$$(\nabla_X S)Y = A_{LY}X + \phi h^l(X, Y).$$

Hence, we obtain

$$-S\nabla_X Y = A_{LY}X + \phi h^l(X, Y). \tag{3.14}$$

Interchanging the role of X and Y in (3.14), we obtain

$$-S\nabla_Y X = A_{LX}Y + \phi h^l(Y, X). \tag{3.15}$$

Thus, from (3.14) and (3.15), we get

$$S\nabla_X Y - S\nabla_Y X = A_{LX}Y - A_{LY}X + \phi h^l(Y, X) - \phi h^l(X, Y).$$

Since h^l is symmetric, we obtain

$$S[X,Y] = A_{LX}Y - A_{LY}X. \tag{3.16}$$

□

Theorem 3.5. *Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, radical distribution defines a totally geodesic foliation on M if and only if*

$$\bar{g}(\phi Y, X)\eta(Z) = -g(A_{\phi Y}X, \phi Z)$$

for $X, Y \in \Gamma(Rad\,TM)$, $Z \in \Gamma\,(S(TM))$.

Proof. By the definition of radical transversal lightlike submanifold, $(Rad\,TM)$ is a totally geodesic foliation if and only if $g(\nabla_X Y, Z) = 0$ for $X, Y \in \Gamma(Rad\,TM)$, $Z \in \Gamma(S(TM))$. Since $\bar{\nabla}$ is a metric connection (2.3) implies that $g(\nabla_X Y, Z) = X\bar{g}(Y,Z) - g(Y, \bar{\nabla}_X Z)$. Thus, we get $g(\nabla_X Y, Z) = -g(Y, \bar{\nabla}_X Z)$. Using (2.15), (2.21) and (2.3), we get $g(\nabla_X Y, Z) = -g(\phi Y, X)\eta(Z) - g(\phi Y, \nabla_X \phi Z)$. From (2.8) and (2.11), we have

$$g(\nabla_X Y, Z) = -g(\phi Y, X)\eta(Z) - g(A_{\phi Y}X, \phi Z)$$

which completes the proof. □

Theorem 3.6. *Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, screen distribution defines a totally geodesic foliation if and only if $A^*_{\phi N}X$ has no compenents in $(S(TM))$ for $X, Y \in \Gamma\,(S(TM))$, $N \in \Gamma(ltr(TM))$.*

Proof. By the definition of radical transversal lightlike submanifold $(S(TM))$ is a totally geodesic foliation if and only if $g(\nabla_X Y, N) = 0$ for $X, Y \in \Gamma(S(TM))$, $N \in \Gamma(ltr(TM))$. Using (2.3), we get $g(\nabla_X Y, N) = g(\bar{\nabla}_X Y, N)$. From (2.15), we have $g(\bar{\nabla}_X Y, N) = g(\bar{\nabla}_X \phi Y, \phi N)$. Using (2.3) and (2.10), we obtain

$$g(\bar{\nabla}_X Y, N) = g(A^*_{\phi N}X, \phi Y)$$

which completes the proof. □

4. Totally contact umbilical radical transversal lightlike submanifolds

In this section, we study totally contact umbilical radical transversal submanifolds. First of all, we remind that any totally umbilical lightlike submanifold, tangent to the structure vector field, of an indefinite Sasakian manifold is totally geodesic and invariant [8]. Therefore, the notion of totally umbilical submanifolds [5] of a semi-Riemannian manifolds does not work for lightlike submanifolds of indefinite Sasakian manifold. In [8], the authors introduced the notion of totally contact umbilical submanifolds as follows. According to their definition, a lightlike submanifold of an indefinite Sasakian manifold is totally umbilical if, for $X, Y \in \Gamma(M)$,

$$\begin{aligned} h^l(X,Y) &= [g(X,Y) - \eta(X)\eta(Y)]\alpha_L \\ &\quad + \eta(X)h^l(Y,V) + \eta(Y)h^l(X,V) \end{aligned} \tag{4.1}$$

$$\begin{aligned} h^s(X,Y) &= [g(X,Y) - \eta(X)\eta(Y)]\alpha_S \\ &\quad + \eta(X)h^s(Y,V) + \eta(Y)h^s(X,V), \end{aligned} \tag{4.2}$$

where $\alpha_S \in \Gamma(S(TM^\perp))$ and $\alpha_L \in \Gamma(ltr(TM))$.

Theorem 4.1. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\alpha_L = 0$ if and only if screen distribution is integrable.*

Proof. From (2.15) and (2.3), we get

$$\bar{g}([X,Y],N) = \bar{g}(h^l(X,\phi Y),\phi N) - \bar{g}(h^l(Y,\phi X),\phi N) \tag{4.3}$$

for $X, Y \in \Gamma(S(TM))$ and $N \in \Gamma(ltrTM)$. Using (4.1), (2.3) and (2.20), we obtain

$$h^l(X,\phi Y) = g(X,\phi Y)\alpha_L \tag{4.4}$$

$$h^l(Y,\phi X) = g(Y,\phi X)\alpha_L. \tag{4.5}$$

Here using (2.3), we get

$$\bar{g}([X,Y],N) = 2g(Y,\phi X)g(\alpha_L,\phi N), \tag{4.6}$$

which proves the assertion. □

Theorem 4.2. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $\alpha_L = 0$ if and only if $h^*(X,\phi Y) = 0$ for $X, Y \in \Gamma(S(TM) - \{V\})$.*

Proof. From (2.21), (3.9) and (2.3), we obtain

$$\begin{aligned} g(X,Y)V &= \nabla_X \phi Y + h^l(X,\phi Y) + h^s(X,\phi Y) \\ &\quad - S\nabla_X Y - L\nabla_X Y - \phi h^l(X,Y) - \phi h^s(X,Y). \end{aligned}$$

Using this, we get

$$g(\nabla_X \phi Y, \phi\xi) - g(\phi h^l(X,Y),\phi\xi) = 0.$$

From (2.8), (2.15) and (4.1), we obtain

$$g(h^*(X,\phi Y),\phi\xi) = g(X,Y)g(\alpha_L,\xi),$$

which proves the assertion. □

Theorem 4.3. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the induced connection ∇ on M is a metric connection if and only if $A_{\phi\xi}X = -\eta(X)\xi$ for $X \in \Gamma(TM)$, $\xi \in \Gamma(Rad\,TM)$.*

Proof. Using (2.21), (2.3), (2.4), (4.1), (4.2) and (3.9) we obtain

$$\begin{aligned} -A_{\phi\xi}X + \nabla^l_X \phi\xi + D^s(X, \phi\xi) &= S\nabla_X \xi + L\nabla_X \xi + \eta(X)\phi h^l(\xi, V) \\ &\quad + \eta(X)\phi h^s(\xi, V) \end{aligned}$$

for $X \in \Gamma(TM), \xi \in \Gamma(Rad\,TM)$. Taking the tangential parts of this equation, we have

$$-A_{\phi\xi}X = S\nabla_X \xi + \eta(X)\phi h^l(\xi, V). \tag{4.7}$$

On the other hand, using (2.20) and (2.3), we get $h^l(\xi, V) = -\phi\xi$. Thus, from (4.7) and (2.16), we obtain

$$S\nabla_X \xi = -A_{\phi\xi}X - \eta(X)\xi. \tag{4.8}$$

Hence, $\nabla_X \xi \in \Gamma(Rad\,TM)$ if and only if $A_{\phi\xi}X = -\eta(X)\xi$. □

Theorem 4.4. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then radical distribution is parallel if and only if $A_{\phi\xi_2}\xi_1 = 0$ for $\forall \xi_1, \xi_2 \in \Gamma(Rad\,TM)$.*

Proof. From (2.21), we obtain

$$\bar{\nabla}_{\xi_1}\phi\xi_2 - \phi\bar{\nabla}_{\xi_1}\xi_2 = 0$$

for all $\xi_1, \xi_2 \in \Gamma(Rad\,TM)$. Using (2.3), (2.4) and taking tangential parts, we have

$$-A_{\phi\xi_2}\xi_1 = S\nabla_{\xi_1}\xi_2,$$

which completes the proof. □

Lemma 4.1. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, $\alpha_S = 0$.*

Proof. From (2.21), (2.3), (3.9), we get

$$\begin{aligned} g(X, X)V &= \nabla_X \phi X + h^l(X, \phi X) + h^s(X, \phi X) - S\nabla_X X - L\nabla_X X \\ &\quad -\phi h^l(X, X) - \phi h^s(X, X) \end{aligned}$$

for $X \in \Gamma(S(TM) - \{V\})$. Taking the screen transversal parts of this equation, we have

$$h^s(X, \phi X) = \phi h^s(X, X). \tag{4.9}$$

By using (4.9) and (4.2) for $W \in \Gamma(S(TM^{\perp}))$, we obtain

$$g(X, X)g(\alpha_S, \phi W) = 0.$$

Since $(S(TM))$ is non-dejenere, we get $\alpha_S = 0$. Proof is complete. □

A plane section p in $T_x\bar{M}$ of a Sasakian manifold $\bar{M}$ is called a ϕ-section if it is spanned by a unit vector X orthogonal to V and ϕX, where X is a non-null vector field on $\bar{M}$. The sectional curvature $K(p)$ with respect to p determined by X is called a ϕ-sectional curvature. If $\bar{M}$ has a ϕ-sectional curvature c which does not depend on the ϕ-section at each point, then c is constant in $\bar{M}$. Then, $\bar{M}$ is called a Sasakian space form, denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a Sasakian space form $\bar{M}(c)$ is given by [11]

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{(c+3)}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y\} + \frac{(c-1)}{4}\{\epsilon\eta(X)\eta(Z)Y \\ &\quad -\epsilon\eta(Y)\eta(Z)X + \bar{g}(X,Z)\eta(Y)V - \bar{g}(Y,Z)\eta(X)V \\ &\quad +\bar{g}(\phi Y,Z)\phi X + \bar{g}(\phi Z,X)\phi Y - 2\bar{g}(\phi X,Y)\phi Z\} \qquad (4.10)\end{aligned}$$

for any X, Y and Z vector fields on $\bar{M}$.

In the rest of this section, we check the existence (non-existence) of radical transversal lightlike submanifolds of indefinite Sasakian space form. But, we first give some preparatory lemmas.

Lemma 4.2. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$h^l(\nabla_X \phi X, \xi) = -g(\nabla_X \phi X, V)\phi\xi \qquad (4.11)$$

for $X \in \Gamma(S(TM) - \{V\})$ and $\xi \in \Gamma(Rad\, TM)$.

Proof. From (2.3) and (2.20) we get

$$\nabla_\xi V + h^l(\xi, V) + h^s(\xi, V) = -\phi\xi.$$

Hence

$$h^l(\xi, V) = -\phi\xi. \qquad (4.12)$$

On the other hand, from (4.1) we have

$$h^l(\nabla_X \phi X, \xi) = g(\nabla_X \phi X, V) h^l(\xi, V).$$

Thus, using (4.12) in the above equation, we get (4.11). □

Lemma 4.3. *Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(\nabla_X \phi X, V) = g(\phi X, \phi X). \qquad (4.13)$$

Proof. Since $\bar{\nabla}$ is metric connection, we have

$$\bar{g}(\bar{\nabla}_X \phi X, V) + \bar{g}(\phi X, \bar{\nabla}_X V) = 0.$$

Hence, using (2.3) and (2.20) we get

$$g(\nabla_X \phi X, V) - g(\phi X, \phi X) = 0.$$

Then, (2.15) gives (4.13). □

Lemma 4.4. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$h^l(\phi X, V) = 0. \tag{4.14}$$

Proof. From (2.3), we get

$$\bar{\nabla}_{\phi X} V = \nabla_{\phi X} V + h^l(\phi X, V) + h^s(\phi X, V).$$

Using (2.16) and (2.20) in the above equation, we have (4.14). □

Lemma 4.5. *Let M be a radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(\nabla_{\phi X} X, V) = -g(X, X) \tag{4.15}$$

for $\forall X \in \Gamma(S(TM) - \{V\})$.

Proof. Since $\bar{\nabla}$ is metric connection, we have

$$\bar{g}(\bar{\nabla}_{\phi X} X, V) + \bar{g}(X, \bar{\nabla}_{\phi X} V) = 0.$$

Using (2.3) and (2.20)

$$g(\nabla_{\phi X} X, V) + g(X, -\phi^2 X) = 0.$$

Thus, from (2.16), we have

$$g(\nabla_{\phi X} X, V) + g(X, X) = 0.$$

□

Lemma 4.6. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(X, \nabla_{\phi X} \xi) = -g(h^l(\phi X, X), \xi). \tag{4.16}$$

Proof. Since $\bar{\nabla}$ is metric connection, we have

$$\bar{g}(\bar{\nabla}_{\phi X} X, \xi) + \bar{g}(X, \bar{\nabla}_{\phi X} \xi) = 0.$$

Hence, using (2.3) we obtain (4.16). □

Lemma 4.7. *Let M be a totally contact umbilical radical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, we have*

$$g(\phi X, \nabla_X \xi) = -g(h^l(X, \phi X), \xi). \tag{4.17}$$

Proof. Since $\bar{\nabla}$ is metric connection, we have

$$\bar{g}(\bar{\nabla}_X \phi X, \xi) + \bar{g}(\phi X, \bar{\nabla}_X \xi) = 0.$$

Hence, using (2.3) we obtain (4.17). □

Theorem 4.5. *There exist no totally contact umbilical radical proper transversal lightlike submanifolds in an indefinite Sasakian space form $\bar{M}(c)$ with $c \neq -3$.*

Proof. Suppose M is a totally contact umbilical proper radical transversal lightlike submanifold of $\bar{M}(c)$ such that $c \neq -3$. From (2.14), (4.10) and (4.3) we get

$$\frac{1-c}{2} g(\phi X, \phi X) g(\phi \xi, \xi^{'}) = \bar{g}((\nabla_X h^l)(\phi X, \xi), \xi^{'}) - \bar{g}((\nabla_{\phi X} h^l)(X, \xi), \xi^{'}) \tag{4.18}$$

for $\forall X \in \Gamma(S(TM) - \{V\})$, $\xi, \xi^{'} \in \Gamma(Rad\ TM)$, where

$$(\nabla_X h^l)(\phi X, \xi) = \nabla^l_X h^l(\phi X, \xi) - h^l(\nabla_X \phi X, \xi) - h^l(\phi X, \nabla_X \xi) \tag{4.19}$$

and

$$(\nabla_{\phi X} h^l)(X, \xi) = \nabla^l_{\phi X} h^l(X, \xi) - h^l(\nabla_{\phi X} X, \xi) - h^l(X, \nabla_{\phi X} \xi). \tag{4.20}$$

Since M is totally contact umbilical, (4.1) implies that

$$h^l(\phi X, \xi) = 0. \tag{4.21}$$

From (4.1), (4.11), (4.13) and (2.15) we get

$$h^l(\nabla_X \phi X, \xi) = -g(X, X) \phi \xi. \tag{4.22}$$

From (4.1) and (4.14)

$$h^l(\phi X, \nabla_X \xi) = g(\phi X, \nabla_X \xi) \alpha_L. \tag{4.23}$$

Using (4.21), (4.22) and (4.23) in (4.19), we have

$$(\nabla_X h^l)(\phi X, \xi) = g(X, X) \phi \xi - g(\phi X, \nabla_X \xi) \alpha_L. \tag{4.24}$$

On the other hand, from (4.1), we have

$$h^l(X, \xi) = 0. \tag{4.25}$$

By using (4.1), (4.12) and (4.15), we obtain

$$h^l(\nabla_{\phi X} X, \xi) = g(X, X)\phi\xi. \tag{4.26}$$

From (4.1) and (4.14), we get

$$h^l(X, \nabla_{\phi X}\xi) = g(X, \nabla_{\phi X}\xi)\alpha_L. \tag{4.27}$$

Using (4.25), (4.26) and (4.27) in (4.20), we have

$$(\nabla_{\phi X} h^l)(X, \xi) = -g(X, X)\phi\xi - g(X, \nabla_{\phi X}\xi)\alpha_L. \tag{4.28}$$

Hence, using (4.24) and (4.28) in (4.18), we obtain

$$\begin{aligned} \frac{1-c}{2} g(X, X) g(\phi\xi, \xi^{'}) &= 2g(X, X)g(\phi\xi, \xi^{'}) + g(X, \nabla_{\phi X}\xi)g(\alpha_L, \xi^{'}) \\ &\quad - g(\phi X, \nabla_X \xi)g(\alpha_L, \xi^{'}). \end{aligned} \tag{4.29}$$

Then (4.16) and (4.17) imply that

$$\begin{aligned} \frac{1-c}{2} g(X, X) g(\phi\xi, \xi^{'}) &= 2g(X, X)g(\phi\xi, \xi^{'}) \\ &\quad + g(\alpha_L, \xi^{'})(g(h^l(X, \phi X) - h^l(\phi X, X), \xi)). \end{aligned}$$

Since h^l is symmetric, we obtain

$$\frac{1}{2}(1-c)g(X, X)g(\phi\xi, \xi^{'}) = 2g(X, X)g(\phi\xi, \xi^{'}).$$

Hence we have

$$(3+c)g(X, X)g(\phi\xi, \xi^{'}) = 0.$$

Since $(S(TM))$ and $(Rad\ TM) \oplus ltr(TM)$ are non-degenerate, we can choose a non-null vector field X and $g(\phi\xi, \xi^{'}) \neq 0$, so $c = -3$ which completes the proof. □

5. Transversal Lightlike submanifolds

In this section, we define transversal lightlike submanifolds, investigate the geometry of these submanifolds. As one can see from definition 5.1, transversal lightlike submanifolds can be considered as a lightlike version of totally real submanifolds defined in the Sasakian geometry [15].

Definition 5.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field V, immersed in an indefinite Sasakian manifold $(\bar{M}, \bar{g})$. We say that M is a transversal lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:*

$$\phi(Rad\,TM) = ltr\,(TM) \tag{5.1}$$

$$\phi(S(TM)) \subseteq (S(TM^{\perp})). \tag{5.2}$$

We denote the orthogonal complementary subbundle to $\phi S(TM)$ in $S(TM^{\perp})$ by μ. It is easy to see that μ is invariant. We also have the following results whose proofs are similar to those given in section 3.

Proposition 5.1. *There does not exist 1-lightlike transversal lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$.

Proposition 5.2. *There exist no isotropic or totally lightlike transversal lightlike submanifolds of an indefinite Sasakian manifold* $\bar{M}$.

A transversal lightlike submanifold is called proper if $S(TM) \neq 0$ and $S(TM^{\perp}) \neq 0$. Let M be a transversal lightlike submanifolds of an indefinite Sasakian manifold $\bar{M}$. Then Definition 5.1 and invariant μ imply the following special features:(i): $dim(Rad\,TM) \geq 2$, (ii): Any proper 3-dimensional transversal lightlike submanifolds must be 2-lightlike.

Example 3. Let $\bar{M} = (\mathbf{R}^9_2, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,+,+,+,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}.$$

Suppose M is a submanifold of $\mathbf{R}^9_2$ defined by

$$x^1 = y^2, x^2 = y^1, x^3 = y^4, x^4 = y^3.$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned} Z_1 &= 2(\partial x_1 + \partial y_2 + y^1 \partial z) \\ Z_2 &= 2(\partial x_2 + \partial y_1 + y^2 \partial z) \\ Z_3 &= 2(\partial x_3 + \partial y_4 + y^3 \partial z) \\ Z_4 &= 2(\partial x_4 + \partial y_3 + y^4 \partial z) \\ Z_5 &= V = 2\partial z. \end{aligned}$$

Hence $Rad\,TM = span\{Z_1, Z_2\}$ and screen transversal bundle $S(TM^{\perp})$ is spanned by

$$W_1 = 2(\partial x_3 - \partial y_4 + y^3 \partial z),\ W_2 = 2(-\partial x_4 + \partial y_3 - y^4 \partial z).$$

Thus $\phi Z_3 = -W_2, \phi Z_4 = W_1$. Also, lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = (-\partial x_1 + \partial y_2 - y^1 \partial z),\ N_2 = (\partial x_2 - \partial y_1 + y^2 \partial z).$$

It follows that $\phi Z_1 = \frac{1}{2} N_2$, $\phi Z_2 = -\frac{1}{2} N_1$. Thus, the conditions (5.1) and (5.2) are satisfied. Hence M is a transversal 2-lightlike submanifold.

Example 4. Let $\bar{M} = (\mathbf{R}_4^{11}, \bar{g})$ be a semi-Euclidean space, where $\bar{g}$ is of signature $(-,-,+,+,+,-,-,+,+,+,+)$ with respect to canonical basis

$$\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial z\}.$$

Suppose M is a submanifold of $\mathbf{R}_4^{11}$ defined by

$$x^1 = y^3, x^3 = y^1, x^4 = y^4, x^5 = y^5, x^2 = y^2 = 0$$

It is easy to see that a local frame of TM is given by

$$\begin{aligned} Z_1 &= 2(\partial\, x_1 + \partial\, y_3 + y^1 \partial\, z), \\ Z_2 &= 2(\partial\, x_3 + \partial\, y_1 + y^3 \partial\, z) \\ Z_3 &= 2(\partial\, x_4 + \partial\, y_4 + y^4 \partial\, z) \\ Z_4 &= 2(\partial\, x_5 + \partial\, y_5 + y^5 \partial\, z) \\ Z_5 &= V = 2\partial\, z. \end{aligned}$$

Hence $Rad\, TM = span\{Z_1, Z_2\}$ and lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = (-\partial\, x_1 + \partial\, y_3 - y^1 \partial\, z),\ N_2 = (\partial\, x_3 - \partial\, y_1 + y^3 \partial\, z).$$

It follows that $\phi Z_1 = \frac{1}{2} N_2$, $\phi Z_2 = -\frac{1}{2} N_1$. Hence, $\phi S(TM)$ is spanned by

$$W_1 = 2(\partial\, x_4 - \partial\, y_4 + y^4 \partial\, z),\ W_2 = 2(\partial\, x_5 - \partial\, y_5 + y^5 \partial\, z).$$

It follows that $\phi Z_3 = W_1$, $\phi Z_4 = W_2$. Consequently, M is a transversal lightlike submanifold. Note that the invariant bundle μ is spanned by

$$H_1 = 2(\partial\, x_2 + \partial\, y_2 + y^2 \partial\, z),\ H_2 = 2(\partial\, x_2 - \partial\, y_2 + y^2 \partial\, z).$$

Let M be a transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Let Q and T be the projection morphisms on $Rad\ TM$ and $S(TM) - \{V\}$, respectively. Then, for $X \in \Gamma(TM)$, we have

$$X = TX + QX + \eta(X)V, \tag{5.3}$$

where $TX + \eta(X) \in \Gamma(S(TM))$, $QX \in \Gamma(Rad\ TM)$. Applying ϕ to (5.3) we obtain:

$$\phi X = \phi TX + \phi QX, \tag{5.4}$$

If we put $\phi TX = WX$ and $\phi QX = LX$, we rewrite (5.4) as:

$$\phi X = WX + LX \tag{5.5}$$

where $WX \in \Gamma(S(TM^\perp))$ and $LX \in \Gamma(ltrTM)$. For $W \in \Gamma(S(TM^\perp))$, we can write

$$\phi W = BW + CW, \tag{5.6}$$

where $BW \in \Gamma(S(TM))$ and $CW \in \Gamma(\mu)$.

Let M be a transversal lightlike submanifolds of an indefinite Sasakian manifold $\bar{M}$. Then, from (2.21) and using (5.5), (2.3)-(2.5) and (5.6) for $X, Y \in \Gamma(TM)$, we have

$$-A_{WY}X - A_{LY}X - \phi h^l(X,Y) - Bh^s(X,Y) - g(X,Y)V + \eta(Y)X = 0 \tag{5.7}$$

$$\nabla^s_X WY + D^s(X, LY) - W\nabla_X Y - Ch^s(X,Y) = 0 \tag{5.8}$$

$$D^l(X, WY) + \nabla^l_X LY - L\nabla_X Y = 0. \tag{5.9}$$

Now, we investigate the integrability of distributions on transversal lightlike submanifolds.

Theorem 5.1. *Let M be a transversal lightlike submanifolds of an indefinite Sasakian manifold $\bar{M}$. Then $(Rad\, TM)$ is integrable if and only if*

$$D^s(X, LY) = D^s(Y, LX)$$

for $\forall X, Y \in \Gamma(Rad\, TM)$.

Proof. If we interchange the role of X and Y in (5.8), we obtain

$$\nabla^s_Y WX + D^s(Y, LX) - W\nabla_Y X - Ch^s(Y,X) = 0. \tag{5.10}$$

From (5.8) and (5.10), we get

$$D^s(Y, LX) - D^s(X, LY) + W(\nabla_X Y - \nabla_Y X) + Ch^s(X,Y) - Ch^s(Y,X) = 0.$$

Since h^s is symmetric, we obtain

$$W[X,Y] = D^s(X, LY) - D^s(Y, LX). \tag{5.11}$$

□

Theorem 5.2. *Let M be a transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $(S(TM))$ is integrable if and only if*

$$D^l(X, WY) = D^l(Y, WX)$$

for $\forall X, Y \in \Gamma(S(TM))$.

Proof. Interchanging role of X and Y in (5.9), then, we obtain

$$D^l(X, WY) + \nabla^l_X LY - L\nabla_X Y = 0. \tag{5.12}$$

From (5.9) and (5.12) , we get

$$D^l(Y, WX) - D^l(X, WY) + L[X,Y] = 0. \tag{5.13}$$

Thus, proof follows from (5.13). □

From (5.10) and (5.12) we have the following results.

Corollary 5.1. *Let M be a transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ .Then screen distribution $S(TM)$ defines a totally geodesic foliation if and only if $D^l(Y,WX)=0$ for $\forall X,Y\in\Gamma(S(TM))$.*

Corollary 5.2. *Let M be a transversal lightlike submanifolds of an indefinite Sasakian manifold $\bar{M}$. Then, radical distribution $Rad(TM)$ defines a totally geodesic foliation if and only if $D^s(X,LY)=Ch^s(X,Y)$ for $\forall X,Y\in\Gamma(Rad\,TM)$.*

From Corollary 5.1 and Corollary 5.2, we have the following corollary.

Corollary 5.3. *A transversal lightlike submanifold of an indefinite Sasakian manifold is a lightlike product manifold if and only if $D^l(Y,WX)=0$ and $D^s(X,LY)\in\Gamma(\mu)$.*

Now, we find a necessary and sufficient condition for ∇ to be metric connection.

Theorem 5.3. *Let M be a transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the induced connection ∇ on M is a metric connection if and only if $BD^s(X,\phi Y)=\eta(\nabla_X Y)$ for $X\in\Gamma(TM)$ and $Y\in\Gamma(Rad\,TM)$.*

Proof. From (2.21), we have

$$\bar{\nabla}_X\phi Y-\phi\bar{\nabla}_X Y=0.$$

Then, using (2.3), (2.4), (2.16) and (5.5) we get

$$\begin{aligned} -\nabla_X Y &= -WA_{\phi Y}X-LA_{\phi Y}X+\phi\nabla^l_X\phi Y+BD^s(X,\phi Y)\\ &\quad +CD^s(X,\phi Y)-\eta(\nabla_X Y)V+h^l(X,Y)+h^s(X,Y). \end{aligned}$$

Taking the tangential parts of the above equation, we obtain

$$-\nabla_X Y = \phi\nabla^l_X\phi Y+BD^s(X,\phi Y)-\eta(\nabla_X Y)V \tag{5.14}$$

for any $X\in\Gamma(TM)$ and $Y\in\Gamma(Rad\,TM)$. Then, from (5.14), $\nabla_X Y\in\Gamma(Rad\,TM)$ if and only if

$$BD^s(X,\phi Y)=\eta(\nabla_X Y)V,$$

which completes the proof. □

If M is totally contact umbilical, we have the following result.

Theorem 5.4. *Let M be a totally contact umbilical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then, the induced connection ∇ on M is a metric connection if and only if $D^s(X,\phi\xi)=0$ for $X\in\Gamma(TM)$, $\xi\in\Gamma(Rad\,TM)$.*

Proof. From (2.21), we have

$$\bar{\nabla}_X\phi\xi-\phi\bar{\nabla}_X\xi=0.$$

Then, using (2.3), (2.4), (4.1), (4.2) and (5.5)

$$\begin{aligned} -A_{\phi\xi}X + \nabla^l_X \phi\xi + D^s(X, \phi\xi) &= W\nabla_X\xi + L\nabla_X\xi + \eta(X)\phi h^l(\xi, V) \\ &\quad + \eta(X)Bh^s(\xi, V) + \eta(X)Ch^s(\xi, V) \end{aligned}$$

Taking the screen transversal parts of the above equation, we obtain:

$$D^s(X, \phi\xi) = W\nabla_X\xi. \tag{5.15}$$

Then, our assertion follows from (5.15). □

Corollary 5.4. *Let M be a totally contact umbilical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Rad TM is parallel if and only if $D^s(\xi_1, \phi\xi_2) = 0$ for $\forall \xi_1, \xi_2 \in \Gamma(Rad\ TM)$.*

Proof. From (2.21), we have

$$(\bar{\nabla}_{\xi_1}\phi)\xi_2 = \bar{\nabla}_{\xi_1}\phi\xi_2 - \phi\bar{\nabla}_{\xi_1}\xi_2 = 0$$

Hence,using (2.3), (2.4), (4.1), (4.2) and (5.5), we have

$$-A_{\phi\xi_2}\xi_1 + \nabla^l_{\xi_1}\phi\xi_2 + D^s(\xi_1, \phi\xi_2) = W\nabla_{\xi_1}\xi_2 + L\nabla_{\xi_1}\xi_2$$

Then, taking the screen tranversal parts of the above equation, we obtain:

$$D^s(\xi_1, \phi\xi_2) = W\nabla_{\xi_1}\xi_2.$$

Thus the proof is complete. □

Finally, we give a classification theorem for totally contact umbilical transversal lightlike submanifolds, but we first give a lemma which will be useful:

Lemma 5.1. *Let M be a totally contact umbilical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $\alpha_L = 0$ if and only if $D^s(X, \phi\xi)$ has no components in $\phi S(TM)$ for $X \in \Gamma(S(TM) - \{V\}), \xi \in \Gamma(Rad\ TM)$.*

Proof. Let M be a totally contact umbilical transversal lightlike submanifold. From (2.21), we have

$$g(X, X)V = \bar{\nabla}_X\phi X - \phi\bar{\nabla}_X X.$$

Then, using (2.3), (2.5), (5.5) and 5.6) we get

$$\begin{aligned} g(X, X)V &= -A_{\phi X}X + \nabla^s_X\phi X + D^l(X, \phi X) - W\nabla_X X - L\nabla_X X \\ &\quad -\phi h^l(X, X) - Bh^s(X, X) - Ch^s(X, X). \end{aligned}$$

Then, taking the tangential parts of the above equation, we obtain

$$g(X, X)V = -A_{\phi X}X - \phi h^l(X, X) - Bh^s(X, X). \tag{5.16}$$

Hence, we get

$$g(A_{\phi X}X, \phi\xi) + g(\phi h^l(X, X), \phi\xi) = 0.$$

Then, using (2.15) in this equation, we have

$$g(A_{\phi X}X, \phi\xi) + g(h^l(X, X), \xi) = 0. \tag{5.17}$$

Using (4.1) and (2.7) in (5.17), we obtain

$$g(D^s(X, \phi\xi), \phi X) + g(X, X)g(\alpha_L, \xi) = 0. \tag{5.18}$$

Since $(S(TM))$ is non-degenere, $\alpha_L = 0$ if and only if $D^s(X, \phi\xi)$ has no components in $\phi(S(TM))$. □

Theorem 5.5. *Let M be a totally contact umbilical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ such that $\phi S(TM) = S(TM^{\perp})$. Then $\alpha_S = 0$ or $dim(S(TM)) = 1$.*

Proof. From (5.16) for $Z \in \Gamma(S(TM) - \{V\})$, we have

$$g(A_{\phi X}X, Z) = g(h^s(X, X), \phi Z). \tag{5.19}$$

On the other hand, using (2.6) we get

$$g(A_{\phi X}X, Z) = g(h^s(X, Z), \phi X). \tag{5.20}$$

From (5.19) and (5.20), we obtain

$$g(h^s(X, X), \phi Z) = g(h^s(X, Z), \phi X). \tag{5.21}$$

Thus, from (5.21) and (4.2) we have

$$g(X, X)g(\alpha_S, \phi Z) = g(X, Z)g(\alpha_S, \phi X). \tag{5.22}$$

Interchanging the role X and Z, we derive

$$\bar{g}(\alpha_S, \phi X) = \frac{g(X, Z)^2}{g(X, X)g(Z, Z)}\,\bar{g}(\alpha_S, \phi\, X). \tag{5.23}$$

Thus either $S(TM)$ is one dimensional or $\alpha_S = 0$. Thus proof is complete. □

Corollary 5.5. *Let M be a totally contact umbilical transversal lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. If $D^s(X, \phi X) \in \Gamma(\phi(S(TM))$ for $X \in \Gamma(S(TM))$, then $dim(S(TM)) = 1$ or $\alpha_S = 0$.*

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Cumali YILDIRIM
Adıyaman University
Faculty of Science and Arts
Department of Mathematics
02210 Adıyaman-TURKEY
e-mail:cyildirim@adiyaman.edu.tr

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Bayram ŞAHİN
İnönü University
Faculty of Science and Arts
Department of Mathematics
44280, Malatya-TURKEY
e-mail:bsahin@inonu.edu.tr