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Number of pseudo–Anosov elements in the mapping class group of a four–holed sphere

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Abstract

We compute the growth series and the growth functions of reducible and pseudo-Anosov elements of the pure mapping class group of the sphere with four holes with respect to a certain generating set. We prove that the ratio of the number of pseudo-Anosov elements to that of all elements in a ball with center at the identity tends to one as the radius of the ball tends to infinity.

Key Words: Mapping class group, growth series, growth functions.

1. Introduction

A finitely generated group can be seen as a metric space after fixing a finite generating set. The metric is the so called word metric. As is well-known, the mapping class group of a compact surface is finitely generated, thus a metric space.

One of the purposes of this note is to prove that, after fixing a certain set of generators, in a ball centered at the identity in the pure mapping class group of a four holed sphere (which is a free group of rank two), almost all elements are pseudo–Anosov. More precisely, in a ball with center at the identity, the ratio of the number of pseudo–Anosov elements to the number of all elements tends to one as the radius of the ball tends to infinity. In fact, we prove more: We give the growth series of reducible and of pseudo–Anosov elements with respect to a fixed set of generators. It turns out that the growth functions of these elements are rational. This gives a partial answer to Question 3.13 and verifies Conjecture 3.15 in [2] in a special case. Similar results are proved in [5] and [6] by using different methods, which do not immediately imply the results of this paper.

2. Preliminaries

Let G be a finitely generated group with a finite generating set A, so that every element of G can be written as a product of elements in $A \cup A^{-1}$. The *length* of an element $g \in G$ (with respect to A) is defined as

$$||g||_A = \min\{k :\ g = a_1 a_2 \cdots a_k,\ a_i \in A \cup A^{-1}\}.$$

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The *distance* between two elements g and h is defined as $d_A(g,h) = ||h^{-1}g||_A$. The function d_A is a metric on G, called the *word metric*. Of course, this metric depends heavily on the generating set. The choice of different generating sets give rise to equivalent metrics. We will always fix a finite generating set A and drop A from the notation.

For a subset P of G, the *growth series* of P relative to the generating set A is the formal power series $\sum c_n x^n$, where the coefficient c_n of x^n is the number of elements of length n in P. The *growth function* of P is the function represented by the growth series. In the mapping class group, we may take P to be periodic, reducible or pseudo-Anosov elements.

Let S be a compact connected orientable surface of genus g with $r \geq 0$ holes (= boundary components). The *mapping class group* $\mathrm{Mod(S)} = \mathrm{Mod(g,r)}$ of S is defined as the group of isotopy classes of orientation-preserving homeomorphisms $S \to S$. The subgroup $\mathrm{PMod(g,r)}$ of $\mathrm{Mod(g,r)}$ consisting of isotopy classes of homeomorphisms preserving each boundary component of S is the *pure mapping class group*.

Thurston's classification of surface diffeomorphisms says that, for a mapping class f which is not the identity, one of the following holds: (1) f is periodic, i.e. $f^m = 1$ for some $m \geq 2$; (2) f is reducible, i.e. there is a (closed) one-dimensional submanifold C of S such that $f(C) = C$; (3) f is pseudo-Anosov (Anosov if S is a torus). It is well known that f is pseudo-Anosov if and only if f is neither periodic nor reducible.

It is well known that the mapping class group $\mathrm{Mod}(1,0)$ of a torus is isomorphic to $SL(2,\mathbb{Z})$. The elements of the group $\mathrm{Mod}(1,0)$ are classified by the traces of the corresponding matrices; if f is an element of $\mathrm{Mod}(1,0)$, then it is periodic if $|\mathrm{trace(f)}| < 2$, reducible if $|\mathrm{trace(f)}| = 2$, and Anosov if $|\mathrm{trace(f)}| > 2$ (cf. see [1]). In [7], Takasawa computed the growth series of periodic, reducible and Anosov elements of $\mathrm{Mod}(1,0)$ and found their growth functions. He proved that almost all elements of the mapping class group of the torus are Anosov. That is, with respect to a certain generating set, the ratio of the number of Anosov elements to the number of all elements in a ball centered at the identity tends to one as the radius of the ball tends to infinity.

Now let S be a sphere with four holes and let a and b be two distinct nonisotopic simple closed curves on S such that each of a and b separates S into two pairs of pants and that a intersects b precisely at two points (c.f. Figure 1). It is well known that $\mathrm{PMod}(0,4)$ is isomorphic to the free group F_2 and freely generated by the Dehn twists t_a and t_b about a and b respectively. We will always take this generating set below.

3. The number of reducible and pseudo-Anosov elements in the mapping class group $\mathrm{PMod}(0,4)$

3.1. Counting certain elements in the free group of rank two

We begin by counting certain type of elements in the free group of rank two. Let F_2 be the free group of rank two freely generated by $\{\alpha, \beta\}$. We fix this set of generators throughout this subsection.

The next lemma is elementary and is easy to prove.

Lemma 3.1 *The growth series of F_2 is*

$$h(x) \quad = \quad 1 + 4x + 4 \cdot 3x^2 + 4 \cdot 3^2 x^3 + \cdots + 4 \cdot 3^{n-1} x^n + \cdots .$$

For an element $\gamma \in F_2$, let $C(\gamma, n)$ denote the set of elements in F_2 of length n of the form $w\gamma^k w^{-1}$, where k is an integer and $w \in F_2$. Let $|C(\gamma, n)|$ denote the cardinality of $C(\gamma, n)$.

Lemma 3.2 *1. If $w\alpha^k w^{-1}$ and $v\alpha^l v^{-1}$ are reduced, then $w\alpha^k w^{-1} = v\alpha^l v^{-1}$ if and only if $w = v$ and $k = l$.*

2. For each nonnegative integer r, $|C(\alpha, 2r+1)| = |C(\alpha, 2r+2)| = |C(\beta, 2r+1)| = |C(\beta, 2r+2)| = 2\cdot 3^r$.

3. For each nonnegative integer r, $|C(\alpha\beta, 2r+1)| = 0$ and $|C(\alpha\beta, 2r+2)| = 4\cdot 3^r$.

Proof. If $w\alpha^k w^{-1} = v\alpha^l v^{-1}$, then $\alpha^{k-l} = w^{-1}v\alpha^l v^{-1}w\alpha^{-l}$, a commutator. Hence, $k = l$. Now, by looking at the lengths of each side of $\alpha^k = w^{-1}v\alpha^k v^{-1}w$, we deduce that $w = v$. The converse is clear, proving (1).

Define a function $\phi\colon C(\alpha, 2r+1) \to C(\alpha, 2r+2)$ by

$$\phi(w\alpha^k w^{-1}) = \begin{cases} w\alpha^{k+1}w^{-1}, & \text{if } k > 0 \\ w\alpha^{k-1}w^{-1}, & \text{if } k < 0, \end{cases}$$

where $w\alpha^k w^{-1}$ is reduced. Clearly, the function ϕ is onto. It follows from (1) that it is also one-to-one. Consider also the automorphism $\psi\colon F_2 \to F_2$ given by $\psi(\alpha) = \beta$ and $\psi(\beta) = \alpha$. The map ψ is an isometry and $\psi(C(\alpha, n)) = C(\beta, n)$. Thus, the first three equalities in (2) are proved. In order to complete the proof of (2), we show $|C(\alpha, 2r+1)| = 2\cdot 3^r$. The proof of this claim is by induction on r.

Note that if k is even then the length of $w\alpha^k w^{-1}$ is even for any $w \in F_2$. Hence, $C(\alpha, 2r+1)$ contains the conjugates of odd powers of α. Note also that if $w\alpha^k w^{-1}$ is a reduced word of length n, then $-n \le k \le n$.

The set $C(\alpha, 1)$ contains only two elements, α and α^{-1}. Hence, the claim holds in the case $r = 0$.

Assume that $|C(\alpha, 2r+1)| = 2\cdot 3^r$. Define a function φ from $C(\alpha, 2r+1)$ to the subsets of $C(\alpha, 2r+3)$ as follows:

- $\varphi(\alpha^{2r+1}) = \{\alpha^{2r+3}, \beta\alpha^{2r+1}\beta^{-1}, \beta^{-1}\alpha^{2r+1}\beta\}$;
- $\varphi(\alpha^{-(2r+1)}) = \{\alpha^{-(2r+3)}, \beta\alpha^{-(2r+1)}\beta^{-1}, \beta^{-1}\alpha^{-(2r+1)}\beta\}$;
- $\varphi(\alpha w\alpha^{-1}) = \{\alpha^2 w\alpha^{-2}, \beta\alpha w\alpha^{-1}\beta^{-1}, \beta^{-1}\alpha w\alpha^{-1}\beta\}$;
- $\varphi(\alpha^{-1}w\alpha) = \{\alpha^{-2}w\alpha^2, \beta\alpha^{-1}w\alpha\beta^{-1}, \beta^{-1}\alpha^{-1}w\alpha\beta\}$;
- $\varphi(\beta w\beta^{-1}) = \{\beta^2 w\beta^{-2}, \alpha\beta w\beta^{-1}\alpha^{-1}, \alpha^{-1}\beta w\beta^{-1}\alpha\}$;
- $\varphi(\beta^{-1}w\beta) = \{\beta^{-2}w\beta^2, \alpha\beta^{-1}w\beta\alpha^{-1}, \alpha^{-1}\beta^{-1}w\beta\alpha\}$.

It is easy to check that the set

$$\{\varphi(x) \,:\, x \in C(\alpha, 2r+1)\}$$

is a partition of $C(\alpha, 2r+3)$. That is, elements of this set are pairwise disjoint and their union is equal to $C(\alpha, 2r+3)$. We deduce from this that $|C(\alpha, 2r+3)| = 3|C(\alpha, 2r+1)| = 2\cdot 3^{r+1}$, completing the proof of (2).

It is clear that $|C(\alpha\beta, 2r+1)| = 0$ for all $r \ge 0$. Note that for any $w \in F_2$, the word length of $w(\alpha\beta)^k w^{-1}$ is at least $2|k|$. That is, the set $C(\alpha\beta, 2r+2)$ does not contain any conjugate of $(\alpha\beta)^k$ for $|k| > r+1$.

The element $(\beta\alpha)^n$ is conjugate to $(\alpha\beta)^n$ and any element in $C(\alpha\beta, 2r+2)$ is of the form $w(\alpha\beta)^n w^{-1}$ or $w(\beta\alpha)^n w^{-1}$ for some $w \in F_2$ with $||w|| = r+1-n$. Hence, we will only consider the (reduced) words in these two forms.

The only conjugates of $(\alpha\beta)^k$ for $|k| = r+1$ contained in $C(\alpha\beta, 2r+2)$ are elements of

$$A_{r+1} = \{(\alpha\beta)^{r+1}, (\beta\alpha)^{r+1}, (\alpha\beta)^{-(r+1)}, (\beta\alpha)^{-(r+1)}\}.$$

All other elements of $C(\alpha\beta, 2r+2)$ are conjugates of $(\alpha\beta)^k$ for $|k| \leq r$, hence they are conjugates of elements of $C(\alpha\beta, 2r)$.

Consider the subset of $C(\alpha\beta, 2r)$ consisting of the conjugates of $(\alpha\beta)^{\pm r}$. They form the set

$$A_r = \{(\alpha\beta)^r, (\beta\alpha)^r, (\alpha\beta)^{-r}, (\beta\alpha)^{-r}\}.$$

Each element of A_r gives rise two elements of length $2r+2$ by conjugation. For instance, one may conjugate $(\alpha\beta)^r$ only with α and β^{-1} in order to get an element of length $2r+2$. Therefore, there are eight such elements in $C(\alpha\beta, 2r+2)$.

The elements of the difference $C(\alpha\beta, 2r) - A_r$ are of the form $\alpha w \alpha^{-1}$, $\alpha^{-1} w \alpha$, $\beta w \beta^{-1}$ or $\beta^{-1} w \beta$. The number of such elements is $|C(\alpha\beta, 2r)| - 4$ and each gives rise to three elements of length $2r+2$ by conjugation (if there is cancellation, we do not need to take them).

It follows that

$$|C(\alpha\beta, 2r+2)| = 4 + 8 + 3\,(|C(\alpha\beta, 2r)| - 4) = 3|C(\alpha\beta, 2r)|.$$

Now, (3) follows from the fact that $C(\alpha\beta, 2)$ consists of four elements; namely,

$$C(\alpha\beta, 2) = \{\alpha\beta, \beta\alpha, (\alpha\beta)^{-1}, (\beta\alpha)^{-1}\}.$$

This finishes the proof of the lemma. $\square$

Corollary 3.3 *The number of elements of length n conjugate to a power of α, β or $\alpha\beta$ is $4 \cdot 3^r$ if $n = 2r+1$ and $8 \cdot 3^r$ if $n = 2r+2$ $(r \geq 0)$.*

Proof. The set of elements of length n conjugate to the given elements is $C(\alpha, 2r+1) \cup C(\beta, 2r+1)$ if $n = 2r+1$ and $C(\alpha, 2r+2) \cup C(\beta, 2r+2) \cup C(\alpha\beta, 2r+2)$ if $n = 2r+2$. These sets are pairwise disjoint. The result now follows from Lemma 3.2. $\square$

3.2. The mapping class group $\mathrm{PMod}(0,4)$

Since $\mathrm{PMod}(0,4)$ is isomorphic to F_2, there are no periodic elements in $\mathrm{PMod}(0,4)$. Elements of $\mathrm{PMod}(0,4)$ different from the identity are either reducible or pseudo-Anosov. In this section, we compute the growth series and the growth functions of these elements in $\mathrm{PMod}(0,4)$.

Let S be a sphere with four holes. A simple closed curve a on S is called *trivial* if either it bounds a disc or it is parallel to a boundary component. Otherwise, it is called nontrivial.

Let us fix two nontrivial simple closed curves a and b on S intersecting transversely twice as in Figure 1. It is well known that the Dehn twists t_a and t_b generate the group $\mathrm{PMod}(0,4)$ freely. By the lantern relation, there is a unique simple closed curve c on S separating S into two pairs of pants and intersecting both a and b twice such that the Dehn twists t_a, t_b and t_c satisfy $t_a t_b t_c = 1$ (c.f. Figure 1). Thus, we have $t_c = (t_a t_b)^{-1}$, and hence conjugates of powers t_a, t_b and $t_a t_b$ are reducible. In fact, they are the only reducible elements in $\mathrm{PMod}(0,4)$.

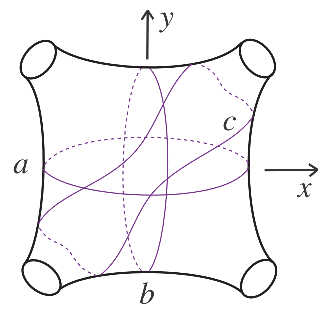

Figure 1. The Dehn twists about a, b, c satisfy $t_a t_b t_c = 1$ in $\mathrm{PMod}(0,4)$ by the lantern relation.

Lemma 3.4 *The reducible elements of* $\mathrm{PMod}(0,4)$ *consist of conjugates of nonzero powers of* t_a, t_b *and* $t_a t_b$.

Proof. Let f be a reducible element $\mathrm{PMod}(0,4)$. Then $F(d) = d$ for some nontrivial simple closed curve d and $F \in f$. Thus, $t_d f = f t_d$, since $f t_d f^{-1} = t_{F(d)} = t_d$. Since $\mathrm{PMod}(0,4)$ is a nonabelian free group and t_d can be completed to a free basis of $\mathrm{PMod}(0,4)$, we conclude that $f = t_d^k$ for some nonzero integer k.

It follows from the classification of simple closed curves on S (c.f. see [4]) that there is a homeomorphism $H : S \to S$ preserving each boundary component of S such that $H(d) \in \{a, b, c\}$.

Let h denote the isotopy class of H in $\mathrm{PMod}(0,4)$. If $H(d) = a$ then $f = t_d^k = h^{-1} t_a^k h$, if $H(d) = b$ then $f = h^{-1} t_b^k h$, and if $H(d) = c$ then $f = h^{-1} t_c^k h = h^{-1}(t_a t_b)^{-k} h$, proving the lemma. $\square$

We are now ready to state and prove the main result of this paper.

Theorem 3.5 *With respect to the generating set* $\{t_a, t_b\}$ *of* $\mathrm{PMod}(0,4)$,

1. *the growth series of reducible elements is*

$$\begin{aligned} r(x) &= 4(x + 3x^3 + 3^2x^5 + 3^3x^7 + \cdots + 3^r x^{2r+1} + \cdots) \\ &\quad +8(x^2 + 3x^4 + 3^2x^6 + 3^3x^8 + \cdots + 3^r x^{2r+2} + \cdots). \end{aligned}$$

Hence, the growth function of reducible elements is

$$r(x) = \frac{4x + 8x^2}{1 - 3x^2}.$$

2. *the growth series of pseudo-Anosov elements is*

$$4\sum_{r=0}^{\infty} 3^r(3^{r+1} - 2)x^{2r+2} + 4\sum_{r=1}^{\infty} 3^r(3^r - 1)x^{2r+1}$$

and the growth function of pseudo-Anosov elements is

$$p(x) = \frac{4x^2(1 + 3x)}{(1 - 3x)(1 - 3x^2)}.$$

3. *if* p_n *and* h_n *denote the number of pseudo-Anosov and all elements of length at most* n *respectively, then we have*

$$\lim_{n\to\infty}\frac{p_n}{h_n}=1\,.$$

Proof. By Lemma 3.4, reducible elements in $\mathrm{PMod}(0,4)$ are conjugates of nonzero powers of t_a, t_b and t_at_b. By Corollary 3.3, the number of such elements of length $n>0$ in $\mathrm{PMod}(0,4)$ is $4\cdot 3^r$ if $n=2r+1$ and $8\cdot 3^r$ if $n=2r+2$.

Therefore the growth series of reducible elements is

$$\begin{aligned} r(x) &= 4x+4\cdot 3x^3+4\cdot 3^2x^5+4\cdot 3^3x^7+\cdots+4\cdot 3^r x^{2r+1}+\cdots \\ &\quad +8x^2+8\cdot 3x^4+8\cdot 3^2x^6+8\cdot 3^3x^8+\cdots+8\cdot 3^r x^{2r+2}+\cdots \\ &= (4x+8x^2)(1+3x^2+3^2x^4+3^3x^6+\cdots+3^r x^{2r}+\cdots). \end{aligned}$$

It follows that the growth function is given by

$$r(x)=\frac{4x+8x^2}{1-3x^2}.$$

This proves (1).

The growth series and the growth function of all elements are

$$\begin{aligned} h(x) &= 1+4x+4\cdot 3x^2+4\cdot 3^2x^3+\cdots+4\cdot 3^{n-1}x^n+\cdots \\ &= \frac{1+x}{1-3x}\,. \end{aligned}$$

The growth series of pseudo-Anosov elements follows from this and (1). The growth function of pseudo-Anosov elements is

$$\begin{aligned} p(x) &= h(x)-1-r(x) \\ &= \frac{4x}{1-3x}-\frac{4x+8x^2}{1-3x^2} \\ &= \frac{4x^2(1+3x)}{(1-3x)(1-3x^2)}. \end{aligned}$$

This proves (2).

Let r_n denote number of reducible elements of length at most n. By (1), we have

$$\begin{aligned} r_n &= 4(1+3+3^2+\cdots+3^r)+8(1+3+3^2+\cdots+3^{r-1}) \\ &= 10\cdot 3^r-6 \end{aligned}$$

if $n=2r+1$ and

$$\begin{aligned} r_n &= 4(1+3+3^2+\cdots+3^{r-1})+8(1+3+3^2+\cdots+3^{r-1}) \\ &= 2\cdot 3^{r+1}-6 \end{aligned}$$

if $n = 2r$. By Lemma 3.1, we get

$$\begin{aligned} h_n &= 1 + 4(1 + 3 + 3^2 + \cdots + 3^{n-1}) \\ &= 2 \cdot 3^n - 1. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \frac{r_n}{h_n} = 0\,.$$

Since $p_n = h_n - r_n - 1$, the proof of (3) follows. $\square$

3.3. A little more

Let $\imath$ (resp. $\jmath$) denote the isotopy class of the rotation about the x-axis (resp. y-axis) by π. (We assume that the surface lie in the three space and is invariant under these rotations, as in Figure 1.) Let Γ denote the subgroup of the mapping class group $\mathrm{Mod}(0,4)$ generated by $\mathrm{PMod}(0,4)$, $\imath$ and $\jmath$. Then Γ is isomorphic to $\mathrm{PMod}(0,4) \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and is of index 6 in $\mathrm{Mod}(0,4)$.

Since $\imath, \jmath$ and $\imath\jmath$ preserve each nonboundary parallel simple closed curve up to isotopy, it can be shown that an element f in $\mathrm{PMod}(0,4)$ is pseudo–Anosov if and only if $f\imath, f\jmath$ and $f\imath\jmath$ are pseudo–Anosov. It follows that, with respect to the generating set $\{t_a, t_b, \imath, \jmath\}$ of Γ, the ratio of the number of pseudo–Anosov elements to that of all elements in a ball of radius n centered at the identity tends to one as n tends to infinity. It would be good to extend this result to $\mathrm{Mod}(0,4)$ and to all $\mathrm{Mod}(0,\mathrm{n})$.

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