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Chaos in product maps

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Abstract

We discuss how chaos conditions on maps carry over to their products. First we give a counterexample showing that the product of two chaotic maps (in the sense of Devaney) need not be chaotic. We then remark that if two maps (or even one of them) exhibit sensitive dependence on initial conditions, so does their product; likewise, if two maps possess dense periodic points, so does their product. On the other side, the product of two topologically transitive maps need not be topologically transitive. We then give sufficient conditions under which the product of two chaotic maps is chaotic in the sense of Devaney [6].

Key Words: Devaney's chaos, topological transitivity, sensitive dependence on initial conditions.

1. Introduction

Let X and Y be two metric spaces and $f: X \to X$, $g: Y \to Y$ two maps, which we assume not to be continuous in general, but chaotic in the sense of Devaney (which we explain instantly). It is natural to ask whether their product $f \times g: X \times Y \to X \times Y$ is also chaotic (in the same sense). We show by counter-example that the answer is in the negative. We then discuss the transfer of the sub-conditions of chaos and finally give some simple sufficient conditions making the product chaotic. These conditions are satisfied for many known chaotic maps.

Now we first recall the chaos conditions for a not-necessarily continuous map $f: X \to X$, X being a metric space with metric d. The discrete dynamical system (X, f) and the map f are used as synonyms in this work, so that phrases such as "The map f is chaotic" or "The discrete dynamical system (X, f) exhibits chaos" are used in the same sense.

Definition 1 *Sensitive dependence on initial conditions:*

A (not-necessarily continuous) map $f: X \to X$ is called sensitively dependent on initial conditions, if there exists $\varepsilon > 0$ such that, for any $x \in X$, and for any neighborhood U of x, there exists $x' \in U$ and an integer $n > 0$ such that $d(f^n(x), f^n(x')) > \varepsilon$.

Definition 2 *Topological transitivity:*

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A (not-necessarily continuous) map $f : X \to X$ is called to be topologically transitive if for any pair of non-empty open sets $U, V \subset X$ there exists an integer $n > 0$ such that $f^n(U) \cap V \neq \emptyset$.

Definition 3 *Chaos in the sense of Devaney [6]*

A (not-necessarily continuous) map $f : X \to X$ is called chaotic, if it is sensitively dependent on initial conditions, topologically transitive and, additionally, its periodic points are dense in X, i.e. every non-empty open subset of X contains a periodic point. (A point $x \in X$ is called periodic if there exists $n > 0$ with $f^n(x) = x$.)

Remark 1 *For a non-finite metric space X and a continuous map $f : X \to X$, topological transitivity and denseness of periodic points imply sensitive dependence on initial conditions (see [2],[7]).*

Remark 2 *The condition "topological transitivity" is sometimes replaced by or used falsely as synonym for the condition "existence of a dense orbit". They are nevertheless equivalent for complete separable metric spaces without isolated points [8]. For a discussion of this matter see [9] or [5]. There is a vast literature on transitivity and about a dozen related notions, of which we will use only a few in the sequel.*

Now, given two maps $f : X \to X$ and $g : Y \to Y$ on metric spaces X and Y with metrics d_1 and d_2 respectively, consider their product $f \times g : X \times Y \to X \times Y$, $(f \times g)(x, y) = (f(x), g(y))$, with product metric on $X \times Y$ (i.e. $d((x, y), (x', y')) = d_1(x, x') + d_2(y, y')$).

The following example shows that the product of two chaotic maps need not be chaotic.

Example 1 *Let* $f : [0, 2] \to [0, 2]$

$$f(x) = \begin{cases} 2x + 1 & \text{for} \quad 0 \leq x \leq 1/2 \\ -2x + 3 & \text{for} \quad 1/2 \leq x \leq 1 \\ -x + 2 & \text{for} \quad 1 \leq x \leq 2. \end{cases}$$

Then, the map f is chaotic, but $f \times f : [0, 2] \times [0, 2] \to [0, 2] \times [0, 2]$ is not chaotic.

Proof. In Figure 1 the graphs of f and $f^2 = f \circ f$ are shown.

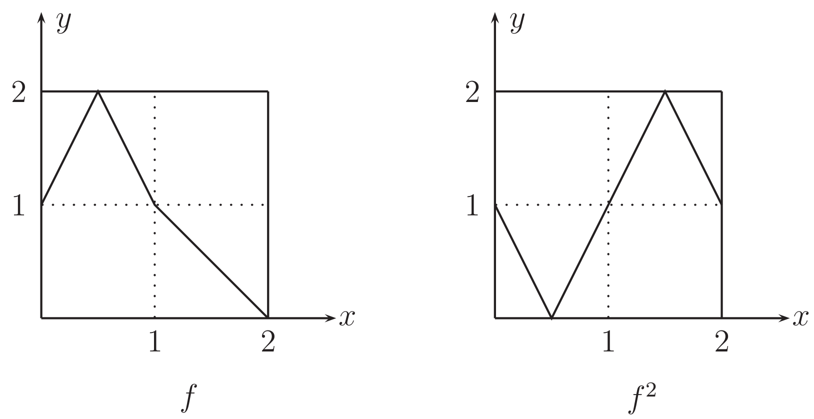

Figure 1

Firstly, we show that the map f is chaotic. By [2] it is enough to see that f is topologically transitive and the periodic points of f are dense. Put $g_1 = f^2|_{[0,1]}$ and $g_2 = f^2|_{[1,2]}$. Using the fact that g_1 and g_2 are chaotic maps (they are tent maps), one can see that some iterate of a non-empty open set $U \subset [0,2]$ intersects any other non-empty open set $V \subset [0,2]$.

Now let us see that periodic points of f are dense. Given any interval $(a,b) \subset [0,2]$ one can find a periodic point in (a,b). If $1 \in (a,b)$, we are done as 1 is periodic. Otherwise $(a,b) \subset (0,1)$ or $(a,b) \subset (1,2)$. In the first case, there is a periodic point of g_1 in (a,b), which is also a periodic point of f. In the other case use g_2.

The map f is now shown to be chaotic, but $f \times f$ is not chaotic, because it is not topologically transitive. If we take $U = (0,1) \times (0,1)$ and $V = (0,1) \times (1,2)$, then there is no k with $(f \times f)^k(U) \cap V \neq \varnothing$, because $(f \times f)(U) \subset (1,2) \times (1,2)$ and $(f \times f)((1,2) \times (1,2)) \subset U$, so that $(f \times f)^k(U) \subset U$ for k even and $(f \times f)^k(U) \subset (1,2) \times (1,2)$ for k odd. Thus $(f \times f)^k(U)$ never intersects $V = (0,1) \times (1,2)$ for any k. □

Remark 3 *The essence of the above example lies in the fact that f is topologically transitive but f^2 is not. Maps of this type (i.e. topologically transitive maps $f : X \to X$ with a non-transitive power f^n) and related decompositions of the domain, have been thoroughly investigated by Banks [3].*

In the next section we will discuss the sub-conditions of chaos and in the last section we will give some sufficient conditions for a product map to be chaotic.

2. Sub-conditions of chaos

In this section we discuss how the sub-conditions of chaos carry over to the products and vice versa.

Lemma 1 *Let X and Y be metric spaces with metrics d_1 and d_2, respectively, $f : X \to X$ and $g : Y \to Y$ be not-necessarily continuous maps.*

i) If f or g is sensitively dependent on initial conditions, then $f \times g : X \times Y \to X \times Y$ is sensitively dependent on initial conditions.

ii) If $f \times g : X \times Y \to X \times Y$ is sensitively dependent on initial conditions, then at least one of f or g is sensitively dependent on initial conditions.

Proof. i) Let us assume f is sensitively dependent on initial conditions. Then we will show that same is true for $f \times g$. Let $p = (x,y) \in X \times Y$ be any point and W any neighborhood of p. Then there exist open neighborhoods U of x in X and V of y in Y such that $U \times V \subset W$. As f is sensitively dependent on initial conditions, there exists $\varepsilon > 0$ such that for a certain $x' \in U$ and an integer $n > 0$ the inequality $d_1(f^n(x), f^n(x')) > \varepsilon$ holds. Then for any $y' \in V$, $p' = (x', y')$ belongs to W and

$$\begin{aligned} d((f \times g)^n(p), (f \times g)^n(p')) &= d_1(f^n(x), f^n(x')) + d_2(g^n(y), g^n(y')) \\ &\geq d_1(f^n(x), f^n(x')) > \varepsilon. \end{aligned}$$

This means that $f \times g$ is sensitively dependent on initial conditions.

ii) Let us assume that both f and g are not sensitively dependent on initial conditions. This means that, given any $\varepsilon > 0$ there exists $x \in X$ such that for a certain open set $U \subset X$ containing x, the inequality

$$d_1(f^n(x), f^n(x')) < \varepsilon/2$$

holds for every $x' \in U$ and positive integer n. Similarly, there exists $y \in Y$ such that for a certain open set $V \subset Y$ containing y, the inequality

$$d_1(g^n(y), g^n(y')) < \varepsilon/2$$

holds for every $y' \in V$ and positive integer n. Then we get

$$d((f \times g)^n(p), (f \times g)^n(p')) = d_1(f^n(x), f^n(x')) + d_2(g^n(y), g^n(y')) < \varepsilon$$

for $(x', y') \in U \times V$. This means that $f \times g$ is not sensitively dependent on initial conditions, contradicting the hypothesis. □

We now show that denseness of periodic points carry over to products and vice versa:

Lemma 2 *Let X and Y be metric spaces with metrics d_1 and d_2 respectively, $f : X \to X$ and $g : Y \to Y$ (not-necessarily continuous) maps. The set of periodic points of $f \times g$ is dense in $X \times Y$ if and only if, for both of f and g the sets of periodic points in X and Y are dense (in X, resp. Y).*

Proof. Let us assume that the set of periodic points of f is dense in X and the set of periodic points of g is dense in Y. Let us see that the set of periodic points of $f \times g$ is dense in $X \times Y$. Let $W \subset X \times Y$ be any non-empty open set. Then there exist non-empty open sets $U \subset X$ and $V \subset Y$ with $U \times V \subset W$. By assumption, there exists a point $x \in U$ such that $f^n(x) = x$ with $n > 0$. Similarly, there exists $y \in V$ such that $g^m(y) = y$ with $m > 0$. For $p = (x, y) \in W$ and $k = mn$ we get

$$(f \times g)^k(p) = (f \times g)^k(x, y) = (f^k(x), g^k(y)) = (x, y).$$

This means that W contains a periodic point and thus the set of periodic points of $f \times g$ is dense in $X \times Y$.

Conversely let $U \subset X$ and $V \subset Y$ be non-empty open subsets. Then $U \times V$ is a non-empty open subset of $X \times Y$. As the set of the periodic points of $f \times g$ is dense in $X \times Y$, there exists a point $p = (x, y)$ in $U \times V$ such that $(f \times g)^n(x, y) = (f^n(x), g^n(y)) = (x, y)$ for some n. From the last equality we obtain $f^n(x) = x$ for $x \in U$ and $g^n(y) = y$ for $y \in Y$. □

By Lemma 1 and Lemma 2, sensitive dependence on initial conditions and denseness of periodic points carry over from factors to products. But, topological transitivity may not carry over to products as Example 1 shows. The converse of this situation is however true:

Lemma 3 *Let $f : X \to X$ and $g : Y \to Y$ be (not-necessarily continuous) maps and let us assume that the product $f \times g$ is topologically transitive on $X \times Y$. Then the maps f and g are both topologically transitive on X and Y respectively.*

Proof. We show the transitivity of f; the transitivity of g can be shown similarly. Let U_1, V_1 be non-empty open sets in X. Then the sets $U = U_1 \times Y$ and $V = V_1 \times Y$ are open in $X \times Y$. As $f \times g$ is transitive, there exists a positive integer k such that $(f \times g)^k(U) \cap V \neq \emptyset$. From the equalities

$$\begin{aligned}(f \times g)^k(U) \cap V &= \left[f^k(U_1) \times g^k(Y)\right] \cap [V_1 \times Y] \\ &= \left[f^k(U_1) \cap V_1\right] \times \left[g^k(Y) \cap Y\right]\end{aligned}$$

it follows $\left[f^k(U_1) \cap V_1\right] \times \left[g^k(Y) \cap Y\right] \neq \emptyset$, so $f^k(U_1) \cap V_1 \neq \emptyset$. Thus f is topologically transitive. □

By Lemma 1 and Lemma 2, given two chaotic maps f and g, their product $f \times g$ is sensitively dependent on initial conditions, it possesses dense periodic points, but one could have trouble with topological transitivity as we have seen in Example 1. We now give some sufficient conditions for topological transitivity of the product. First we recall a definition.

Definition 4 *Let $f : X \to X$ be a (not-necessarily continuous) map on the metric space X. If for every non-empty open subsets $U, V \subset X$ there exists a positive integer n_0 such that for every $n \geq n_0$, $f^n(U) \cap V \neq \emptyset$, then f is called topologically mixing.*

It is clear that topological mixing implies topological transitivity.

There is an even stronger notion that implies topological mixing.

Definition 5 *Let $f : X \to X$ be a (not-necessarily continuous) map on the metric space X. If for every non-empty open subset $U \subset X$ there exists a positive integer n_0 such that for every $n \geq n_0$, $f^n(U) = X$, then f is called locally eventually onto.*

Remark 4 *The best known chaotic maps are locally eventually onto and hence topologically mixing. The following are some examples of such maps.*

- The logistic map: $f : [0,1] \to [0,1]$, $f(x) = 4x(1-x)$.
- The baker map: $B : [0,1] \to [0,1]$, $B(x) = \begin{cases} 2x & , \text{ if } \ 0 \leq x < 1/2 \\ 2x - 1 & , \text{ if } \ 1/2 \leq x \leq 1. \end{cases}$
- The map doubling the circle: $D : S^1 \to S^1 \quad D(\theta) = 2\theta$
- The shift map: $S : \sum_2 \to \sum_2 \quad S(s_0 s_1 s_2 ...) = s_1 s_2 ....$
- More generally, any complex polynomial on its Julia set (see [6] on page 288).

Lemma 4 *The product of two topologically mixing maps is topologically mixing.*

Proof. Let $f : X \to X$ and $g : Y \to Y$ be topologically mixing maps. Given W_1, $W_2 \subset X \times Y$, there exists open sets $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$, such that $U_1 \times V_1 \subset W_1$ and $U_2 \times V_2 \subset W_2$. By assumption there exist n_1 and n_2 such that $f^n(U_1) \cap U_2 \neq \emptyset$ for $n \geq n_1$ and $g^n(V_1) \cap V_2 \neq \emptyset$ for $n \geq n_2$. For $n \geq n_0 = \max\{n_1, n_2\}$ we get

$$\begin{aligned}[(f \times g)^n(U_1 \times V_1)] \cap (U_2 \times V_2) &= [f^n(U_1) \times g^n(V_1)] \cap (U_2 \times V_2) \\ &= [f^n(U_1) \cap U_2] \times [g^n(V_1) \cap V_2] \neq \emptyset\end{aligned}$$

which means that $f \times g$ is topologically mixing. □

3. Chaos in products

In this section we give some sufficient conditions for a product map to be chaotic.

Theorem 1 *Let $f : X \to X$ and $g : Y \to Y$ be not-necessarily continuous, chaotic and topologically mixing maps on the metric spaces X and Y. Then $f \times g : X \times Y \to X \times Y$ is chaotic.*

Proof. The map $f \times g$ is sensitively dependent on initial conditions by Lemma 1, it has dense periodic points by Lemma 2 and it is topologically mixing by Lemma 4 and hence topologically transitive. Thus all three conditions of Devaney chaos are satisfied. □

Example 2 *Multiply any two of the maps under Remark 1. (They are locally eventually onto and hence topologically mixing.)*

Remark 5 *The product map $D \times D : S^1 \times S^1 \to S^1 \times S^1$, where $D : S^1 \to S^1$ is the doubling map $D(\theta) = 2\theta$, is used to construct the Lattes-Böetcher function on the Riemann sphere, whose Julia set is the entire sphere [4]. One defines $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $R = P \circ (D \times D) \circ P^{-1}$, where $P : S^1 \times S^1 \to \mathbb{C}_\infty$ is the Weierstrass elliptic function with respect to the integer lattice. It turns out that R is the rational function $R(z) = \frac{(z^2+1)^2}{4z(z^2-1)}$.*

We can demand the topological mixing property for only one of the functions, at the price of requiring continuity for the other:

Theorem 2 *Let X be a metric space, $f : X \to X$ a continuous and chaotic map; $g : Y \to Y$ a not-necessarily continuous, chaotic and topologically mixing map on the metric space Y. Then $f \times g : X \times Y \to X \times Y$ is chaotic.*

Proof. It is enough to show that $f \times g$ is topologically transitive. It is obviously enough to show this for open sets of the form $U \times V$. So, let be given two sets $U_1 \times V_1$ and $U_2 \times V_2$ with U_1, U_2 open in X and V_1, V_2 open in Y. As g is topologically mixing, there exists $n_0 > 0$ with $g^n(V_1) \cap V_2 \neq \emptyset$ for all $n \geq n_0$. On the other hand, there exists a periodic point $x \in U_1$ whose orbit enters U_2 (see e.g. [10]). Thus, if we denote the period of x by p, there exists k with $0 \leq k < p$ and $f^k(x) \in U_2$. This implies $f^{mp+k}(x) \in U_2$ for any positive integer m. Now choose m such that $l = mp + k \geq n_0$. Then we have $g^l(V_1) \cap V_2 \neq \emptyset$ and there exists a point $y \in V_1$ with $g^l(y) \in V_2$. Now, for the point $(x, y) \in U_1 \times V_1$ we get $(f \times g)^l(x, y) \in U_2 \times V_2$. Hence $f \times g$ is topologically transitive. □

We can give yet another sufficient condition (for the product to be chaotic) with the help of following property (used in the proof above):

Definition 6 *Given a metric space X and a not-necessarily continuous map $f : X \to X$, we say that f has Touhey property on X if given U and V, non-empty open subsets of X, there exists a periodic point $x \in U$ and* *a non-negative k such that $f^k(x) \in V$, that is, if every pair of non-empty open subsets of X shares a periodic orbit. (If f is continuous and X non-finite, then this property implies chaos in the sense of Devaney by [2]; see also [10]).*

Theorem 3 *Let X be any metric space and assume that the (not-necessarily continuous) map $f : X \to X$ has the Touhey property. Let $g : Y \to Y$ be a not-necessarily continuous, chaotic and topologically mixing map on the metric space Y. Then $f \times g : X \times Y \to X \times Y$ is chaotic.*

Proof. As g is sensitively dependent on initial conditions, so is $f \times g$. On the other hand, the Touhey property implies denseness of periodic points of f, hence, as the periodic points of g are also dense, we have denseness of periodic points of $f \times g$. The transitivity of $f \times g$ can be seen as in the preceding proof. □

Remark 6 *We note that product maps are a simple and useful source for examples and counter-examples in chaos. In fact, in [1] the counter-example showing that density of periodic points and sensitive dependence on initial conditions do not imply topological transitivity is defined in product form. The other counter-example in [1] which shows that topological transitivity and sensitive dependence on initial conditions do not imply density of periodic points could more easily and naturally be constructed in this way also: Let $T_\lambda : S^1 \to S^1, T_\lambda(\theta) = \theta + 2\pi\lambda$ mod 2π be irrational rotation of the circle and $D : S^1 \to S^1$, $D(\theta) = 2\theta$ mod 2π, the map doubling the circle. Then the product map*

$$T_\lambda \times D : S^1 \times S^1 \to S^1 \times S^1$$

has no periodic points, but it is topologically transitive and sensitively dependent on initial conditions.

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