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Pseudo simplicial groups and crossed modules

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Abstract

In this paper, we define the notion of pseudo 2-crossed module and give a relation between the pseudo 2-crossed modules and pseudo simplicial groups with Moore complex of length 2.

Key Words: Crossed modules, Pseudo simplicial groups, Moore complex.

1. Introduction

Simplicial groups occupy a place somewhere between homological group theory, homotopy theory, algebraic K-theory and algebraic geometry. In each sector they have played a significant part in developments over quite a lengthy period of time and there is an extensive literature on their homotopy theory.

Crossed modules were introduced by Whitehead in [15] with a view to capturing the relationship between π_1 and π_2 of a space. Homotopy systems (which would now be called free crossed complexes [5] or totally free crossed chain complexes [3], [4]) were introduced, again by Whitehead, to incorporate the action of π_1 on the higher relative homotopy groups of a CW-complex. They consist of a crossed module at the base and a chain complex of modules over π_1 further up.

Conduché [6] defined the notion of 2-crossed module, as a model of connected 3-types and showed how to obtain a 2-crossed module from a simplicial group.

Inasaridze(c.f. [8],[9]) constructed homotopy groups of pseudosimplicial groups and nonabelian derived functors with values in the category of groups.

In this paper we analysis the low dimensional parts of the Moore complex of a pseudosimplicial group. We prove that the category of crossed modules is equivalent to the category of pseudosimplicial groups with Moore complex of length 1. We extend this result to 2-dimension by defining pseudo 2-crossed modules and give the relation between the category of pseudo 2-crossed modules and the category of pseudosimplicial groups with Moore complex of length 2.

The above theorems, in some sense, are well known. We give details of the proofs as analogous proofs can be found in the literature [1], [2], [6], [10] and [13].

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2. Pseudo simplicial groups

A *pseudo simplicial group* $\mathbf{G}$ consists of $\{G_n\}$ together with boundary homomorphisms $\partial_i^n : G_n \to G_{n-1}$, $0 \leq i \leq n$, $(n \neq 0)$ and pseudo degeneracies $s_i^n : G_n \to G_{n+1}$, $0 \leq i \leq n$, satisfying the following pseudosimplicial identities:

$$\begin{array}{lcll} \partial_i^{n-1}\partial_j^n & = & \partial_{j-1}^{n-1}\partial_i^n & \text{for } i<j \\ \partial_i^{n+1}s_j^n & = & s_{j-1}^{n-1}\partial_i^n & \text{for } i<j \\ \partial_j^{n+1}s_j^n & = & 1=\partial_{j+1}^{n+1}s_j^n & \\ \partial_i^{n+1}s_j^n & = & s_j^{n-1}\partial_{i-1}^n & \text{for } i>j+1, \end{array}$$

The groups G_n can be nonabelian. To obtain the definition of *simplicial group*, we must add the condition that $s_i^{n+1}s_j^n = s_{j+1}^{n+1}s_i^n$ for $i \leq j$ (see [11]).

A topological interpretation is, for example, the F-construction of Milnor [12], which gives the simplicial group of loops of the suspension of a complex. For an arbitrary simplicial set $\mathbf{K}$ with pole ψ, the group of n-simplicies FK_n is the free group on a family of generators σ in one-to-one correspondence with the n-simplicies $\sigma \in K_n$, with the single relation $(s_{n-1}s_{n-2}\ldots s_0(\psi)) = e_n$, while the boundary and degeneracy homomorphisms are induced by the corresponding mappings of the set $\mathbf{K}$.

For any pseudosimplicial group $\boldsymbol{G}$, put $NG_n = G_n \cap Ker\ \partial_0^n \cap \ldots Ker\ \partial_{n-1}^n$, $n \geq 0$, and let d_n be the restriction of ∂_n^n to NG_n, $n > 0$. Then $im\ d_n$ is a normal subgroup of G_{n-1}, and $im\ d_{n+1} \subset Ker\ d_n$ for $n > 0$. This determines the Moore complex $\mathbf{NG} = \{NG_n, d_n\}$. Clearly $\mathbf{NG}$ is independent of the pseudodegeneracies, depending only on the boundary homomorphisms.

The n-dimensional homology group of the Moore complex $\mathbf{NG}$ is called the *n-dimensional homotopy group* $\pi_n(\mathbf{G})$ of the pseudosimplicial group $\mathbf{G}$, $n \geq 0$.

A mapping $\mathbf{f} : \mathbf{G} \longrightarrow \mathbf{G}'$ induces, in a natural fashion, homomorphisms $\pi_n(\mathbf{f}) : \pi_n(\mathbf{G}) \longrightarrow \pi_n(\mathbf{G}')$, $n \geq 0$.

Let $\mathbf{f}$ and $\mathbf{g}$ be two mappings from $\mathbf{G}$ to $\mathbf{G}'$. The following definition is due to Inassaridze [9]. $\mathbf{f}$ is *pseudohomotopic* to $\mathbf{g}$ if there exist homomorphisms $h_i^n : G_n \longrightarrow G'_{n+1}$, $0 \leq i \leq n$, such that

$$\begin{array}{rcccc} \partial_0^{n+1}h_0^n & = & f_n & & \partial_{n+1}^{n+1}h_n^n = g_n, \\ \partial_i^{n+1}h_j^n & = & h_{j-1}^{n-1}\partial_i^n & \text{for} & i<j, \\ \partial_{j+1}^{n+1}h_{j+1}^n & = & \partial_{j+1}^{n+1}h_j^n, & & \\ \partial_i^{n+1}h_j^n & = & h_j^{n-1}\partial_{i-1}^n & \text{for} & i>j+1. \end{array}$$

To obtain the definition of homotopy of $\mathbf{f}$ to $\mathbf{g}$, we must add the following conditions:

$$s_i^{n+1}h_j^n = h_{j+1}^{n+1}s_i^n \text{ for } i \leq j, \text{ and } s_i^{n+1}h_j^n = h_j^{n+1}s_{i-1}^n \text{ for } i > j.$$

Theorem 2.1 *[9]The homotopy groups $\pi_n(\mathbf{G})$ are abelian for $n \geq 1$. If the mapping $\mathbf{f} : \mathbf{G} \longrightarrow \mathbf{G}'$ is pseudohomotopic to a mapping, then $\pi_n(\mathbf{f}) = \pi_n(\mathbf{g})$, $n \geq 0$.*

A mapping $\mathbf{f} : \mathbf{G} \longrightarrow \mathbf{G}'$ of pseudosimplicial groups is called *simplicial* if it satisfies the condition $f_{n+1}s_i^n = s_i^n f_n$ for $n \geq 0$, $0 \leq i \leq n$. A simplicial map $\mathbf{f} : \mathbf{G} \longrightarrow \mathbf{G}'$ is called a *weak equivalence* if it induces isomorphisms $\pi_n(\mathbf{G}) \cong \pi_n(\mathbf{G}')$ for $n \geq 0$. A simplicial map $\mathbf{f}: \mathbf{G} \longrightarrow \mathbf{G}'$ called a *fibration* if $f_n : G_n \longrightarrow G'_n$ is surjective for $n \geq 0$.

By a k-truncated pseudosimplicial group we mean a collection of groups $\{G_0, \ldots, G_k\}$ and boundary homomorphisms $\partial_i^n : G_n \longrightarrow G_{n-1}$ for $0 \leq i \leq n$, $0 \leq n \leq k$ and pseudodegeneracies $s_i^n : G_n \longrightarrow G_{n+1}$ for $0 \leq i \leq n$, $0 \leq n \leq k$ which satisfy the pseudosimplicial identities. Clearly by forgetting higher dimensions, any pseudosimplicial group $\mathbf{G}$ yields a k-truncated pseudosimplicial group $tr^k\mathbf{G}$. The functor tr^k admits a right adjoint $\cos k^k$, called the *k-coskeleton functor*, and a left adjoint functor sk^k called the *k-skeleton functor.* We recall from [7] a brief description of these functors.

Suppose $tr^k(\mathbf{G}) = \{G_0, \ldots, G_k\}$ is a pseudosimplicial group. A family of homomorphisms

$$(\delta_0, \ldots, \delta_{k+1}) : X_{k+1} \begin{array}{c} \xrightarrow{\delta_{k+1}} \\ \vdots \\ \xrightarrow[\delta_0]{} \end{array} G_k$$

is the *simplicial kernel* of the family of boundary homomorphisms $(\partial_0, \ldots, \partial_k)$ if it has the following universal property: given any family $(\partial_0, \ldots, \partial_{k+1})$ of $k+2$ homomorphisms $\partial_i : Y \longrightarrow G_k$ satisfying the identities $\partial_i \partial_j = \partial_{j-1}\partial_i$ $(0 \leq i < j \leq k+1)$ with the last part of the truncated pseudosimplicial group, there exists a unique homomorphism $f : Y \longrightarrow X_{k+1}$ such that $\delta_i f = \partial_i$. Given the simplicial kernel X_{k+1} the family of

homomorphisms $(\alpha_{n+1j}, \ldots, \alpha_{1j}, \alpha_{0j})$, defined by

$$\alpha_{ij} = \begin{cases} s_{j-1} & i < j \\ id & i = j,\, i = j+1 \\ s_j d_{i-1} & i > j+1, \end{cases}$$

satisfies the pseudosimplicial identities with the last part of the truncated pseudosimplicial group; hence there exists a unique $s_j : G_k \longrightarrow X_{k+1}$ such that $\delta_i s_j = \alpha_{ij}$. We thus have a $(k+1)$-truncated pseudosimplicial group $\{G_0, \ldots, G_k, X_{k+1}\}$. By iterating this construction we get a pseudosimplicial group $\cos k^k(tr^k(\mathbf{G})) = \{G_0, \ldots, G_k, X_{k+1}\}$ called the *coskeleton* of the truncated pseudosimplicial group. If $\mathbf{G}$, $\mathbf{G}'$ are any pseudosimplicial groups, than any truncated simplicial map $\mathbf{f} : tr^k\mathbf{G} \longrightarrow tr^k\mathbf{G}'$ extends uniquely to a simplicial map $\mathbf{f} : \mathbf{G} \longrightarrow \cos k^k(tr^k(\mathbf{G}'))$.

The k-skeleton functor can be constructed by a dual process involving pseudosimplicial cokernels

$$(s_0, \ldots, s_k) : G_k \begin{array}{c} \xrightarrow{s_k} \\ \vdots \\ \xrightarrow[s_0]{} \end{array} X_{k+1}.$$

(That is, universal systems of $k+1$ arrows which satisfy pseudosimplicial identities.)

3. Crossed modules and pseudo 2-crossed modules

3.1. Crossed modules

J.H.C. Whitehead (1949) [15] described crossed modules in various contexts, especially in his investigation into the group structure of relative homotopy groups.

Definition 3.1 *Let P be a group. A pre-crossed module of groups is a P-group M, and a group homomorphism*

$$\partial : M \longrightarrow P$$

such that

$$CM1)\ \partial({}^{p}m) = p\partial(m)p^{-1}$$

for all $m \in M$, $p \in P$. This is a crossed P-module if, in addition,

$$CM2)\ {}^{\partial(m)}m' = mm'm^{-1}$$

for all $m, m' \in M$. The last condition is called the Peiffer identity. We denote such a crossed module by (M, P, ∂). A map of crossed modules

$$(\partial : M \longrightarrow P\ ) \longrightarrow (\partial' : M' \longrightarrow P')$$

is a pair of homomorphisms $f_0 : P \longrightarrow P'$, $f_1 : M \longrightarrow M'$ such that $f_0\partial = \partial' f_1$ and $f_1\left({}^{p}m\right) = {}^{(f_0 p)} f_1\left(m\ \right)$ for all $m \in M$, $p \in P$.

The Moore complex

$$\cdots \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0$$

of a pseudosimplicial group is of length k, if $M_n = 0$ for all $n \geq k+1$ (so a Moore complex of length k is also of length r for $r \geq k$).

The following lemma is a straightforward modification of Theorem 1.3 in [6].

Lemma 3.2 *Let $\mathbf{G}$ be a pseudosimplicial group. The Moore complex of its k -coskeleton $\cos k^k(tr^k\mathbf{G})$ is of length $k+1$, and is identical to the Moore complex of $\mathbf{G}$ in dimensions $\leq k$. Moreover, in dimensions $k-1$ to $k+2$ the Moore complex of $\cos k^k(tr^k\mathbf{G})$ is an exact sequence*

$$1 \longrightarrow N(\cos k^k(tr^k\mathbf{G}))_{k+1} \overset{\partial_{k+1}}{\longrightarrow} NG_k \overset{\partial_k}{\longrightarrow} NG_{k-1}.$$

where N_k is the k th term of the Moore complex of G.

Proof. The $(k+1)$-dimensional part of $\cos k^k\left(tr^k G\right)$ can be identified with the subgroup of the $(k+2)$-fold direct sum G_k^{k+2} consisting of those elements $(x_0, ..., x_{k+1})$ such that $d_j x_k = d_{k-1} x_j$ for $j < k$; the face maps are given by $d_j\left(x_0, ..., x_{k+1}\right) = x_j$ Thus $N(\cos k^k(tr^k\mathbf{G}))_{k+1}$consists of elements $(1, ..., 1, x_{k+1})$ such that $d_j x_{k+1} = 1$ for all j. In other words $N(\cos k^k(tr^k\mathbf{G}))_{k+1}$ is the kernel of $\partial_k : NG_k \longrightarrow NG_{k-1}$, and hence we have the exact sequence of the lemma.

The injectivity of ∂_{k+1} and the isomorphism

$$\cos k^{n-1}\left(tr^{n-1}\left(\cos k^{k}\left(tr^{k}G\right)\right)\right) \simeq \cos k^{k}\left(tr^{k}G\right)$$

for $n \geq k+2$ shows that the Moore complex of $\cos k^{k}\left(tr^{k}G\right)$ is of length $k+1$. □

The following theorem is well known. In [6] and [10] this theorem was proved.

Theorem 3.3 *([6], [10]) The category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length 1.*

Now, we shall give the pseudo version of this theorem.

Theorem 3.4 *The category of crossed modules is equivalent to the category of pseudosimplicial groups with Moore complex of length 1.*

Proof. Let **G** be a pseudosimplicial group with Moore complex of length 1. Put $P = NG_0 = G_0$, $M = NG_1 = \ker(d_0 : G_1 \longrightarrow G_0)$ and $\partial = d_1$ (restricted to M). Then $p \in P$ acts on $m \in M$ by ${}^{p}m = s_0(p)\, m s_0(p)^{-1}$, and $\partial({}^{p}m) = d_1\left(s_0(p)\, m s_0(p)^{-1}\right)$. Since the Moore complex $\cdots \longrightarrow 1 \longrightarrow M \xrightarrow{\partial} P \longrightarrow 1$ is of length 1, we have $\partial_2 NG_2 = 1$. It then follows that for all $m, m' \in M$ and $p \in P$,

$$\begin{aligned}
(i)\quad \partial_1({}^{p}m) &= d_1({}^{p}m)\\
&= d_1\left(s_0(p)\, m s_0(p)^{-1}\right)\\
&= d_1 s_0(p)\, d_1(m)\, d_1 s_0(p)^{-1}\\
&= p\partial_1(m)\, p^{-1}
\end{aligned}$$

$$\begin{aligned}
(ii)\quad {}^{(\partial_1 m)}m' &= s_0\partial_1(m)\, m' s_0\partial_1(m)^{-1}\\
&= s_0 d_1(m)\, m' s_0 d_1(m)^{-1}\\
&= s_0 d_1(m)\, m' s_0 d_1(m)^{-1}\,[\left(m\,(m')^{-1} m^{-1}\right)\left(mm'm^{-1}\right)]\\
&= d_2 s_0(m)\, d_2 s_1(m') d_2 s_0(m)^{-1}\, d_2 s_1(m) d_2 s_1(m')^{-1}\\
&\qquad d_2 s_1(m^{-1})\left(mm'm^{-1}\right)\\
&= d_2\left(s_0(m)\, s_1(m')\, s_0(m)^{-1}\, s_1(m)\, s_1(m')^{-1}\, s_1(m)^{-1}\right)\\
&\qquad \left(mm'm^{-1}\right)\\
&= mm'm^{-1}
\end{aligned}$$

for $m, m' \in M$, because $s_0(m)\, s_1(m')\, s_0(m)^{-1}\, s_1(m)\, s_1(m')^{-1}\, s_1(m)^{-1}$ lies in $\partial_2 NG_2$. Thus $\partial : M \longrightarrow P$ is a crossed module.

Conversely, let $\partial : M \longrightarrow P$ be a crossed module. By using the action of P on M we can form the semi-direct product $M \rtimes P = \{(m,p) : m \in M\ ,\ p \in P\}$, in which multiplication

$$(m,p)\cdot(m',p') = (m\ {}^{p}m', pp')$$

for $m, m' \in M$, $p, p' \in P$. There are homomorphisms

$$\begin{aligned}
d_0 &: M \rtimes P \longrightarrow P, && (m,p) \longmapsto p,\\
d_1 &: M \rtimes P \longrightarrow P, && (m,p) \longmapsto (\partial m)\, p,\\
s_0 &: P \longrightarrow M \rtimes P, && p \longmapsto (1,p)\,.
\end{aligned}$$

Let $G_0 = P$, $G_1 = M \rtimes P$. We have a 1-truncated pseudosimplicial group $\{G_0, G_1\}$ whose 1-coskeleton we denote by G^1. The group $M \rtimes P$ acts on M via the action of P on M and the homomorphism d_1. We can thus form the semi-direct product $M \rtimes (M \rtimes P)$ and construct homomorphisms

$$\begin{array}{ll} d_0 : M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, & (m, m', p) \longmapsto (m', p)\,, \\ d_1 : M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, & (m, m', p) \longmapsto (mm', p)\,, \\ d_2 : M \rtimes (M \rtimes P) \longrightarrow M \rtimes P, & (m, m', p) \longmapsto (m, (\partial m')\, p)\,, \\ s_0 : M \rtimes P \longrightarrow M \rtimes (M \rtimes P)\,, & (m, p) \longmapsto (1, m, p)\,, \\ s_1 : M \rtimes P \longrightarrow M \rtimes (M \rtimes P)\,, & (m, p) \longmapsto (m, 1, p)\,. \end{array}$$

Conditions (i) and (ii) of a crossed module ensure that these are homomorphisms (Condition (ii) is needed for d_2). Let $G_2 = M \rtimes (M \rtimes P)$. We then have a 2-truncated pseudosimplicial group $\{G_0, G_1, G_2\}$ whose 2-coskeleton we denote by G^2. There is a unique simplicial map $G^2 \longrightarrow G^1$ which in dimensions 0 and 1 is the identity. We let $\overline{G}^2$ denote the image of G^2 in G^1. It is readily checked that the Moore complex of G^2 is trivial in dimension 2; it follows from Lemma 3.2 that $\overline{G}^2$ is a pseudosimplicial group whose Moore complex is of length 1. □

3.2. Pseudo 2-crossed modules

Conduché [6] in 1984 described the notion of 2-crossed module as a model for (homotopy connected) 3-types.

Definition 3.5 *A pseudo 2-crossed module of groups consists of a complex of P-groups*

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$$

and ∂_2, ∂_1 morphisms of P-groups, where the group P acts on itself by conjugation, such that

$$L \xrightarrow{\partial_2} M$$

is a crossed module. Thus M acts on L and we require that for all $l \in L$, $m \in M$ and $p \in P$ that ${}^{p_m}({}^{p}l) = {}^{p}({}^{m}l)$. Further, there is a P-equivariant function,

$$\{,\} : M \rtimes M \longrightarrow L$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{lrcl} \mathbf{P-2CM1)} & \partial_2\{m, m'\} & = & \left({}^{\partial_1 m}m'\right) mm'^{-1}m^{-1} \\ \mathbf{P-2CM2)} & \{\partial_2 l, \partial_2 l'\} & = & [l', l] \\ \mathbf{P-2CM3)} & (i)\quad \{mm', m''\} & = & {}^{\partial_1 m}\{m', m''\}\{m, m'm''m'^{-1}\} \\ & (ii)\quad \{m, m'm''\} & = & \{m, m'\}^{mm'm^{-1}}\{m, m''\} \\ \mathbf{P-2CM4)} & (a)\quad \{\partial_2 l, m\} & = & {}^{m}l(l)^{-1}, \\ & (b)\quad \{m, \partial_2 l\} & = & ({}^{\partial_1 m}l)({}^{m}(l)^{-1}). \\ \mathbf{P-2CM5)} & \{m, \partial_2 l\}\{\partial_2 l, m\} & = & {}^{\partial_1 m}l(l)^{-1} \end{array}$$

for all $l, l' \in L$, $m, m', m'' \in M$ *and* $p \in P$. *We denote such a pseudo 2-crossed module of groups by* $\{L, M, P, \partial_2, \partial_1\}$. *To obtain the definition of 2-crossed modules, we must add the condition that:*

$$\textbf{2CM6)} \qquad {}^{p}\{m, m'\} = \{{}^{p}m, {}^{p}m'\}.$$

A morphism of pseudo 2-crossed modules of groups may be pictured by diagram

$$\begin{array}{ccccc} L & \xrightarrow{\partial_2} & M & \xrightarrow{\partial_1} & P \\ {\scriptstyle f_2}\Big\downarrow & & {\scriptstyle f_1}\Big\downarrow & & {\scriptstyle f_0}\Big\downarrow \\ L' & \xrightarrow{\partial_2} & M' & \xrightarrow{\partial_1} & P' \end{array}$$

of groups and homomorphisms such that $f_0\partial_1 = \partial_1' f_1$, $f_1\partial_2 = \partial_2' f_2$ *and such that*

$$f_1\left({}^{p}m\right) = {}^{(f_0(p))} f_1(m), \; f_2\left({}^{p}l\right) = {}^{(f_0(p))} f_2(l)$$

and

$$\{,\} = f_1 \times f_1 = f_2 \{,\},$$

for all $l \in L$, $m \in M$, $p \in P$. *We thus define the category of pseudo 2-crossed modules, denoting it by* $\mathfrak{pX}_2\mathfrak{Mod}$. *Morphisms* f_1 *and* f_2 *are called equivariant if* $P = P'$ *with* $f_0 =$ *identity of* P.

The category of simplicial groups with Moore complex of length 2 is equivalant to that of 2-crossed modules. This equivalence was proved by Conduché in [6]. Now, we shall give the pseudo version of this equivalance in the following theorem.

Theorem 3.6 *The category of pseudo 2-crossed modules is equivalent to that of category of pseudosimplicial groups with Moore complex of length 2.*

Proof. Let $\mathbf{G}$ be a pseudosimplicial group with Moore complex of length 2. We construct a pseudo 2-crossed module as follows: $P = G_0$, $M = \ker(d_0 : G_1 \longrightarrow G_0)$, and $L = \ker(d_0 : G_2 \longrightarrow G_1) \cap \ker(d_1 : G_2 \longrightarrow G_1)$. Then $p \in P$ acts on $m \in M$ by ${}^{p}m = s_0(p)ms_0(p)^{-1}$, and on $l \in L$ by ${}^{\partial_1(m)}l = s_0(m)ls_0(m)^{-1}$ and $m \in M$ acts on $l \in L$ by ${}^{m}l = s_1(m)ls_1(m)^{-1}$. For $m, m' \in M$, set $\{m, m'\} = s_0(m)s_1(m')s_0(m)^{-1}s_1(m)s_1(m')^{-1}s_1(m)^{-1}$. Let $\partial_1 = d_1$ (restricted to M) and $\partial_2 = d_2$ (restricted to L).

$$\begin{array}{llcl} \textbf{P}-\textbf{2CM1)} & \partial_2\{m, m'\} & = & \partial_2\left(s_0(m)s_1(m')s_0(m)^{-1}s_1(m)s_1(m')^{-1}s_1(m)^{-1}\right) \\ & & = & d_2s_0(m)d_2s_1(m')d_2s_0(m)^{-1}d_2s_1(m)d_2s_1(m')^{-1} \\ & & & \qquad d_2s_1(m)^{-1} \\ & & = & d_2s_0(m)m'd_2s_0(m)^{-1}m(m')^{-1}(m)^{-1} \\ & & = & s_0d_1(m)m's_0d_1(m)^{-1}m(m')^{-1}(m)^{-1} \\ & & = & \left({}^{\partial_1 m}m'\right)mm'^{-1}m^{-1}. \end{array}$$

$$\begin{array}{llcl} \textbf{P}-\textbf{2CM2)} & \{\partial_2 l, \partial_2 l'\} & = & \{d_2 l, d_2 l'\} \\ & & = & s_0d_2(l)s_1d_2(l')s_0d_2(l)^{-1}s_1d_2(l)s_1d_2(l')^{-1}s_1d_2(l)^{-1} \\ & & = & d_3s_0(l)d_3s_1(l')d_3s_0(l)^{-1}d_3s_1(l)d_3s_1(l')^{-1}d_3s_1(l)^{-1} \\ & & = & d_3s_0(l)d_3s_1(l')d_3s_0(l)^{-1}d_3s_1(l)d_3s_1(l')^{-1}d_3s_1(l)^{-1} \end{array}$$

$$\begin{aligned}
&[l,l'][l',l]\\
=\ & d_3s_0(l)\,d_3s_1(l')\,d_3s_0(l)^{-1}\,d_3s_1(l)\,d_3s_1(l')^{-1}\,d_3s_1(l)^{-1}\\
&\left(ll'(l)^{-1}(l)^{-1}\right)[l',l]\\
=\ & d_3s_0(l)\,d_3s_1(l')\,d_3s_0(l)^{-1}\,d_3s_1(l)\,d_3s_1(l')^{-1}\,d_3s_1(l)^{-1}\\
& d_3s_2(l)\,d_3s_2(l')\,d_3s_2'(l)^{-1}\,d_3s_2(l)^{-1}[l',l]\\
=\ & d_3(s_0(l)\,s_1(l')\,s_0(l)^{-1}\,s_1(l)\,s_1(l')^{-1}\,s_1(l)^{-1}\,s_2(l)\,s_2(l')\\
& s_2(l)^{-1}\,s_2(l)^{-1})[l',l]\\
=\ & [l',l],
\end{aligned}$$

where $s_0(l)\,s_1(l')\,s_0(l)^{-1}\,s_1(l)\,s_1(l')^{-1}\,s_1(l)^{-1}\,s_2(l)\,s_2(l')\,s_2(l)^{-1}\,s_2(l)^{-1}$ lies in ∂_3NG_3.

P − 2CM3) (i)

$$\begin{aligned}
\{mm',m''\} =\ & s_0(mm')\,s_1(m'')\,s_0(mm')^{-1}\,s_1(mm')\,s_1(m'')^{-1}\\
& s_1(mm')^{-1}\\
=\ & s_0(m)\,s_0(m')\,s_1(m'')\,s_0(m')^{-1}\,s_0(m)^{-1}\,s_1(m)\\
& s_1(m')\,s_1(m'')^{-1}\,s_1(m')^{-1}\,s_1(m)^{-1}\\
=\ & s_0(m)\,s_0(m')\,s_1(m'')\,s_0(m')^{-1}\,(s_1(m)\,s_1(m'')^{-1}\\
& s_1(m')^{-1}\,s_0(m)^{-1})\left(s_0(m)\,s_1(m')\,s_1(m'')\,s_1(m)^{-1}\right)\\
& s_0(m)^{-1}\,s_1(m)\,s_1(m')\,s_1(m'')^{-1}\,s_1(m')^{-1}\,s_1(m)^{-1}\\
=\ & (s_0(m)\,s_0(m')\,s_1(m'')\,s_0(m')^{-1}\,s_1(m)\,s_1(m'')^{-1}\\
& s_1(m')^{-1}\,s_0(m)^{-1})(s_0(m)\,s_1(m')\,s_1(m'')\,s_1(m)^{-1}\\
& s_0(m)^{-1}\,s_1(m)\,s_1(m')\,s_1(m'')^{-1}\,s_1(m')^{-1}\,s_1(m)^{-1})\\
=\ & {}^{\partial_1 m}\{m',m''\}(s_0(m)\,s_1\left(m'm''(m')^{-1}\right)s_0(m)^{-1}\,s_1(m)\\
& s_1\left(m'm''(m')^{-1}\right)^{-1}s_1(m)^{-1})\\
=\ & {}^{\partial_1 m}\{m',m''\}\{m,m'm''(m')^{-1}\}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\{m,m'm''\} =\ & s_0(m)\,s_1(m'm'')\,s_0(m)^{-1}\,s_1(m)\,s_1(m'm'')^{-1}\,s_1(m)^{-1}\\
=\ & s_0(m)\,s_1(m')\,s_1(m'')\,s_0(m)^{-1}\,s_1(m)\,s_1(m'')^{-1}\,s_1(m')^{-1}\,s_1(m)^{-1}\\
=\ & s_0(m)\,s_1(m')\left(s_0(m)^{-1}\,s_1(m)\,s_1(m')^{-1}\,s_1(m)^{-1}\right)\\
& \left(s_1(m)\,s_1(m')\,s_1(m)^{-1}\,s_0(m)\right)s_1(m'')\,s_0(m)^{-1}\,s_1(m)\\
& s_1(m'')^{-1}\,s_1(m')^{-1}\,s_1(m)^{-1}\\
=\ & \left(s_0(m)\,s_1(m')\,s_0(m)^{-1}\,s_1(m)\,s_1(m')^{-1}\,s_1(m)^{-1}\right)\\
& (s_1(m)\,s_1(m')\,s_1(m)^{-1}\,s_0(m)\,s_1(m'')\,s_0(m)^{-1}\,s_1(m)\\
& s_1(m'')^{-1}\,s_1(m')^{-1}\,s_1(m)^{-1})\\
=\ & \{m,m'\}(s_1(m)\,s_1(m')\,s_1(m)^{-1}\,s_0(m)\,s_1(m'')\,s_0(m)^{-1}\,s_1(m)\\
& s_1(m'')^{-1}\left(s_1(m)^{-1}\,s_1(m)\right)s_1(m')^{-1}\,s_1(m)^{-1})\\
=\ & \{m,m'\}(s_1(m)\,s_1(m')\,s_1(m)^{-1}\,s_0(m)\,s_1(m'')\,s_0(m)^{-1}\,s_1(m)\\
& s_1(m'')^{-1}\,s_1(m)^{-1}\,s_1(m)\,s_1(m')^{-1}\,s_1(m)^{-1})\\
=\ & \{m,m'\}(s_1\left(mm'(m)^{-1}\right)s_0(m)\,s_1(m'')\,s_0(m)^{-1}\,s_1(m)\\
& s_1(m'')^{-1}\,s_1(m)^{-1}\,s_1\left(mm'(m)^{-1}\right)^{-1})\\
=\ & \{m,m'\}^{mm'm^{-1}}\{m,m''\}.
\end{aligned}$$

$$\begin{array}{lllll}
\mathbf{P-2CM4)} & (a) & \{\partial_2 l, m\} & = & s_0\left(\partial_2(l)\right)s_1(m)\,s_0\left(\partial_2(l)^{-1}\right)s_1\left(\partial_2(l)\right)s_1(m)^{-1}s_1\left(\partial_2(l)^{-1}\right)\\
&&& = & s_0d_2(l)\,s_1(m)\,s_0d_2(l)^{-1}s_1d_2(l)\,s_1(m)^{-1}s_1d_2(l)^{-1}\\
&&& = & d_3s_0(l)\,s_1(m)\,d_3s_0(l)^{-1}d_3s_1(l)\,s_1(m)^{-1}d_3s_1(l)^{-1}\\
&&&& \left(ls_0(m)\,l^{-1}s_0(m)^{-1}\right)\left(s_0(m)\,ls_0(m)^{-1}l^{-1}\right)\\
&&& = & (d_3s_0(l)\,d_3s_2s_1(m)\,d_3s_0(l)^{-1}d_3s_1(l)\,d_3s_2s_1(m)^{-1}d_3s_1(l)^{-1}\\
&&&& d_3s_2(l)\,d_3s_2s_0(m)\,d_3s_2(l^{-1})d_3s_2s_0(m)^{-1})\left(s_0(m)\,ls_0(m)^{-1}l^{-1}\right)\\
&&& = & d_3(s_0(l)\,s_2s_1(m)\,s_0(l)^{-1}s_1(l)\,s_2s_1(m)^{-1}s_1(l)^{-1}\\
&&&& s_2(l)\,s_2s_0(m)\,s_2(l^{-1})s_2s_0(m)^{-1})\left(s_0(m)\,ls_0(m)^{-1}l^{-1}\right)\\
&&& = & {}^{m}l(l)^{-1},
\end{array}$$

where $s_0(l)\,s_2s_1(m)\,s_0(l)^{-1}s_1(l)\,s_2s_1(m)^{-1}s_1(l)^{-1}s_2(l)\,s_2s_0(m)\,s_2(l^{-1})s_2s_0(m)^{-1}$ lies in ∂_3NG_3.

$$\begin{array}{llll}
(b) & \{m, \partial_2 l\} & = & s_0(m)\,s_1\partial_2(l)\,s_0(m)^{-1}s_1(m)\,s_1\partial_2(l)^{-1}s_1(m)^{-1}\\
&& = & s_0(m)\,s_1d_2(l)\,s_0(m)^{-1}s_1(m)\,s_1d_2(l)^{-1}s_1(m)^{-1}\\
&& = & \left(s_0(m)\,ls_0(m)^{-1}\right)\left(s_1(m)\,(l)^{-1}s_1(m)^{-1}\right)\\
&& = & ({}^{\partial_1 m}l)({}^{m}(l)^{-1}).
\end{array}$$

$$\begin{array}{llll}
\mathbf{P-2CM5)} & \{m,\partial_2 l\}\{\partial_2 l, m\} & = & \left(s_0(m)\,s_1d_2(l)\,s_0(m)^{-1}s_1(m)\,s_1d_2(l)^{-1}s_1(m)^{-1}\right)\\
&&& \left(s_0d_2(l)\,s_1(m)\,s_0d_2(l)^{-1}s_1d_2(l)\,s_1(m)^{-1}s_1d_2(l)\right)\\
&& = & \left(s_0(m)\,d_3s_1(l)\,s_0(m)^{-1}s_1(m)\,d_3s_1(l)^{-1}s_1(m)^{-1}\right)\\
&&& \left(d_3s_0(l)\,s_1(m)\,d_3s_0(l)^{-1}d_3s_1(l)\,s_1(m)^{-1}d_3s_1(l)\right)\\
&&& \left(ls_0(m)\,(l)^{-1}s_0(m)^{-1}\right)\left(s_0(m)\,ls_0(m)^{-1}(l)^{-1}\right)\\
&& = & (d_3s_2s_0(m)\,d_3s_1(l)\,d_3s_2s_0(m)^{-1}d_3s_2s_1(m)\\
&&& d_3s_1(l)^{-1}d_3s_2s_1(m)^{-1})(d_3s_0(l)\,d_3s_2s_1(m)\,d_3s_0(l)^{-1}\\
&&& d_3s_1(l)\,d_3s_2s_1(m)^{-1}d_3s_1(l))(d_3s_2(l)\,d_3s_2s_0(m)\\
&&& d_3s_2(l)^{-1}d_3s_2s_0(m)^{-1}\left(s_0(m)\,ls_0(m)^{-1}(l)^{-1}\right)\\
&& = & d_3(s_2s_0(m)\,s_1(l)\,s_2s_0(m)^{-1}s_2s_1(m)\,s_1(l)^{-1}s_2s_1(m)^{-1}\\
&&& s_0(l)\,s_2s_1(m)\,s_0(l)^{-1}s_1(l)\,s_2s_1(m)^{-1}s_1(l)\,s_2(l)\\
&&& s_2s_0(m)\,s_2(l)^{-1}s_2s_0(m)^{-1})\left(s_0(m)\,ls_0(m)^{-1}(l)^{-1}\right)\\
&& = & {}^{\partial_1 m}l(l)^{-1},
\end{array}$$

where

$$\begin{array}{l}
s_2s_0(m)\,s_1(l)\,s_2s_0(m)^{-1}s_2s_1(m)\,s_1(l)^{-1}s_2s_1(m)^{-1}s_0(l)\,s_2s_1(m)\,s_0(l)^{-1}\\
\qquad s_1(l)\,s_2s_1(m)^{-1}s_1(l)\,s_2(l)\,s_2s_0(m)\,s_2(l)^{-1}s_2s_0(m)^{-1}
\end{array}$$

lies in ∂_3NG_3.

Conversely we start with a pseudo 2-crossed module $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} P$. Set $G_0 = P$. Using the action of P on M, we can form the semi-direct product $G_1 = M \rtimes P$. There are homomorphisms

$$\begin{array}{ll}
d_0 : M \rtimes P \longrightarrow P\ , & (m,p) \longmapsto p\ ,\\
d_1 : M \rtimes P \longrightarrow P\ , & (m,p) \longmapsto (\partial m)\,p\ ,\\
s_0 : P \longrightarrow M \rtimes P\ , & p \longmapsto (1,p)\,.
\end{array}$$

There is an action of $m \in M$ on $l \in L$ given by

$$m.l = \{m, \partial_2 l\}^{\partial_1 m} l.$$

Using this action we can form the semi-direct product $L \rtimes M$. There is an action of $(m,p) \in M \rtimes P$ on $(l,m') \in L \rtimes M$ given by

$$(m,p) \cdot (l,m') = ({}^{m}l\ {}^{p}l,\ {}^{m}m'\ {}^{p}m).$$

Using this action, we form the semi-direct product $G_2 = (L \rtimes M) \rtimes (M \rtimes P)$. There are homomorphisms

$$\begin{array}{ll}
d_0 : (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (M \rtimes P), & (l,m',m,p) \longmapsto (m',p), \\
d_1 : (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (M \rtimes P), & (l,m',m,p) \longmapsto ({}^{m}m',p), \\
d_2 : (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (M \rtimes P), & (l,m',m,p) \longmapsto \left(m, {}^{\partial_1 m'} p\right), \\
s_0 : (M \rtimes P) \longrightarrow (L \rtimes M) \rtimes (M \rtimes P), & (m,p) \longmapsto (1,1,m',p), \\
s_1 : (M \rtimes P) \longrightarrow (L \rtimes M) \rtimes (M \rtimes P), & (m,p) \longmapsto (1,m',1,p).
\end{array}$$

There is an action of $(l,m) \in L \rtimes M$ on $l' \in L$ given by

$${}^{(l,m)}l' = (ll'l)^{m}\, l'$$

and we can construct the semi-direct product $L \rtimes (L \rtimes M)$. There is an action of $(m,p) \in M \rtimes P$ on $(l,l',m') \in L \rtimes (L \rtimes M)$ given by

$$m \cdot l = \{m, \partial l\}^{m}\, l.$$

There is also an action of $(l',m) \in L \rtimes M$ on $(l,l',m') \in L \rtimes (L \rtimes M)$ given by

$$(m,p) \cdot (l,m') = ({}^{m}l\ {}^{p}l, {}^{m}m'\ {}^{p}m).$$

These last two actions combine to give an action of $(L \rtimes M) \rtimes (M \rtimes P)$ on $L \rtimes (L \rtimes M)$, from which we construct the semi-direct product $G_3 = (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P)$. There are homomorphisms

$$\begin{array}{l}
d_0 : (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\
\qquad (l,l',m,l'',m',m'',p) \longmapsto (l',m',m'',p), \\
d_1 : (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\
\qquad (l,l',m,l'',m',m'',p) \longmapsto (l'l'',mm',m'',p), \\
d_2 : (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\
\qquad (l,l',m,l'',m',m'',p) \longmapsto (ll',m,m'm'',p), \\
d_3 : (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow ((L \rtimes M) \rtimes (M \rtimes P)) \\
\qquad (l,l',m,l'',m',m'',p) \longmapsto (l,m,(\partial_2 l'')\,m',(\partial m')\,p), \\
s_1 : (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \\
\qquad (l,m,m',p) \longmapsto (1,1,1,1,m',m,p), \\
s_2 : (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \\
\qquad (l,m,m',p) \longmapsto (1,1,m',1,1,m,p), \\
s_3 : (L \rtimes M) \rtimes (M \rtimes P) \longrightarrow (L \rtimes (L \rtimes M)) \rtimes (L \rtimes M) \rtimes (M \rtimes P) \\
\qquad (l,m,m',p) \longmapsto (1,1,m',1,m,1,p).
\end{array}$$

Axioms (1) – (5) ensure that these are indeed homomorphisms. Let G_2 be the 2-coskeleton of the 2-truncated pseudosimplicial groups $\{G_0, G_1, G_2\}$; let G_3 be the 3-coskeleton of the 3-truncated pseudosimplicial groups

$\{G_0, G_1, G_2, G_3\}$. There is a unique simplicial map $G^3 \longrightarrow G^2$ which in dimensions 0, 1 and 2 is the identity. We let $\overline{G}^3$ denote the image of G^3 in G^2. It is readily checked that the Moore complex of G^3 is trivial in dimension 3; it follows from Lemma 3.2 that $\overline{G}^3$ is a pseudosimplicial group whose Moore complex is of length 2.

The above constructions yield the required equivalence.

We now associate to the each pseudosimplicial groups G a simplicial inclusion $U^kG \longrightarrow G$ and quotient $G \longrightarrow V^kG$ such that the following Proposition 4 holds. The inclusion and quotient are described carefully in the proof of Proposition 4, but in essence can be described in terms of Moore complexes as follows. Suppose that (M_n, ∂_n) is the Moore complex of G. Then U^kG will have the Moore complex

$$\cdots \longrightarrow M_{k+3} \longrightarrow M_{k+2} \longrightarrow \ker(\partial_{k+1}) \longrightarrow 1 \longrightarrow 1 \cdots ,$$

and V^kG will have the Moore complex

$$1 \longrightarrow 1 \longrightarrow im\,(\partial_{k+1}) \longrightarrow M_k \longrightarrow M_{k-1} \longrightarrow \cdots .$$

□

Proposition 3.7 *For any pseudosimplicial group G and integer $k \geq 0$, there is a functorial short exact sequence of pseudosimplicial groups*

$$1 \longrightarrow U^kG \xrightarrow{I} G \xrightarrow{\phi} V^kG \longrightarrow 1$$

such that:

(i) the Moore complex of U^kG is trivial in dimensions $0, 1, \ldots, k$, and identical with the Moore complex of G in dimensions $\geq k+2$;

(ii) the map $\imath$ induces isomorphisms on homotopy groups $\pi_n\left(U^kG\right) \cong \pi_n(G)$ for $n \geq k+1$;

(iii) the Moore complex of V^kG is trivial in dimensions $\geq k+2$, and in dimensions $\leq k$ is identical with the Moore complex of G.

Proof. First construct the k-coskeleton $\cos k^k\left(tr^kG\right)$ of G. By lemma the Moore complex of $\cos k^k\left(tr^kG\right)$ is

$$1 \longrightarrow 1 \longrightarrow K \overset{\partial_{k+1}}{\rightarrowtail} M_k \xrightarrow{\partial_k} M_{k-1} \longrightarrow \cdots \xrightarrow{\partial_1} M_0 .$$

Here ∂_{k+1} is an inclusion, K is the kernel of ∂_k, and in dimensions $\leq k$ this complex coincides with the Moore complex of G. Since K is a normal subgroup of G_k, we can quotient tr^kG by K to obtain a k-truncated pseudosimplicial groups $tr^kG = \{G_0, \ldots, G_{k-1}, G_k/K\}$. We now construct the k-coskeleton $\cos k^k\left(tr^kG/K\right)$. There is a unique simplicial map $G \longrightarrow \cos k^k\left(tr^kG/K\right)$ which in dimensions less than k is the identity, and in dimension k is the quotient $G_k \twoheadrightarrow G_k/K$; we let U^kG denote the kernel of this map, and V^kG denote the image of G in $\cos k^k\left(tr^kG/K\right)$. We thus have a short exact sequence of pseudosimplicial groups

$$1 \longrightarrow U^kG \longrightarrow G \longrightarrow V^kG \longrightarrow 1$$

which induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_n\left(U^kG\right) \longrightarrow \pi_n\left(G\right) \longrightarrow \pi_n\left(V^kG\right) \to \pi_{n-1}\left(U^kG\right) \to \cdots .$$

The assertions of the proposition are easily checked. □

Proposition 3.8 *Let G be a pseudosimplicial groups such that $\pi_n G = 1$, for $n = 0, \ldots, k$. Then there exists a weak homotopy equivalence $F \simeq G$ with F a free pseudosimplicial groups such that $F_n = 1$ for $n = 0, \ldots, k$.*

Proposition 3.9 *It follows from Proposition 2.7(i) and the simplicial identities that the pseudosimplicial groups U^kG is trivial in dimensions $\leq k$. From axiom $(M2)$ of a model category (see [14]) there is a weak equivalence $F \simeq U^kG$ with F a free pseudosimplicial groups. We can construct F so that it meets the requirements of the propositions.*

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