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On weakly $\mathcal{M}$-supplemented primary subgroups of finite groups*

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Abstract

A subgroup H of a group G is said to be weakly $\mathcal{M}$-supplemented in G if there exists a subgroup B of G provided that (1) $G = HB$, and (2) if H_1/H_G is a maximal subgroup of H/H_G, then $H_1B = BH_1 < G$. where H_G is the largest normal subgroup of G contained in H. In this paper we will prove that: Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F^*(H)$ having no supersolvable supplement in G, is weakly $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.

Key Words: Primary subgroups,weakly $\mathcal{M}$-supplemented subgroups, supersolvable groups, formation.

1. Introduction

A subgroup H of a group G is called *supplemented* in G if there exists a subgroup K of G such that $HK = G$ and K is called a *supplement* of H in G. Obviously every subgroup of G is supplemented in G as G can be one of its supplements. Hence we should give some restricting conditions.

The relationship between the properties of primary subgroups and the structure of finite groups has been investigated extensively by many authors. For instance, in 1937 Hall [6] proved that a group G is solvable if and only if every Sylow subgroup of G is complemented in G. In 1982 Arad and Ward [1] proved that a group G is solvable if and only if every *Sylow* 2-subgroup and *Sylow* 3-subgroup of G are complemented in G. In 1999 A.Ballester-Bolinches and X.Guo [2] proved that the class of all supersolvable groups with elementary abelian Sylow subgroups is just the class of all finite groups for which every minimal subgroup is complemented. In 1980 Srinivassan[13] proved that a finite group is supersolvable if every maximal subgroup of Sylow subgroup is normal. By considering c-supplement of some primary subgroups, in 2000 Wang [14] obtained some new conditions for the solvability and supersolvability of a finite group. In 2005 Miao and Guo [9]proved that G is supersolvable if and only if every maximal subgroups of the Sylow subgroup of G is supersolvable s-supplemented in G. In 2007, Miao [10] introduced the concept of Q-supplemented subgroups and obtained some sufficient condition for supersolvability of finite groups. Recently, Miao and W. Lempken [11] introduced

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the definition of $\mathcal{M}$-supplemented subgroup, and get some new information on the structure of finite groups. Now, we continue these work and introduce the concept of weakly $\mathcal{M}$-supplemented subgroups.

Definition 1.1 *A subgroup H of a group G is said to be weakly $\mathcal{M}$-supplemented in G if there exists a subgroup B of G such that* (1) $G = HB$, *and* (2) *if H_1/H_G is a maximal subgroup of H/H_G, then $H_1B = BH_1 < G$. where H_G is the largest normal subgroup of G contained in H.*

We recall that a subgroup H is called $\mathcal{M}$-supplemented in a finite group G, if there exists a subgroup B of G such that $G = HB$ and H_1B is a proper subgroup of G for any maximal subgroup H_1 of H. Moreover, a subgroup H is called c-normal in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$ where H_G is the largest normal subgroup of G contained in H.

It is clear that every $\mathcal{M}$-supplemented subgroup and every c-normal subgroup are weakly $\mathcal{M}$-supplemented. In this paper, we shall investigate the properties of the weakly $\mathcal{M}$-supplemented subgroups in a finite group G.

Throughout this paper, all groups are finite. Our terminology and notation are standard, see [4] and [12]. In particular, let G denote a finite group, $M < \cdot G$ indicates that M is a maximal subgroup of G.$|G|$ denotes the order of G. $\mathcal{U}$ denotes the class of all supersolvable groups. $\pi(G)$ denotes the set of all prime divisors of G. For the group G, $G = [H]K$ denotes the fact that G is the semi-direct product of H and K where H is normal in G.

Let π be a set of primes. We say that $G \in E_\pi$ if G has a Hall π-subgroup. We say that $G \in C_\pi$ if $G \in E_\pi$ and any two Hall π-subgroups of G are conjugate in G. We say that $G \in D_\pi$ if $G \in C_\pi$ and every π-subgroup of G is contained in a Hall π-subgroup of G.

Let $\mathcal{F}$ be a class of groups. $\mathcal{F}$ is said to be a formation provided that (1) *if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and* (2) *if G/M and G/N are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$. It is clear that for a formation, every group G has a smallest normal subgroup (denoted by $G^{\mathcal{F}}$) whose quotient $G/G^{\mathcal{F}}$ is in $\mathcal{F}$. The normal subgroup $G^{\mathcal{F}}$ is called the $\mathcal{F}$-residual of G. A formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. It is well known that the class of all supersolvable groups and the class of all p-nilpotent groups are saturated formations(cf.[5]).*

2. Preliminaries

For the sake of convenience, we first list here some known results which will be useful in the sequel.

Lemma 2.1 *Let G be a group. Then*

(1) *If H is weakly $\mathcal{M}$-supplemented in G, $H \leq M \leq G$, then H is weakly $\mathcal{M}$-supplemented in M.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is weakly $\mathcal{M}$-supplemented in G if and only if H/N is weakly $\mathcal{M}$-supplemented in G/N.*

(3) *Let π be a set of primes. Let K be a normal π'-subgroup and H be a π-subgroup of G. If H is weakly $\mathcal{M}$-supplemented in G, then HK/K is weakly $\mathcal{M}$-supplemented in G/K.*

(4) *Let R be a solvable minimal normal subgroup of a group G and R_1 be a maximal subgroup of R. If R_1 is weakly $\mathcal{M}$-supplemented in G, then R is a cyclic group of prime order.*

(5) *Let P be a p-subgroup of G where p is a prime divisor of $|G|$. If P is weakly $\mathcal{M}$-supplemented in*

G, then there exists a subgroup B of G such that $|G:TB| = p$ for any maximal subgroup T of P containing P_G.

Proof. (1) If H is weakly $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for any maximal subgroup H_1 of H with $H_G \leq H_1$. Since $H \leq M \leq G$, we have $H_G \leq H_M$. So we may set $L = M \cap B$. Clearly, $L = M \cap B \leq M$ and $M = M \cap HB = H(M \cap B) = HL$. Since $TB < G$ for every maximal subgroup T of H with $H_M \leq T$, we easily see that $TL = T(M \cap B) = M \cap TB$ is a proper subgroup of M.

(2) It is obvious by the definition of weakly $\mathcal{M}$-supplemented subgroups.

(3) If H is weakly $\mathcal{M}$-supplemented in G, then there exists a subgroup B such that $G = HB$ and $H_1B = BH_1 < G$ for any maximal subgroup H_1 of H with $H_G \leq H_1$. Clearly, $(HK/K)(BK/K) = G/K$. For any maximal subgroup T/K of HK/K with $(HK/K)_{G/K} \leq T/K$, since K is a π'-subgroup and H is a π-subgroup, we have $T = T_1K$ where T_1 is a maximal subgroup of H with $H_G \leq T_1$. Therefore

$$(T_1K/K)(BK/K) = T_1BK/K = (BK/K)(T_1K/K) < G/K.$$

Otherwise, if $T_1BK = G$, then $|G : T_1B| = |K : K \cap T_1B|$ is a π'-number; on the other hand, $|G : T_1B| = |HB : T_1B|$ is a π- number, a contradiction.

(4) If R_1 is weakly $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = R_1B$ and $TB = BT < G$ for any maximal subgroup T of R_1 with $(R_1)_G \leq T$. On the other hand, since R is a minimal normal subgroup of G, we have $G = RB$ and $R \cap B \in \{1, R\}$. If $R \cap B = R$, then $B = G$, a contradiction. If $R \cap B = 1$, then R is a cyclic subgroup of prime order.

(5) If P is weakly $\mathcal{M}$-supplemented in G, then there exists a subgroup B of G such that $G = PB$ and $TB = BT < G$ for any maximal subgroup T of P with $P_G \leq T$. Since $|P : T| = p$, we get

$$|G| = |PB| = p|T||B|/|P \cap B| = (p/|(P \cap B) : (T \cap B)|) \cdot |TB|.$$

As p is a prime and $TB < G$, we conclude that $P \cap B = T \cap B$ and $|G : TB| = p$. Now the claim follows. □

Lemma 2.2 [5, *Theorem* 1.8.17] *Let N be a nontrivial solvable normal subgroup of a group G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.3 [17] *If H is a subgroup of G with $|G : H| = p$, where p is the smallest prime divisor of $|G|$, then $H \trianglelefteq G$.*

Lemma 2.4 [3] *Suppose a finite group G has a Hall π-subgroup where π is a set of primes not containing 2. Then all Hall π-subgroups of G are conjugate.*

Lemma 2.5 *Let G be a finite group and P a Sylow p-subgroup of G where p is the smallest prime divisor of $|G|$. If every maximal subgroup of P having no p-nilpotent supplement in G, is weakly $\mathcal{M}$-supplemented in G, then $G/O_p(G)$ is solvable p-nilpotent.*

Proof. Assume that the claim is false and choose G to be a counterexample of smallest order. Clearly, G is not a nonabelian simple group. Furthermore we have:

(1) $O_p(G) \neq 1$.

If $O_p(G) = P$, then $G/O_p(G)$ is a p'-group and of course it is p-nilpotent, a contradiction. If $1 < O_p(G) < P$, then $G/O_p(G)$ satisfies the hypotheses and the minimal choice of G implies that $G/O_p(G) \cong G/O_p(G)/O_p(G/O_p(G))$ is p-nilpotent, a contradiction.

(2) $O_p(G) = 1$.

Let P_1 be a maximal subgroup of the Sylow p-subgroup P of G where p is the smallest prime divisor of $|G|$. If $|G|_p = p$, then G is p-nilpotent by Burnside's p-nilpotence Theorem, a contradiction. So we may assume that $|G|_p \geqslant p^2$. By hypotheses, if P_1 has a p-nilpotent supplement in G, then there exists a subgroup K of G such that $G = P_1K$ and K is p-nilpotent. Therefore we have $K_{p'} \trianglelefteq K$ where $K_{p'}$ is a Hall $p^{'}$-subgroup of K and of course is the Hall $p^{'}$-subgroup of G. Hence $G = P_1N_G(K_{p'})$. If $P \cap N_G(K_{p'}) = P$, then $K_{p'} \trianglelefteq G$, a contradiction. If $P \cap N_G(K_{p'}) = L$, where $L < \cdot P$, then

$$|G : N_G(K_{p'})| = |P : P \cap N_G(K_{p'})| = |P : L| = p$$

and hence $N_G(K_{p'}) \trianglelefteq G$ by Lemma 2.3, a contradiction. So we may assume that $P \cap N_G(K_{p'}) \leq L_2 < L_1$ where L_1 is the maximal subgroup of P and L_2 is the maximal subgroup of L_1. If L_1 has a p-nilpotent supplement in G, then there exists a p-nilpotent subgroup H such that $G = L_1H$. With the similar discussion we have $G = L_1N_G(H_{p'})$ where $H_{p'}$ is the Hall $p^{'}$-subgroup of H and of course of G. By Lemma 2.4, there exists an element x of P such that $N_G(K_{p'}) = (N_G(H_{p'}))^x$. Therefore $G = L_1N_G(H_{p'}) = (L_1N_G(H_{p'}))^x = L_1N_G(K_{p'})$. Furthermore,

$$P = P \cap L_1N_G(K_{p'}) = L_1(P \cap N_G(K_{p'})) = L_1$$

a contradiction. So we may assume L_1 is weakly $\mathcal{M}$-supplemented in G, there exists a subgroup B of G such that $G = L_1B$ and $TB < G$ for any maximal subgroup T of L_1 with $(L_1)_G \leq T$. Moreover, $(L_1)_G \leq O_p(G) = 1$ and hence L_1 is $\mathcal{M}$-supplemented in G in this case. Therefore $L_2B < G$ and $|G : L_2B| = p$ by Lemma 2.1(5). Since p is the smallest prime divisor of $|G|$, Lemma 2.3 implies that $L_2B \trianglelefteq G$. We have $G = L_1B = PB = PL_2B$ and $P \cap L_2B = L_2(P \cap B)$ is the Sylow p-subgroup of L_2B. Clearly, $L_2(P \cap B)$ is the maximal subgroup of P. By hypotheses if $L_2(P \cap B)$ is weakly $\mathcal{M}$-supplemented in G, then $L_2(P \cap B)$ is $\mathcal{M}$-supplemented in G and hence is $\mathcal{M}$-supplemented in L_2B by Lemma 2.1. So L_2B is p-nilpotent by [11, Lemma 2.11]. Therefore G is p-nilpotent, a contradiction.

So we may assume $L_2(P \cap B)$ has a p-nilpotent supplement in G. With the similar discussion as above, there exists a p-nilpotent subgroup S of G such that $G = L_2(P \cap B)S = L_2(P \cap B)N_G(S_{p'})$ where $S_{p'}$ is a normal Hall $p^{'}$-subgroup of S and also of G. By Lemma 2.4, there exists an element g of P such that $N_G(K_{p'}) = (N_G(S_{p'}))^g$. Therefore

$$G = L_2(P \cap B)N_G(S_{p'}) = (L_2(P \cap B)N_G(S_{p'}))^g = L_2(P \cap B)N_G(K_{p'}).$$

Furthermore,

$$P = P \cap L_2(P \cap B)N_G(K_{p'}) = L_2(P \cap B)(P \cap N_G(K_{p'})) = L_2(P \cap B),$$

a contradiction.

Thereby we get $G/O_p(G)$ is p-nilpotent. □

Lemma 2.6 [7, 8] *Let G be a group and N a subgroup of G. The generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. Then*

(1) *If N is normal in G, then $F^*(N) \leq F^*(G)$;*

(2) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G)$;*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$;*

(4) *$C_G(F^*(G)) \leq F(G)$;*

(5) *Let $P \trianglelefteq G$ and $P \leq O_p(G)$; then $F^*(G/\Phi(P)) = F^*(G)/\Phi(P)$;*

(6) *If K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$.*

Lemma 2.7 [11, *Lemma* 2.7] *Let G be a finite group with normal subgroups H and L and let $p \in \pi(G)$. Then the following hold:*

1) *If $L \leq \Phi(G)$, then $F(G/L) = F(G)/L$.*

2) *If $L \leq H \cap \Phi(G)$, then $F(H/L) = F(H)/L$.*

3) *If H is a p-group and $L \leq \Phi(H)$, then $F^*(G/L) = F^*(G)/L$.*

4) *If $L \leq \Phi(G)$ with $|L| = p$, then $F^*(G/L) = F^*(G)/L$.*

5) *If $L \leq H \cap \Phi(G)$ with $|L| = p$, then $F^*(H/L) = F^*(H)/L$.*

Lemma 2.8 [15, *Theorem*3.1] *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

3. Main results

Theorem 3.1 *Let G be a group and H be a normal subgroup of G such that G/H is supersolvable. If every maximal subgroups of every noncyclic Sylow subgroup of H having no supersolvable supplement in G, is weakly $\mathcal{M}$-supplemented in G, then G is supersolvable.*

Proof. Assume that the theorem is false and let G be a counterexample with minimal order. Then we have the following claims:

(1) G is solvable.

By hypotheses and Lemma 2.5, $H/O_r(H)$ is solvable r-nilpotent where r is the smallest prime divisor of $|H|$ and hence G is solvable. Let L be a minimal normal subgroup of G contained in H. Clearly, L is an elementary abelian p-group for some prime divisor of $|G|$.

(2) G/L is supersolvable and L is the unique minimal normal subgroup of G contained in H such that $H \cap \Phi(G) = 1$. Furthermore, $L = F(H) = C_H(L)$.

Firstly, we check $(G/L, H/L)$ satisfies the hypotheses for (G, H). We know that $H/L \trianglelefteq G/L$ and $(G/L)/(H/L) \cong G/H$ is supersolvable. Let $\overline{Q} = QL/L$ be a Sylow q-subgroup of H/L. We may assume that Q is a Sylow q-subgroup of H. If $p = q$, we may assume that $L \leq P$, where P is a Sylow p-subgroup of H. If $L \leq P = Q$, then every maximal subgroup of P/L is of the form P_1/L with P_1 a maximal subgroup of P. If P_1/L has no supersolvable supplement in G/L, then P_1 does not admit supersolvable supplement in G, by hypotheses, P_1 is weakly $\mathcal{M}$-supplemented in G and hence P_1/L is weakly $\mathcal{M}$-supplemented in G/L by Lemma 2.1. Now we assume that $p \neq q$. Let $\overline{Q_1}$ be a maximal subgroup of a Sylow q-subgroup of $\overline{H}$. Without loss of generality, we may assume that $\overline{Q_1} = Q_1L/L$ with Q_1 a maximal subgroup of a Sylow q-subgroup of H. Clearly, if Q_1L/L has no supersolvable supplement in G/L, then Q_1L/L is weakly $\mathcal{M}$-supplemented in G/L by Lemma 2.1. So G/L satisfies the hypotheses of the theorem. The minimal choice of G implies that G/L is supersolvable. Since the class of all supersolvable groups is a saturated formation, we know that L is the unique minimal normal subgroup of G which is contained in H and $L \nleq \Phi(G)$. By Lemma 2.2 we have $F(H) = L$. The solvability of H implies that $L \leq C_H(L) = C_H(F(H)) \leq F(H)$ and so $C_H(L) = L = F(H)$.

(3) L is a Sylow subgroup of H.

Let q be the largest prime divisor of $|H|$ and Q be a Sylow q-subgroup of H. Since G/L is supersolvable, we have that H/L is supersolvable. Consequently, LQ/L char $H/L \trianglelefteq G/L$ and hence $LQ \trianglelefteq G$. If $p = q$, then $L \leq Q \trianglelefteq G$. and $Q \leq F(H) = L$. So L is a Sylow q-subgroup of H as desired.

Now we assume $p < q$. Let P be a Sylow p-subgroup of H. Clearly, P is not cyclic. Otherwise, $G/L \in \mathcal{U}$ implies that $G \in \mathcal{U}$. Then $L \leq P$ and $PQ = PLQ$ is a subgroup of H. Note that every maximal subgroup of noncyclic Sylow subgroup of PQ having no supersolvable supplement in PQ, is weakly $\mathcal{M}$-supplemented in PQ by Lemma 2.1. Therefore PQ satisfies the hypotheses for G. If $PQ < G$, the minimal choice of G implies that PQ is supersolvable; in particular, $Q \trianglelefteq PQ$. Hence $LQ = L \times Q$ and $Q \leq C_H(L) \leq L$, a contradiction.

Now we may assume that $G = PQ = H$ and $L < P$. Since G/L is supersolvable, $LQ \trianglelefteq G$. By the Frattini argument, $G = LN_G(Q)$. Note that $L \cap N_G(Q)$ is normalized by $N_G(Q)$ and L. We have that $L \cap N_G(Q) = 1$ since L is the unique minimal normal subgroup of G and Q is not normal in G in this case. Therefore $G = [L]N_G(Q)$. Let P_2 be a Sylow p-subgroup of $N_G(Q)$. Then LP_2 is a Sylow p-subgroup of G. Choose a maximal subgroup P_1 of LP_2 such that $P_2 \leq P_1$. Clearly, $L \nleq P_1$ and hence $(P_1)_G = 1$. Assume P_1 is weakly $\mathcal{M}$-supplemented in G. Then there exists a subgroup B of G such that $G = P_1B$ and $TB < G$ for any maximal subgroup T of P_1 such that $T \geq (P_1)_G = 1$. We may assume $P_2 \leq T$ for some maximal subgroup T of P_1. Otherwise, $P_2 = P_1$, then we have $|L| = p$ and hence G/L is supersolvable implies that G is supersolvable, a contradiction. By Lemma 2.1, $|G : TB| = p$. Therefore $L \leq TB$ or $L \cap TB = 1$. If $L \cap TB = 1$, then $|G : TB| = |L| = p$, a contradiction. So we may assume $L \leq TB$. Since $P_2 \leq T$, we have $LP_2 \leq TB$, contrary to $|G : TB| = p$.

Now we may assume that P_1 has a supersolvable supplement in G, that is, there exists a supersolvable subgroup K of G such that $G = P_1K$. In fact, K has a normal p-complement Q_1 which is also a Sylow q-subgroup of G. By Sylow's theorem, there exists an element $g \in L$ such that $Q_1^g = Q$. Since $P_1 \trianglelefteq LP_2$, we have that $G = P_1K = (P_1K)^g = P_1K^g$. Since $K^g \cong K$ has a normal Sylow q-subgroup and $Q = Q_1^g \leq K^g$, it follows that $K^g \leq N_G(Q)$. Since $LP_2 = LP_2 \cap G = LP_2 \cap P_1K^g = P_1(LP_2 \cap K^g)$, we have that $LP_2 \cap K^g \nleq P_2$. Otherwise $LP_2 \leq P_1P_2 = P_1$, a contradiction. Therefore P_2 is a proper subgroup of $P_3 =< P_2, LP_2 \cap K^g >$.

On the other hand, since both P_2 and K^g are contained in $N_G(Q)$, P_3 is a p-subgroup of $N_G(Q)$ which contains a Sylow p-subgroup P_2 of $N_G(Q)$ as a proper subgroup, a contradiction.

(4) G is supersolvable.

Let L_1 be a maximal subgroup of L. If L_1 has a supersolvable supplement in G, then there exists a supersolvable subgroup K of G such that $G = L_1K$. Since L is a minimal normal subgroup of G, we have $L \cap K \in \{1, L\}$. If $L \cap K = L$, then $G = L_1K = K$, a contradiction. If $L \cap K = 1$, then $|L| = p$, also a contradiction. So we have that L_1 is weakly $\mathcal{M}$-supplemented in G. In this case we know that L is a cyclic subgroup of order p by Lemma 2.1, a contradiction.

The final contradiction completes our proof. □

Corollary 3.2 *Let G be a group. If every maximal subgroup of every noncyclic Sylow subgroup of G not admitting a supersolvable supplement, is weakly $\mathcal{M}$-supplemented in G, then G is supersolvable.*

Theorem 3.3 *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a finite group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of H having no supersolvable supplement in G, is weakly $\mathcal{M}$-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Assume that the claim is false and choose G to be a counterexample of minimal order. Since the pair (H, H) satisfies the hypotheses for the pair (G, H) with $H/H \in \mathcal{U}$, H is supersolvable by Theorem 3.1.

Now let p be the largest prime divisor of $|H|$ and $P \in Syl_p(H)$; so we get $P = O_p(H) \trianglelefteq G$. Let L be a minimal normal subgroup of G contained in P. Using similar arguments as for the proof of Claim (2) in the proof of Theorem 3.1 we easily establish that $G/L \in \mathcal{F}$ and that L is the unique minimal normal subgroup of G contained in H; moreover, $L = F(H) = C_H(L)$ is noncyclic and $H \cap \Phi(G) = 1$.

Clearly, $\Omega_1(Z(P)) \trianglelefteq G$ and so $L \leq \Omega_1(Z(P))$; hence $P \leq C_H(L) = L$ and thus $L = P \in Syl_p(H)$. The same arguments as in the last step of the proof of Theorem 3.1 now yield a final contradiction. □

Theorem 3.4 *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F(H)$ having no supersolvable supplement in G, is weakly $\mathcal{M}$-supplemented in G. Then $G \in \mathcal{F}$.*

Proof. Assume that the assertion is false and choose G to be a counterexample of minimal order.

For any maximal subgroup M of G, if $F(H) \leq M$, then $G \in \mathcal{F}$ by Lemma 2.8, a contradiction. So we may assume that there at least exists a maximal subgroup M of G not containing $F(H)$.

Actually, since $F(H) \nleq M$, there at least exists a prime p of $\pi(|H|)$ with $O_p(H) \nleq M$. Then $G = O_p(H)M$ and $O_p(H) \cap M \trianglelefteq G$. If $|O_p(H)| = p$, then $|G : M| = p$ and hence $G \in \mathcal{F}$ by Lemma 2.8, a contradiction. Let M_p be a Sylow p-subgroup of M. Then we know that $G_p = O_p(H)M_p$ is a Sylow p-subgroup of G. Now, let P_1 be a maximal subgroup of G_p containing M_p and set $P_2 = P_1 \cap O_p(H)$. Then $P_1 = P_2M_p$. Moreover, $P_2 \cap M_p = O_p(H) \cap M_p$, so $|O_p(H) : P_2| = |O_p(H)M_p : P_2M_p| = |G_p : P_1| = p$, that is, P_2 is a maximal subgroup of $O_p(H)$. Hence $P_2(O_p(H) \cap M)$ is a subgroup of $O_p(H)$. By the maximality of P_2 in $O_p(H)$, we have $P_2(O_p(H) \cap M) = P_2$ or $O_p(H)$.

1) If $P_2(O_p(H)\cap M)=O_p(H)$, then $G=O_p(H)M=P_2M$. Notice that $O_p(H)\cap M=P_2\cap M$. So $O_p(H)=P_2$, a contradiction.

2) $P_2(O_p(H)\cap M)=P_2$, that is, $O_p(H)\cap M\leq P_2$. Clearly, $O_p(H)\cap M\trianglelefteq G$, so $O_p(H)\cap M\leq (P_2)_G$. On the other hand, by hypotheses, if P_2 has a supersolvable supplement in G, then there exists a subgroup N of G such that $G=P_2N$ and N is a supersolvable group. Set $K=(P_2)_GN$, then $G=P_2N=P_2K$ and $K/K\cap (P_2)_G=K/(P_2)_G=(P_2)_GN/(P_2)_G\cong N/N\cap (P_2)_G\in\mathcal{U}\subseteq\mathcal{F}$.

Now, we consider the following cases.

a) $K<G$.

Suppose that K_1 is a maximal subgroup of G containing K. Then $O_p(H)\cap K_1\trianglelefteq G$, which implies that $(O_p(H)\cap K_1)M$ is a subgroup of G. If $(O_p(H)\cap K_1)M=G=O_p(H)M$, then $O_p(H)\cap K_1=O_p(H)$ since $(O_p(H)\cap K_1)\cap M=O_p(H)\cap M$. This implies that $O_p(H)\leq K_1$, and hence $G=O_p(H)K_1=K_1$, which is contrary to the above hypotheses on K_1. Thus $(O_p(H)\cap K_1)M=M$ and $O_p(H)\cap K_1\leq M$. Furthermore,

$$P_2\cap K\leq O_p(H)\cap K\leq O_p(H)\cap M\leq (P_2)_G\leq P_2\cap K.$$

So $O_p(H)\cap K=O_p(H)\cap M=P_2\cap K$. This is contrary to $G=P_2K=O_p(H)K$.

b) $K=(P_2)_GN=G$.

In this case, if $(P_2)_G=1$, then $N=G\in\mathcal{F}$, a contradiction. So we may assume that $(P_2)_G\neq 1$. Thus $(P_2)_GM=M$ or $(P_2)_GM=G$. If $(P_2)_GM=G$, then $G=(P_2)_GM=O_p(H)M=P_2M$. Note that $O_p(H)\cap M=P_2\cap M$, so $O_p(H)=P_2$, a contradiction. Therefore $(P_2)_GM=M$, that is $(P_2)_G\leq M$, then $(P_2)_G\leq O_p(H)\cap M\leq (P_2)_G$ and hence $O_p(H)\cap M=(P_2)_G$. By hypotheses, $G/(P_2)_G\in\mathcal{U}$ implies that $|G/(P_2)_G:M/(P_2)_G|=|G:M|=p$. This means that $F(H)\cap M$ has a prime index in $F(H)$ and hence $G\in\mathcal{F}$ also by Lemma 2.8, a contradiction.

So we may assume that P_2 is weakly $\mathcal{M}$-supplemented in G. There exists a subgroup B of G such that $P_2B=G$ and $TB<G$ for any maximal subgroup T containing $(P_2)_G$. If P_2 is normal in G. The maximality of M in G implies $P_2M=M$ or $P_2M=G$. If $P_2M=G$, then we have $G=O_p(H)M=P_2M$ and hence $O_p(H)=P_2$ since $O_p(H)\cap M=P_2\cap M$, a contradiction. So $P_2M=M$, that is, $P_2\leq M$. Thus $O_p(H)\cap M=P_2\cap M=P_2$ and hence

$$|F(H):F(H)\cap M|=|G:M|=|O_p(H):O_p(H)\cap M|=p.$$

This indicates that $F(H)\cap M$ is a maximal subgroup of $F(H)$. By Lemma 2.8, $G\in\mathcal{F}$.

Next we may assume $(P_2)_G<P_2$. For any maximal subgroup T of P_2 containing $(P_2)_G$, we have $|G:TB|=p$ by Lemma 2.1(5). Clearly, TB is a maximal subgroup of G. Then $O_p(H)\cap TB\trianglelefteq G$, which implies that $(O_p(H)\cap TB)M$ is a subgroup of G. If $(O_p(H)\cap TB)M=G=O_p(H)M$, then $O_p(H)\cap TB=O_p(H)$ since $(O_p(H)\cap TB)\cap M=O_p(H)\cap M$. This implies that $O_p(H)\leq TB$, and hence $G=O_p(H)TB=TB$, which is contrary to the above hypotheses on TB. Thus $O_p(H)\cap TB\leq M$. Furthermore, $P_2\cap TB\leq O_p(H)\cap TB\leq O_p(H)\cap M\leq (P_2)_G\leq P_2\cap TB$, from this, $O_p(H)\cap TB=O_p(H)\cap M=P_2\cap TB$. This is contrary to $G=P_2B=O_p(H)B$.

The final contradiction completes our proof. □

Corollary 3.5 *Let G be a group with a solvable normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of every noncyclic Sylow subgroup of $F(H)$ having no supersolvable supplement in G, is weakly $\mathcal{M}$-supplemented in G. Then $G \in \mathcal{U}$.*

Theorem 3.6 *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G be a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every maximal subgroup of a noncyclic Sylow subgroup of $F^*(H)$ having no supersolvable supplement in G, is weakly $\mathcal{M}$-supplemented in G. Then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is false and choose G to be a counterexample of minimal order; so in particular, $H \neq 1$. We consider the following two case.

Case 1. $\mathcal{F} = \mathcal{U}$.

By Corollary 3.2 we easily verify that $F^*(H)$ is supersolvable and hence $F(H) = F^*(H) \neq 1$. Since H satisfies the hypotheses of the theorem, the minimal choice of G implies that H is supersolvable if $H < G$. Then $G \in \mathcal{U}$ by Corollary 3.5, a contradiction. Thus we have

(1) $H = G$, $F^*(G) = F(G) \neq 1$.

Let S be a proper normal subgroup of G containing $F^*(G)$. By Lemma 2.6, $F^*(G) = F^*(F^*(G)) \leq F^*(S) \leq F^*(G)$, so $F^*(S) = F^*(G)$. And every maximal subgroup of every noncyclic Sylow subgroup of $F^*(S)$ having no supersolvable supplement in S, is weakly $\mathcal{M}$-supplemented in S by Lemma 2.1. Hence S is supersolvable by the minimal choice of G and we get

2) Every proper normal subgroup of G containing $F^*(G)$ is supersolvable.

Suppose now that $\Phi(O_p(G)) \neq 1$ for some $p \in \pi(F(G))$. By Lemma 2.6 we have $F^*(G/\Phi(O_p(G))) = F^*(G)/\Phi(O_p(G))$. Using Lemma 2.1 observe that the pair $(G/\Phi(O_p(G)), F^*(G)/\Phi(O_p(G)))$ satisfies the hypotheses of the theorem. The minimal choice of G then implies $G/\Phi(O_p(G)) \in \mathcal{U}$. Since $\mathcal{U}$ is a saturated formation we then get $G \in \mathcal{U}$, a contradiction. Thus we have

(3) If $p \in \pi(F(G))$, then $\Phi(O_p(G)) = 1$ and so $O_p(G)$ is elementary abelian; in particular, $F^*(G) = F(G)$ is abelian and $C_G(F(G)) = F(G)$.

If L is a minimal normal subgroup of G contained in $F(G)$ and $|L| = p$ where $p \in \pi(F(G))$, then set $C = C_G(L)$. Clearly, $F(G) \leq C \trianglelefteq G$. If $C < G$, then C is solvable by (2). On the other hand, since G/C is cyclic, then we have G is solvable, a contradiction. So we may assume $C = G$. Now we have $L \leq Z(G)$. Then we consider subgroup G/L. By Lemma 2.6, we have $F^*(G/L) = F^*(G)/L = F(G)/L$. In fact, G/L satisfies the condition of the theorem by Lemma 2.1. Therefore the minimal choice of G implies that $G/L \in \mathcal{U}$ and hence G is supersolvable, a contradiction. This proves:

(4) There is no normal subgroup of prime order in G contained in $F(G)$.

If every Sylow subgroup of $F(G)$ is cyclic, then $F(G) = H_1 \times \cdots H_r$ where $H_i\,(i = 1, \cdots, r)$ is the cyclic Sylow subgroup of $F(G)$ and hence $G/C_G(H_i)$ is abelian for any $i \in \{1 \cdots r\}$. Moreover, we have $G/\bigcap_{i=1}^{r} C_G(H_i) = G/C_G(F(G))$ is abelian and hence $G/F(G)$ is abelian since $C_G(F(G)) = C_G(F^*(G)) \leq F(G)$. Therefore G is solvable, a contradiction. This proves that

(5) There exist noncyclic Sylow subgroup $O_p(G)$ of $F(G)$ for some prime $p \in \pi(F(G))$.

Let P_1 be a maximal subgroup of $O_p(G)$. If P_1 has a supersolvable supplement in G, then there exists a supersolvable subgroup K of G such that $G = P_1K = O_p(G)K$. Clearly, $G/O_p(G) \cong K/K \cap O_p(G)$ is supersolvable and hence G is solvable, a contradiction. So we obtain that

(6) Every maximal subgroup of every noncyclic Sylow subgroup of $F(G)$ has no supersolvable supplement in G.

Set $R = O_p(G) \cap \Phi(G)$. If $R = 1$, then by Lemma 2.2, $O_p(G)$ is the direct product of some minimal normal subgroup of G. So we may assume that $O_p(G) = R_1 \times \ldots \times R_t$, where R_i is a minimal normal subgroup of G, $i = 1.2...t$. Consider the maximal subgroup P_1 of P, P_1 has the form

$$P_1 = R_1 \times \ldots \times R_{i-1} \times {R_i}^* \times R_{i+1} \times \ldots \times R_t,$$

where ${R_i}^*$ is a maximal subgroup of R_i for some i. Let T denote the normal subgroup $R_1 \times \ldots \times R_{i-1} \times R_{i+1} \times \ldots R_t$ of G, then $P_1 = {R_i}^*T$.By hypotheses and (6), P_1 is weakly $\mathcal{M}$-supplemented in G. We claim that $(P_1)_G = T$. Clearly, $T \leq (P_1)_G$. On the other hand, if $(P_1)_G > T$, then we have $(P_1)_G \cap {R_i}^* > 1$ by $(P_1)_G = (P_1)_G \cap P_1 = (P_1)_G \cap ({R_i}^*T) = T((P_1)_G \cap {R_i}^*)$. Hence we have that $1 < {R_i}^* \cap (P_1)_G \leq R_i \cap (P_1)_G < R_i$ and $R_i \cap (P_1)_G$ is normal in G. Since R_i is a minimal normal subgroup of G, we have $R_i \cap (P_1)_G = R_i$, a contradiction. By hypotheses P_1 is weakly $\mathcal{M}$-supplemented in G. There exists a subgroup B of G such that $G = P_1B$ and $SB = BS < G$ for any maximal subgroup S of P_1 containing $(P_1)_G = T$. So by Lemma 2.1(5) we have $|G : SB| = p$. Since R_i is the minimal normal subgroup of G, we have $R_i \cap SB \in \{1, R_i\}$. Clearly, if $R_i \leq SB$, then we have $SB = R_iSB = G$, a contradiction. So we have $R_i \nleq SB$, we know that $|R_i| = p$, contrary to (4). This contradiction leads to

(7) $R = O_p(G) \cap \Phi(G) \neq 1$.

Let Q be a Sylow q-subgroup of $F(G)$, and let L be a minimal normal subgroup of G contained in R, where $q \neq p$. Then Q is elementary abelian by (3). By the definition of a generalized Fitting subgroup, $F^*(G/L) = F(G/L)E(G/L)$ and $[F(G/L), E(G/L)] = 1$, where $E(G/L)$ is the layer of G/L. Since $L \leq \Phi(G)$, $F(G/L) = F(G)/L$. Now set $E/L = E(G/L)$. Since Q is normal in G and $[F(G)/L, E/L] = 1$, $[Q, E] \leq Q \cap L = 1$, i.e., $[Q, E] = 1$. Therefore we have $F(G)E \leq C_G(Q)$. If $C_G(Q) < G$, then $C_G(Q)$ is supersolvable by (2). Thus $E(G/L) = E/L$ is supersolvable. The semisimplicity of $E(G/L)/Z(E(G/L))$ implies $E(G/L) = Z(E(G/L))$. So $E(G/L) \leq F(G/L)$ and $F^*(G/L) = F(G)/L$, with the same argument in (3), we have that G/L satisfies the hypotheses of the theorem. By the minimal choice of G, G/L is supersolvable and so is G, a contradiction. If $C_G(Q) = G$, then $Q \leq Z(G)$. By Lemma 2.6, $F^*(G/Q) = F^*(G)/Q = F(G)/Q$. Similarly, G/Q is supersolvable and so is G by Corollary 3.5, a contradiction. This verifies

(8) $F(G) = O_p(G)$, and G has a unique minimal normal subgroup L contained in R.

If $R = O_p(G)$, then by hypotheses any maximal subgroups P_1 of $O_p(G)$ is weakly $\mathcal{M}$-supplemented in G. That is, there exists a subgroup B such that $G = P_1B$ and $TB < G$ for any maximal subgroup T of P_1 containing $(P_1)_G$. Then $G = P_1B = B$ since $P_1 \leq \Phi(G)$, a contradiction. Hence $R \neq O_p(G)$. Now $\Phi(G/R) = 1$. Then by Lemma 2.2, $O_p(G)/R = (H_1/R) \times \cdots \times (H_m/R)$, where $H_i/R (i = 1, \cdots m)$ are minimal normal in G/R. With the same argument as in (7), we know that $H_i/R (i = 1, \cdots m)$ are all of order p because all maximal subgroups of $O_p(G)/R$ are weakly $\mathcal{M}$-supplemented in G/R by Lemma 2.1. Again, since $O_p(G)$ is an elementary abelian p-group, H_i is of the form $< x_i > R, (i = 1, \cdots m)$. This proves

(9) $O_p(G) =< x_1 > \times \cdots \times < x_m > \times R$ where $< x_i > R \trianglelefteq G, (i = 1, \cdots m)$.

Let R^* be a maximal subgroup of R. Clearly, TR^* is a maximal subgroup of $O_p(G)$ where $T =< x_1 > \times \cdots < x_m >$. If $(TR^*)_G \cap R = 1$ for some maximal subgroup R^* of R, then $(TR^*)_G \cap < x_1 > R$ is of order 1 or p. Observe that $(TR^*)_G \cap < x_1 > R$ is normal in G, and its order is not p by (4). Thus $(TR^*)_G \cap < x_1 > R = 1$, similarly, $(TR^*)_G \cap < x_1 >< x_2 > R = \cdots = (TR^*)_G \cap TR = 1$, i.e., $(TR^*)_G = 1$. By hypotheses, TR^* is weakly $\mathcal{M}$-supplemented in G and there exists a subgroup B of G such that $G = TR^*B$ and $SB = BS < G$ for any maximal subgroup S containing $(TR^*)_G = 1$. By Lemma 2.1, $|G : SB| = p$. Clearly, SB is a maximal subgroup of G, and $L \cap SB \in \{1, L\}$. If $L \cap SB = 1$, then $|L| = p$, this is contrary to (4). So we have $L \leq SB$ for any maximal subgroup S of TR^*. Furthermore, if $L \cap TR^* = 1$, then also we have $|L| = p$, a contradiction. So we get $L \cap TR^* \neq 1$. We claim that $L \cap TR^* \leq S$ for any maximal subgroup S of TR^*. Otherwise, there exists a maximal subgroup S of TR^* such that $L \cap TR^* \nleq S$. So we consider $SB = (L \cap TR^*)SB = TR^*B = G$, a contradiction. Based on the discussion as above, we have $1 < L \cap TR^* \leq \Phi(TR^*) \leq \Phi(O_p(G))$, contrary to (3). Then we get

(10) For any maximal subgroup R^* of R, we have $(TR^*)_G \cap R \neq 1$, where $T =< x_1 > \times \cdots < x_m >$.

From (8) and (10), we have $L \leq (TR^*)_G \cap R$ for any maximal subgroup R^* of R. By hypotheses, TR^* is weakly $\mathcal{M}$-supplemented in G, so there exists a subgroup B such that $G = TR^*B$ and $SB < G$ for any maximal subgroup S of TR^* containing $(TR^*)_G$. Then $L \leq (TR^*)_G \cap R \leq TR^* \cap (TR^*)_G B \cap R = R^* \cap (TR^*)_G B$. Thus $L \leq \bigcap_{R^* <\cdot R}(R^* \cap (TR^*)_G B) = \Phi(R) \cap (\bigcap_{R^* <\cdot R}((TR^*)_G B) = 1$ by (3), a final contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By case 1, H is supersolvable. Particularly, H is solvable and hence $F^*(H) = F(H)$. Therefore $G \in \mathcal{F}$ by Corollary 3.5.

The final contradiction completes our proof. □

Corollary 3.7 *Let G be a group with a normal subgroup H such that $G/H \in \mathcal{U}$. If every maximal subgroup of a noncyclic Sylow subgroup of $F^*(H)$ having no supersolvable supplement in G, is weakly $\mathcal{M}$-supplemented in G. Then $G \in \mathcal{U}$.*

Corollary 3.8 [16, *Theorem* 1.1] *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroup of any Sylow subgroup of $F^*(H)$ is c-normal in G, then $G \in \mathcal{F}$.*

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