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Statistical convergence of max-product approximating operators

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Abstract

In this study, using the notion of statistical convergence, we obtain various statistical approximation theorems for a general sequence of max-product approximating operators, including Shepard type operators, although its classical limit fails. We also compute the corresponding statistical rates of the approximation.

Key Words: Statistical convergence, max-product operators, Shepard operators, statistical rates.

1. Introduction

In the classical approximation theory, many well-known approximating operators obey the linearity condition. In recent years, Bede et al. [3] have shown that it is possible to find some approximating operators that are not linear, such as, max-product and max-min Shepard type approximating operators. Actually, these operators are pseudo-linear which is a quite effective structure in solving the problems in many branches of applied mathematics, such as, image processing [4], differential equations [19, 20], idempotent analysis [18] and approximation theory [3, 5]. However, so far, almost all results regarding approximations by pseudo-linear operators are based on the validity of the classical limit of the operators. Hence, in this paper, we focus on the following problem: is it possible to make an approximation by max-product operators although its classical limit fails? As an answer to this problem we mainly use the concept of statistical convergence, which was first introduced by Fast [13]. Recent studies demonstrate that the notion of statistical convergence provides an important contribution to the improvement of the classical approximation theory (see, for instance, [1, 2, 7, 8, 9, 10, 11, 12]).

This paper is organized as follows: The first section is devoted to basic definitions and notations used in the paper. In the second section, we obtain some statistical approximation results for a general class of max-product operators including Shepard type operators. In the third section, we compute the corresponding statistical rates of the approximation, while, in the last section, we give some quantitative statistical rates.

Let (x_n) be a sequence of numbers. Then, (x_n) is called statistically convergent to a number L if, for every $\varepsilon > 0$,

$$\lim_j \frac{\#\{n \le j : |x_n - L| \ge \varepsilon\}}{j} = 0,$$

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where $\#B$ denotes the cardinality of the subset B (see [13], also [15]). We denote this statistical limit by $st-\lim_n x_n = L$. Now, let $A=(a_{jn})$ be an infinite summability matrix. Then, the A-transform of x, deneted by $Ax := ((Ax)_j)$, is given by $(Ax)_j = \sum_{n=1}^{\infty} a_{jn}x_n$, provided the series converges for each j. We say that A is regular if $\lim_j (Ax)_j = L$ whenever $\lim_j x_j = L$ [16]. Assume now that A is a nonnegative regular summability matrix. Then, a sequence (x_n) is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:\,|x_n-L|\geq\varepsilon} a_{jn} = 0 \tag{1.1}$$

holds (see [14]). It is denoted by $st_A - \lim_n x_n = L$.

Now we recall some basic properties of A-statistical convergence as follows:

- A-statistical convergence method is mainly based on the concept of A-density. Recall that the *A-density* of a subset $K \subset \mathbb{N}$, denoted by $\delta_A(K)$, is given by

$$\delta_A(K) = \lim_j \sum_{n=1}^{\infty} a_{jn}\chi_K(n),$$

provided that the limit exists, where χ_K is the characteristic function of K; or equivalently,

$$\delta_A(K) = \lim_j \sum_{n\in K} a_{jn}.$$

So, by (1.1), we easily see that $st_A - \lim x = L$ if and only if

$$\delta_A(\{n : |x_n - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$.

- If we take $A = C_1 := [c_{jn}]$, where the *Cesáro matrix* is given by

$$c_{jn} := \begin{cases} \frac{1}{j}, & \text{if } 1 \leq n \leq j \\ 0, & \text{otherwise,} \end{cases}$$

then A-statistical convergence reduces to *statistical convergence*, i.e., $st_{C_1} - \lim_n x_n = st - \lim_n x_n = L$.

- Taking $A = I$, the *identity matrix*, A-statistical convergence coincides with the ordinary convergence, i.e., $st_I - \lim x = \lim x = L$.

- Observe that every convergent sequence (in the usual sense) is A-statistically convergent to the same value for any non-negative regular matrix A, but its converse is not always true. Actually, in [17], Kolk proved that A-statistical convergence is stronger than convergence when $A = [a_{jn}]$ is a non-negative regular summability matrix such that $\lim_j \max_n\{a_{jn}\} = 0$. So, one can construct a sequence that is A-statistically convergent but non-convergent.

- Not all properties of convergent sequences are true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however an A-statistically convergent sequence does not need to be bounded.

- A characterization for statistical convergence, i.e., the case of $A = C_1$, was proved by Connor [6]: $st - \lim x = L$ if and only if there exists a subsequence $\{x_{n_k}\}$ of x such that $\delta(\{n_1, n_2, ...\}) = 1$ and $\lim_k x_{n_k} = L$. It is easy to check that a similar characterization is also valid for A-statistical convergence when A is any non-negative regular summability matrix.

2. Approximation properties of max-product operators

Let (X, d) be an arbitrary compact metric space, and let $A = (a_{jn})$ be a non-negative regular summability matrix. By $C(X, [0, \infty))$ we denote the space of all non-negative continuous functions on X. Then we consider the following max-product operators:

$$L_n(f; x) = \bigvee_{k=0}^{n} K_n(x, x_k) \cdot f(x_k), \quad x \in X \text{ and } f \in C(X, [0, \infty)), \tag{2.1}$$

where $x_k \in X$, $k = 0, 1, ..., n$, are the knots; and $K_n(x, x_k)$ are non-negative continuous functions on X having relatively simple expression (algebraic or trigonometric polynomials, rational functions, wavelets, etc.) such that, for any $x \in X$,

$$\delta_A\left(\left\{n \in \mathbb{N} : \bigvee_{k=0}^{n} K_n(x, x_k) = 1\right\}\right) = 1 \tag{2.2}$$

holds. Observe that the operators mapping $C(X, [0, \infty))$ into $C(X, [0, \infty))$ are pseudo-linear, i.e., for every $f, g \in C(X, [0, \infty))$ and for any non-negative numbers α, β,

$$L_n\left(\alpha \cdot f \bigvee \beta \cdot g; x\right) = \alpha \cdot L_n(f; x) \bigvee \beta \cdot L_n(g; x)$$

is satisfied (see [3]).

We first recall the following lemma introduced in [3], which is useful in proving our main results.

Lemma A ([3]). *For any* $a_k, b_k \in [0, \infty)$, $k = 0, 1, ..., n$, *we have*

$$\left|\bigvee_{k=0}^{n} a_k - \bigvee_{k=0}^{n} b_k\right| \leq \bigvee_{k=0}^{n} |a_k - b_k|.$$

Then we obtain the following Korovkin-type result for the max-product operators.

Theorem 2.1 *Let (X,d) be an arbitrary compact metric space, and let $A=(a_{jn})$ be a non-negative regular summability matrix. If, for the operators L_n given by (2.1) and (2.2),*

$$st_A-\lim_n\left\{\bigvee\{|L_n(\varphi_x;x)|:x\in X\}\right\}=0 \quad \textit{with} \;\; \varphi_x(y)=d^2(y,x), \tag{2.3}$$

then, for all $f\in C(X,[0,\infty))$, we have

$$st_A-\lim_n\left\{\bigvee\{|L_n(f;x)-f(x)|:x\in X\}\right\}=0.$$

Proof. Let $x\in X$ and $f\in C(X,[0,\infty))$ be fixed. Then, using the continuity of f and also considering the compactness of X, we immediately see that, for a given $\varepsilon>0$, there exists a positive number δ such that

$$|f(y)-f(x)|\le\varepsilon+\frac{2M_f}{\delta^2}\varphi_x(y) \tag{2.4}$$

holds for all $y\in X$, where $M_f:=\bigvee\{|f(y)|:y\in X\}$. Now put

$$K:=\left\{n\in\mathbb{N}:\bigvee_{k=0}^{n}K_n(x,x_k)=1\right\}. \tag{2.5}$$

Then, by (2.2), we may write that

$$\delta_A(K)=1 \;\text{ and }\; \delta_A(\mathbb{N}\backslash K)=0.$$

Hence, by (2.2), (2.4) and Lemma A, we get, for all $n\in K$, that

$$\begin{aligned}
|L_n(f;x)-f(x)| &= \left|\bigvee_{k=0}^{n}K_n(x,x_k)\cdot f(x_k)-\bigvee_{k=0}^{n}K_n(x,x_k)\cdot f(x)\right| \\
&\le \bigvee_{k=0}^{n}K_n(x,x_k)\cdot|f(x_k)-f(x)| \\
&\le \bigvee_{k=0}^{n}K_n(x,x_k)\cdot\left(\varepsilon+\frac{2M_f}{\delta^2}\varphi_x(x_k)\right) \\
&\le \varepsilon+\frac{2M_f}{\delta^2}\bigvee_{k=0}^{n}K_n(x,x_k)\cdot\varphi_x(x_k) \\
&= \varepsilon+\frac{2M_f}{\delta^2}L_n(\varphi_x;x).
\end{aligned}$$

Now, taking maximum over $x\in X$, the last inequality gives, for all $n\in K$, that

$$\bigvee\{|L_n(f;x)-f(x)|:x\in X\}\le\varepsilon+\frac{2M_f}{\delta^2}\bigvee\{|L_n(\varphi_x;x)|:x\in X\}. \tag{2.6}$$

For a given $r>0$, choose an $\varepsilon>0$ such that $\varepsilon<r$, and then define the sets

$$\begin{aligned}
D &: \left\{n\in\mathbb{N}:\left(\bigvee\{|L_n(f;x)-f(x)|:x\in X\}\right)\ge r\right\}, \\
D' &: \left\{n\in\mathbb{N}:\left(\bigvee\{|L_n(\varphi_x;x)|:x\in X\}\right)\ge\frac{(r-\varepsilon)\delta^2}{2M_f}\right\}.
\end{aligned}$$

So, inequality (2.6) implies

$$D \cap K \subseteq D' \cap K,$$

which yields, for every $j \in \mathbb{N}$, that

$$\sum_{n \in D \cap K} a_{jn} \leq \sum_{n \in D' \cap K} a_{jn} \leq \sum_{n \in D'} a_{jn}. \tag{2.7}$$

Taking limit as $j \to \infty$ on the both-sides of the inequality (2.7) and also using the hypothesis (2.3), we get

$$\lim_j \sum_{n \in D \cap K} a_{jn} = 0. \tag{2.8}$$

On the other hand, since

$$\begin{aligned} \sum_{n \in D} a_{jn} &= \sum_{n \in D \cap K} a_{jn} + \sum_{n \in D \cap (\mathbb{N} \backslash K)} a_{jn} \\ &\leq \sum_{n \in D \cap K} a_{jn} + \sum_{n \in (\mathbb{N} \backslash K)} a_{jn} \end{aligned}$$

holds for every $j \in \mathbb{N}$, letting again $j \to \infty$ in the last inequality and using (2.8) and also the fact that $\delta_A (\mathbb{N} \backslash K) = 0$, we have

$$\lim_j \sum_{n \in D} a_{jn} = 0,$$

which means that

$$st_A - \lim_n \left\{ \bigvee \{ |L_n(f;x) - f(x)| : x \in X \} \right\} = 0.$$

The theorem is proved. □

We immediately obtain the next result from Theorem 2.1 by replacing the matrix $A = (a_{jn})$ with the identity matrix.

Corollary 2.2 *Let (X, d) be an arbitrary compact metric space. Assume that the operators L_n given by (2.1) satisfy the condition*

$$\bigvee_{k=0}^{n} K_n(x, x_k) = 1 \quad (\text{for } n \in \mathbb{N} \text{ and } x \in X).$$

If the sequence $\{L_n(\varphi_x; x)\}_{n \in \mathbb{N}}$ converges uniformly to zero function with respect to $x \in X$, then, for all $f \in C(X, [0, \infty))$, $\{L_n(f; x)\}_{n \in \mathbb{N}}$ is also uniformly convergent to $f(x)$ with respect to $x \in X$.

Remark 2.3 *Observe that Theorem 2.1 gives the statistical approximation to a function $f \in C(X)$ by means of the max-product operators L_n while Corollary 2.2 gives the classical approximation. However, the following example shows that our statistical approximation result is stronger than the classical one.*

Example. Let (X,d) be an arbitrary compact metric space. Consider the Shepard-type max-product operators (see [5]) as follows:

$$S_n^{\lambda}(f;x) = \bigvee_{k=0}^{n} \left(\frac{\frac{1}{d^{\lambda}(x,x_k)}}{\bigvee_{j=0}^{n} \frac{1}{d^{\lambda}(x,x_j)}} \right) \cdot f(x_k) = \frac{\bigvee_{k=0}^{n} \frac{f(x_k)}{d^{\lambda}(x,x_k)}}{\bigvee_{j=0}^{n} \frac{1}{d^{\lambda}(x,x_j)}}, \tag{2.9}$$

where $x \in X$, $\lambda, n \in \mathbb{N}$ and $f \in C(X,[0,\infty))$. In this case, we know from [5] that, for all $f \in C(X,[0,\infty))$, the sequence $\{S_n^{\lambda}(f)\}$ in (2.9) is uniformly convergent to f on X. Now, let (u_n) be a divergent but A-statistically null sequence of positive numbers. Recall that we can construct such a sequence (u_n) due to Kolk [17]. Actually, Kolk [17] proved that A-statistical convergence is stronger than the usual convergence if the matrix $A = (a_{jn})$ is any nonnegative regular summability matrix for which $\lim_j \max_n \{a_{jn}\} = 0$. Then, we define the max-product operators on $C(X,[0,\infty))$ as

$$T_n(f;x) = (1+u_n) S_n^{\lambda}(f;x), \quad x \in X \quad \text{and} \quad f \in C(X,[0,\infty)), \tag{2.10}$$

where the operators S_n are given by (2.9). Observe now that all the conditions of Theorem 2.1 are satisfied for the operators T_n defined by (2.10). Therefore, for all $f \in C(X,[0,\infty))$, we conclude that

$$st_A - \lim_n \left\{ \bigvee \{ |T_n(f;x) - f(x)| : x \in X \} \right\} = 0.$$

However, since (u_n) is divergent, Corollary 2.2 does not work for the operators T_n given by (2.10).

3. Statistical rates of the approximation

This section is devoted to compute the rates of A-statistical convergence in Theorem 2.1. Before starting, we recall that various ways of defining rates of convergence in the A-statistical sense have been introduced in [10] as follows:

Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $(p_n)_{n \in \mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then,

(i) A sequence $x = (x_n)_{n \in N}$ is A-statistically convergent to the number L with the rate of $o(p_n)$ if for every $\varepsilon > 0$,

$$\lim_j \frac{1}{p_j} \sum_{n : |x_n - L| \geq \varepsilon} a_{jn} = 0.$$

In this case we write $x_n - L = st_A - o(p_n)$ as $n \to \infty$.

(ii) $(x_n)_{n \in \mathbb{N}}$ is A-statistically convergent to L with the rate of $o_m(p_n)$, denoted by $x_n - L = st_A - o_m(p_n)$ as $n \to \infty$, if for every $\varepsilon > 0$,

$$\lim_j \sum_{n : |x_n - L| \geq \varepsilon p_n} a_{jn} = 0.$$

Observe that, in definition (i), the "rate" is more controlled by the entries of the summability method rather than the terms of the sequence $(x_n)_{n\in\mathbb{N}}$. For instance, when one takes the identity matrix I, if we choose any non-increasing sequence $(p_n)_{n\in\mathbb{N}}$ satisfying $1/p_n \le M$ for some $M > 0$ and for each $n \in \mathbb{N}$, then $x_n - L = st_A - o(p_n)$ as $n \to \infty$ for any convergent sequence $(x_n - L)_{n\in\mathbb{N}}$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation one may borrow the concept of convergence in measure from measure theory to define the rate of convergence as in definition (ii). So, we use the notation o_m.

We first need the following lemma.

Lemma 3.1 *For every* $a_k, b_k \ge 0$ $(k = 0, 1, ..., n)$, *we have*

$$\bigvee_{k=0}^{n} a_k b_k \le \sqrt{\bigvee_{k=0}^{n} a_k^2}\sqrt{\bigvee_{k=0}^{n} b_k^2}.$$

Proof. Assume that, for some $p, q \in \{0, 1, ..., n\}$,

$$\bigvee_{k=0}^{n} a_k = a_p \quad \text{and} \quad \bigvee_{k=0}^{n} b_k = b_q.$$

Since, for every $k = 0, 1, ..., n$,

$$\bigvee_{k=0}^{n} a_k b_k \le a_p b_q, \quad \bigvee_{k=0}^{n} a_k^2 = a_p^2 \text{ and } \bigvee_{k=0}^{n} b_k^2 = b_q^2,$$

the proof follows immediately. □

Now we are ready to give the corresponding statistical rates.

Theorem 3.2 *Let* (X, d) *be an arbitrary compact metric space, and let* $A = (a_{jn})$ *be a non-negative regular summability matrix. Assume that* (p_n) *is a sequence of positive non-increasing real numbers. If the operators* L_n *given by* (2.1) *and* (2.2) *satisfy that*

$$w(f, \delta_n) = st_A - o(p_n) \quad \text{as} \quad n \to \infty \quad \text{for } f \in C(X, [0, \infty)), \tag{3.1}$$

where (δ_n) *is a sequence whose terms are defined by*

$$\delta_n := \sqrt{\bigvee \{L_n(\varphi_x; x) : x \in X\}} \quad \text{with } \varphi_x(y) = d^2(y, x) \tag{3.2}$$

then, for any sequence (q_n) *of positive non-increasing real numbers satisfying* $q_n \ge p_n$ *and* $q_n \ge 1$ *for all* $n \in \mathbb{N}$, *we have*

$$\bigvee \{|L_n(f; x) - f(x)| : x \in X\} = st_A - o(q_n) \quad \text{as } n \to \infty. \tag{3.3}$$

Proof. Let $x \in X$ and $f \in C(X,[0,\infty))$ be fixed. Considering the set K given by (2.5), we can write, for every $n \in K$ and for any $\delta > 0$, that

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \bigvee_{k=0}^{n} K_n(x,x_k) \cdot |f(x_k) - f(x)| \\ &\leq \bigvee_{k=0}^{n} K_n(x,x_k) \cdot w\left(f, d(x_k,x)\right) \\ &\leq w(f,\delta) \bigvee_{k=0}^{n} K_n(x,x_k) \cdot \left(1 + \frac{d(x_k,x)}{\delta}\right) \\ &= w(f,\delta) \left\{1 + \frac{1}{\delta} \bigvee_{k=0}^{n} K_n(x,x_k) \cdot d(x_k,x)\right\}. \\ &\leq w(f,\delta) \left\{1 + \frac{1}{\delta} \bigvee_{k=0}^{n} \left[K_n^{1/2}(x,x_k)\right] \cdot \left[K_n^{1/2}(x,x_k) d(x_k,x)\right]\right\} \end{aligned}$$

Now, by using Lemma 3.1, we immediately see that

$$|L_n(f;x) - f(x)| \leq w(f,\delta) \left\{1 + \frac{1}{\delta}\sqrt{L_n\left(d^2(\cdot,x);x\right)}\right\}$$

holds for every $n \in K$ and for any $\delta > 0$. Hence, we obtain, for the same n and δ, that

$$\bigvee \{|L_n(f;x) - f(x)| : x \in X\} \leq w(f,\delta) \left\{1 + \frac{\delta_n}{\delta}\right\}. \tag{3.4}$$

Now choosing $\delta := \delta_n$ given by (3.2), it follows from (3.4) that

$$\bigvee \{|L_n(f;x) - f(x)| : x \in X\} \leq 2w(f,\delta_n). \tag{3.5}$$

For a given $\varepsilon > 0$, consider the following sets:

$$\begin{aligned} E &:= \left\{n \in \mathbb{N} : \bigvee \{|L_n(f;x) - f(x)| : x \in X\} \geq \varepsilon\right\}, \\ E' &:= \left\{n \in \mathbb{N} : w(f,\delta_n) \geq \frac{\varepsilon}{2}\right\}. \end{aligned}$$

Then, inequality (3.5) guarantees that

$$E \cap K \subseteq E' \cap K. \tag{3.6}$$

Since $q_j \geq p_j$ for all $j \in \mathbb{N}$, we obtain from (3.6) that

$$\frac{1}{q_j} \sum_{n \in E \cap K} a_{jn} \leq \frac{1}{p_j} \sum_{n \in E' \cap K} a_{jn} \leq \frac{1}{p_j} \sum_{n \in E'} a_{jn}.$$

Letting $j \to \infty$ on the both-sides of the last inequality, and applying the hypothesis (3.1), we get

$$\lim_j \frac{1}{q_j} \sum_{n \in E \cap K} a_{jn} = 0. \tag{3.7}$$

Furthermore, as in the proof of Theorem 2.1, since

$$\begin{aligned} \sum_{n \in E} a_{jn} &= \sum_{n \in E \cap K} a_{jn} + \sum_{n \in E \cap (\mathbb{N} \setminus K)} a_{jn} \\ &\leq \sum_{n \in E \cap K} a_{jn} + \sum_{n \in (\mathbb{N} \setminus K)} a_{jn}, \end{aligned}$$

it is clear that

$$\frac{1}{q_j} \sum_{n \in E} a_{jn} \leq \frac{1}{q_j} \sum_{n \in E \cap K} a_{jn} + \frac{1}{q_j} \sum_{n \in (\mathbb{N} \setminus K)} a_{jn}.$$

Using the fact that $q_j \geq 1$ for all $j \in \mathbb{N}$, the last inequality implies that

$$\frac{1}{q_j} \sum_{n \in E} a_{jn} \leq \frac{1}{q_j} \sum_{n \in E \cap K} a_{jn} + \sum_{n \in (\mathbb{N} \setminus K)} a_{jn}. \tag{3.8}$$

Then, taking limit as $j \to \infty$ in (3.8), and considering (3.7), we conclude that

$$\lim_j \frac{1}{q_j} \sum_{n \in E} a_{jn} = 0.$$

Therefore, we have

$$\bigvee \{|L_n(f;x) - f(x)| : x \in X\} = st_A - o(q_n) \quad \text{as} \quad n \to \infty,$$

which completes the proof. □

In a similar manner, we obtain the following result for the statistical rate o_m.

Theorem 3.3 *Let (X,d) be an arbitrary compact metric space, and let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that (p_n) is a sequence of positive non-increasing real numbers. If the operators L_n given by (2.1) and (2.2) satisfy that*

$$w(f, \delta_n) = st_A - o_m(p_n) \quad \text{as} \quad n \to \infty \quad \text{for } f \in C(X, [0,\infty)),$$

where (δ_n) is the same as in (3.2), then, for any sequence (q_n) of positive non-increasing real numbers satisfying $q_n \geq p_n$ for all $n \in \mathbb{N}$, we have

$$\bigvee \{|L_n(f;x) - f(x)| : x \in X\} = st_A - o_m(q_n) \quad \text{as} \quad n \to \infty.$$

Proof. For any $\varepsilon > 0$, define the sets:

$$F \quad : \quad = \left\{ n \in \mathbb{N} : \bigvee \{ |L_n(f;x) - f(x)| : x \in X \} \geq \varepsilon q_n \right\},$$
$$F' \quad : \quad = \left\{ n \in \mathbb{N} : w(f, \delta_n) \geq \frac{\varepsilon p_n}{2} \right\}.$$

Then, by (3.5), we get

$$F \cap K \subseteq F' \cap K.$$

Hence, we obtain, for every $j \in \mathbb{N}$, that

$$\sum_{n \in F \cap K} a_{jn} \leq \sum_{n \in F' \cap K} a_{jn} \leq \sum_{n \in F'} a_{jn},$$

which gives

$$\lim_j \sum_{n \in F \cap K} a_{jn} = 0. \tag{3.9}$$

As in the proof of Theorem 3.2, it follows from (3.9) that

$$\lim_j \sum_{n \in F} a_{jn} = 0.$$

Therefore, we conclude that

$$\bigvee \{ |L_n(f;x) - f(x)| : x \in X \} = st_A - o_m(q_n) \quad \text{as} \quad n \to \infty,$$

whence the result. □

Remark 3.4 *It is easy to see that our Theorem* 2.1 *can be deduced from Theorem* 3.2 (*or Theorem* 3.3) *by choosing* $p_n = q_n = 1$ *for each* $n \in \mathbb{N}$. *Hence, Theorems* 3.2 *and* 3.3 *give us the statistical rates in the approximation of the max-product operators* L_n *defined by* (2.1) *and* (2.2).

4. Quantitative statistical rates

In order to obtain the statistical rates quantitatively one can consider the following expressions instead of the definitions (i) and (ii) given in Section 3:

Let $A = (a_{jn})$ be a non-negative regular summability matrix and let $(p_n)_{n \in \mathbb{N}}$ be a positive non-increasing sequence of real numbers. Then,

$(i)'$ A sequence $x = (x_n)$ is A-statistically bounded with the rate of $O(p_n)$ if for every $\varepsilon > 0$,

$$\sup_j \frac{1}{p_j} \sum_{n : |x_n| \geq \varepsilon} a_{jn} < \infty.$$

In this case we write $x_n = st_A - O(p_n)$ as $n \to \infty$.

$(ii)'$ $(x_n)_{n\in\mathbb{N}}$ is A-statistically bounded with the rate of $O_m(p_n)$, denoted by $x_n = st_A - O_m(p_n)$ as $n \to \infty$, if for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon p_n} a_{jn} = 0.$$

In this case, by a similar idea used in the proofs of Theorems 3.2 and 3.3, the following results can easily be proved.

Theorem 4.1 *Let (X,d) be an arbitrary compact metric space, and let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that (p_n) is a sequence of positive non-increasing real numbers. If the operators L_n given by (2.1) and (2.2) satisfy that*

$$w(f,\delta_n) = st_A - O(p_n) \quad \text{as} \quad n \to \infty \quad \text{for } f \in C(X,[0,\infty)),$$

where (δ_n) is given by (3.2), then, for any sequence (q_n) of positive non-increasing real numbers satisfying $q_n \geq p_n$ and $q_n \geq 1$ for all $n \in \mathbb{N}$, we have

$$\bigvee \{|L_n(f;x) - f(x)| : x \in X\} = st_A - O(q_n) \quad \text{as } n \to \infty.$$

Theorem 4.2 *Let (X,d) be an arbitrary compact metric space, and let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that (p_n) is a sequence of positive non-increasing real numbers. If the operators L_n given by (2.1) and (2.2) satisfy that*

$$w(f,\delta_n) = st_A - O_m(p_n) \quad \text{as} \quad n \to \infty \quad \text{for } f \in C(X,[0,\infty)),$$

where (δ_n) is given by (3.2), then, for any sequence (q_n) of positive non-increasing real numbers satisfying $q_n \geq p_n$ for all $n \in \mathbb{N}$, we have

$$\bigvee \{|L_n(f;x) - f(x)| : x \in X\} = st_A - O_m(q_n) \quad \text{as} \quad n \to \infty.$$

Now we construct an example satisfying all conditions of Theorem 4.1. Firstly, in (2.9), choosing $X = [0,1]$ and $x_k = \frac{k}{n}$ $(k = 0,1,...,n)$ and also taking the absolute value metric, for all continuous functions $f : [0,1] \to [0,\infty)$, we consider the following Shepard-type max-product operators

$$H_n^{\lambda}(f;x) = \frac{\bigvee_{k=0}^{n} \frac{f(k/n)}{|x-(k/n)|^{\lambda}}}{\bigvee_{j=0}^{n} \frac{1}{|x-(j/n)|^{\lambda}}}, \quad n,\lambda \in \mathbb{N},\ x \in [0,1] \text{ with } x \neq \frac{k}{n}\ (k = 0,1,...,n). \tag{4.1}$$

In this case, we may write from Theorem 6 of [5] that, for every $x \in [0,1]$,

$$H_n^{\lambda}(\varphi_x;x) \leq \frac{3}{2} w\left(\varphi_x, \frac{1}{n}\right) \quad \text{with } \varphi_x(y) := (y-x)^2. \tag{4.2}$$

Now take $A = C_1 := [c_{jn}]$, the Cesáro matrix and defined the sequences (α_n), (p_n) and (q_n) by

$$\alpha_n = \begin{cases} n^2, & \text{if } n = m^2, \\ 0, & \text{if } n \neq m^2. \end{cases} \tag{4.3}$$

and

$$p_n = \frac{1}{\sqrt[3]{n}}, \quad q_n = 1 + \frac{1}{\sqrt[4]{n}}. \tag{4.4}$$

Then, using (4.1) and (4.3) define the max-product operators

$$L_n(f;x) := (1 + \alpha_n) H_n^{\lambda}(f;x), \tag{4.5}$$

where $n, \lambda \in \mathbb{N}$, $x \in [0,1]$ and $f : [0,1] \to [0,\infty)$ is any continuous function on $[0,1]$. Hence, we get from (4.2) that

$$\delta_n := \sqrt{\bigvee \{L_n(\varphi_x; x) : x \in [0,1]\}} \leq \sqrt{\frac{3(1+\alpha_n)}{2}} \sqrt{\bigvee \left\{ w\left(\varphi_x, \frac{1}{n}\right) : x \in [0,1] \right\}}. \tag{4.6}$$

By (4.3) and (4.4), observe that, for every $\varepsilon > 0$,

$$\frac{1}{p_j} \sum_{n:\ |\alpha_n| \geq \varepsilon} c_{jn} = \sqrt[3]{j} \sum_{n:\ |\alpha_n| \geq \varepsilon} \frac{1}{j} \leq \frac{\sqrt[3]{j}\sqrt{j}}{j} = \frac{1}{\sqrt[6]{j}} \leq 1$$

holds for every $j \in \mathbb{N}$. Then, we have

$$\alpha_n = st_{C_1} - O\left(\frac{1}{\sqrt[3]{n}}\right) \quad \text{as} \quad n \to \infty.$$

Combining this with (4.6) we get

$$\delta_n = st_{C_1} - O\left(\frac{1}{\sqrt[3]{n}}\right) \quad \text{as} \quad n \to \infty \tag{4.7}$$

because of the fact that

$$\lim_{n \to \infty} \sqrt{\bigvee \left\{ w\left(\varphi_x, \frac{1}{n}\right) : x \in [0,1] \right\}} = 0.$$

Now using the right continuity at zero of the modulus of continuity, it follows from (4.7) that

$$w(f, \delta_n) = st_{C_1} - O\left(\frac{1}{\sqrt[3]{n}}\right) \quad \text{as} \quad n \to \infty \quad \text{for } f \in C([0,1],[0,\infty)).$$

Therefore, all conditions of our Theorem 4.1 hold. So, for the operators L_n given by (4.5), we get, for all $f \in C([0,1],[0,\infty))$, that

$$\bigvee \{|L_n(f;x) - f(x)| : x \in [0,1]\} = st_A - O\left(q_n\right) \quad \text{as } n \to \infty,$$

since $q_n \geq p_n$ for every $n \in \mathbb{N}$.

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