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# Slant lightlike submanifolds of indefinite Kenmotsu manifolds

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**Abstract**

In this paper, we introduce the notion of a slant lightlike submanifold of an indefinite Kenmotsu manifold. We provide a non-trivial example and obtain necessary and sufficient conditions for the existence of a slant lightlike submanifold. Also, we give an example of a minimal slant lightlike submanifold of $R_2^9$ and prove some characterization theorems.

**Key Words:** Degenerate metric, Slant lightlike submanifolds, Kenmotsu manifold.

## 1. Introduction

In the theory of submanifolds of semi-Riemannian manifolds it is interesting to study the geometry of lightlike submanifolds due to the fact that the intersection of normal vector bundle and the tangent bundle is non-trivial. Thus, the study becomes more interesting and remarkably different from the study of non-degenerate submanifolds. The geometry of lightlike submanifolds of indefinite Kaehler manifolds was presented in a book by Duggal and Bejancu [5]. B. Y. Chen has introduced the notion of slant immersions by generalizing the concept of holomorphic and totally real immersions [3, 4]. Later, it was A. Lotta [9] who introduced the concept of slant immersion of a Riemannian manifold into an almost contact metric manifold. To define the notion of slant submanifolds, one needs to consider the angle between two vector fields. A lightlike submanifold has two (radical and screen) distributions. The radical distribution is totally lightlike and therefore it is not possible to define angle between two vector fields of radical distribution. On the other hand, the screen distribution is non-degenerate. Using these facts the notion of slant lightlike submanifold of an indefinite Hermitian manifold was introduced by B. Sahin [10].

The purpose of the present paper is to introduce the notion of slant lightlike submanifold of an indefinite Kenmotsu manifold.

In Section 2, we have collected the formulae and information which are useful in our subsequent sections. In Section 3, we introduce the concept of slant lightlike submanifold of an indefinite Kenmotsu manifold and provide a non-trivial example. We prove a characterization theorem for the existence of slant lightlike submanifolds and show that co-isotropic $CR$-lightlike submanifolds are slant lightlike submanifolds. Finally, in Section 4, we consider minimal slant lightlike submanifolds and give an example and prove two characterization theorems.

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## 2. Preliminaries

An odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $\{\phi, V, \eta, \overline{g}\}$, where $\phi$ is a (1,1) tensor field, $V$ a vector field, $\eta$ a 1-form and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying

$$\begin{cases} \phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \qquad \phi V = 0, \quad \eta(V) = 1 \\ \overline{g}(\phi X, \phi Y) = \overline{g}(X, Y) - \eta(X)\eta(Y), \quad \overline{g}(X, V) = \eta(X) \end{cases} \tag{2.1}$$

for $X, Y \in T\overline{M}$, where $T\overline{M}$ denotes the Lie algebra of vector fields on $\overline{M}$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite Kenmotsu manifold if [1, 8],

$$(\overline{\nabla}_X \phi)Y = -\overline{g}(\phi X, Y)V + \eta(Y)\phi X, \qquad \text{and} \quad \overline{\nabla}_X V = -X + \eta(X)V. \tag{2.2}$$

for any $X, Y \in T\overline{M}$, where $\overline{\nabla}$ denote the Levi-Civita connection on $\overline{M}$.

A submanifold $M^m$ immersed in a semi-Riemannian manifold $\{\overline{M}^{m+n}, \overline{g}\}$ is called a lightlike submanifold if it admits a degenerate metric $g$ induced from $\overline{g}$ whose radical distribution of Rad(TM) is of rank $r$, where $1 \leq r \leq m$. Now, $Rad(TM) = TM \bigcap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x \in M} \{u \in T_x\overline{M} : \overline{g}(u, v) = 0, \forall v \in T_x M\} \tag{2.3}$$

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $Rad(TM)$ in $TM$, that is, $TM = Rad(TM) \perp S(TM)$.

We consider a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. For any local basis $\{\xi_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\overline{g}(\xi_i, N_j) = \delta_{ij}$ and $\overline{g}(N_i, N_j) = 0$, and therefore, it follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}(cf.[5], page\ 144)$. Let $\mathrm{tr}(TM)$ is complementary (but not orthogonal) vector bundle to $TM$ in $T\overline{M}|_M$. Then

$$\begin{cases} tr(TM) = ltr(TM) \perp S(TM^{\perp}) \\ T\overline{M}|_M = S(TM) \perp [Rad(TM) \bigoplus ltr(TM)] \perp S(TM^{\perp}). \end{cases} \tag{2.4}$$

A submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\overline{M}$ is said to be

(i) r-lightlike if $r < min\{m, n\}$;

(ii) Coisotropic if $r = n < m, S(TM^{\perp}) = \{0\}$;

(iii) Isotropic if $r = m < n, S(TM) = \{0\}$;

(iv) Totally lightlike if $r = m = n, S(TM) = \{0\} = S(TM^{\perp})$.

Let $\overline{\nabla}$ , $\nabla$ and $\nabla^t$ denote the linear connections on $\overline{M}$, $M$ and vector bundle $\mathrm{tr}(TM)$, respectively. Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{2.5}$$

$$\overline{\nabla}_X U = -A_U X + \nabla^t_X U, \forall\, U \in \Gamma(tr(TM)), \tag{2.6}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X,Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively and $A_U$ is the shape operator of $M$ with respect to $U$. Moreover, according to the decomposition (2.4), $h^l$, $h^s$ are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued *lightlike second fundamental form* and *screen second fundamental form* of $M$, respectively. Then

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \forall X, Y \in \Gamma(TM), \tag{2.7}$$

$$\overline{\nabla}_X N = -A_N X + \nabla^l_X(N) + D^s(X,N), N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla^s_X(W) + D^l(X,W), W \in \Gamma(S(TM^{\perp})), \tag{2.9}$$

where $D^l(X,W)$, $D^s(X,N)$ are the projections of $\nabla^t$ on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively and $\nabla^l$, $\nabla^s$ are linear connections on $\Gamma(ltr(TM))$ and $\Gamma(S(TM^{\perp}))$, respectively. We call $\nabla^l$, $\nabla^s$ the lightlike and screen transversal connections on $M$, and $A_N$, $A_W$ are shape operators on $M$ with respect to $N$ and $W$, respectively. Using (2.5) and (2.7)$\sim$(2.9), we obtain

$$\overline{g}(h^s(X,Y), W) + \overline{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.10}$$

$$\overline{g}(D^s(X,N), W) = \overline{g}(N, A_W X). \tag{2.11}$$

Let $\overline{P}$ denote the projection of $TM$ on $S(TM)$ and let $\nabla^*$, $\nabla^{*t}$ denote the linear connections on $S(TM)$ and *Rad* $(TM)$, respectively. Then from the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \overline{P} Y = \nabla^*_X \overline{P} Y + h^*(X, \overline{P} Y), \tag{2.12}$$

$$\nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.13}$$

for $X$, $Y \in \Gamma(TM)$ and $\xi \in \Gamma(\mathit{Rad}\, TM)$, where $h^*$, $A^*$ are the second fundamental form and shape operator of distributions $S(TM)$ and *Rad* $(TM)$, respectively.

From (2.12) and (2.13), we get

$$\overline{g}(h^l(X, \overline{P} Y), \xi) = g(A^*_{\xi} X, \overline{P} Y), \tag{2.14}$$

$$\overline{g}(h^*(X, \overline{P} Y, N) = g(A_N X, \overline{P} Y), \tag{2.15}$$

$$\overline{g}(h^l(X,\xi), \xi) = 0, \qquad A^*_{\xi} \xi = 0. \tag{2.16}$$

In general, the induced connection $\nabla$ on $M$ is not a metric connection. Since $\overline{\nabla}$ is a metric connection, from (2.7), we obtain

$$(\nabla_X g)(Y,Z) = \overline{g}(h^l(X,Y), Z) + \overline{g}(h^l(X,Z), Y). \tag{2.17}$$

However, it is important to note that $\nabla^*$, $\nabla^{*t}$ are metric connections on $S(TM)$ and *Rad* $(TM)$, respectively.

A general notion of a minimal lightlike submanifold in a semi-Riemannian manifold, as introduced by Bejan and Duggal [2], is as follows:

**Definition 2.1** A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is minimal if

(i) $h^s = 0$ on $Rad\ (TM)$;

(ii) trace $h = 0$, where trace is written with respect to $g$ restricted to $S(TM)$.

Similar to definition of contact CR-lightlike submanifolds, invariant submanifolds, screen real submanifolds of Sasakian manifolds given by Duggal and Sahin [6], we state the following definitions [7]:

**Definition 2.2** Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to structure vector field $V$ and immersed in an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$. We say that $M$ is a contact CR-lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:

(a) Rad $TM$ is a distribution on $M$ such that Rad $TM \bigcap \phi(Rad\ TM) = \{0\}$;

(b) there exist vector bundles $D_0$ and $D'$ over $M$ such that

$$\begin{cases} S(TM) = \{\phi(Rad\ TM) \bigoplus D'\} \perp D_0 \perp \{V\}, \\ \phi D_0 = D_0, \phi D' = L_1 \perp ltr(TM) \end{cases} \tag{2.18}$$

where $D_0$ is nondegenerate and $L_1$ is vector subbundle of $S(TM^{\perp})$.

A contact CR-lightlike submanifold is proper if $D_0 \neq \{0\}$ and $L_1 \neq \{0\}$.

**Definition 2.3** A lightlike submanifold $M$, of an indefinite Kenmotsu manifold $\overline{M}$, is screen real submanifold if Rad $(TM)$ and $S(TM)$ are, respectively, invariant and anti-invariant with respect to $\phi$.

The following result is important for our subsequent use.

**Proposition 2.1** [5] *The lightlike second fundamental forms of a lightlike submanifold $M$ do not depend on $S(TM)$, $S(TM^{\perp})$ and $ltr(TM)$.*

## 3. Slant lightlike submanifolds

We prove the following lemma.

**Lemma 3.1** *Let $M$ be an r-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index 2q with structure vector field tangent to $M$. Suppose that $\phi Rad\, TM$ is a distribution on $M$ such that $Rad\, TM \bigcap \phi Rad\, TM = \{0\}$. Then $\phi ltr(TM)$ is a subbundle of the screen distribution $S(TM)$ and $\phi ltr(TM) \bigcap \phi Rad\ TM = \{0\}$.*

**Proof.** Given that $\phi Rad\ TM$ is a distribution on $M$ such that $Rad\ TM \bigcap \phi Rad\ TM = \{0\}$, and hence $\phi Rad\ TM \in S(TM)$. We claim that $ltr(TM)$ is not invariant with respect to $\phi$.

Suppose that $ltr(TM)$ is invariant with respect to $\phi$. Choose $\xi \in Rad\ TM$ and $N \in Rad\ TM$ such that $\overline{g}(N, \xi) = 1$. Then from (2.1), we have

$$0 = \overline{g}(\phi N, \phi \xi) = \overline{g}(N, \xi) - \eta(N)\eta(\xi) = \overline{g}(N, \xi) = 1$$

as $\phi\xi \in S(TM)$ and $\phi N \in ltr(TM)$, and so $ltr(TM)$ is not invariant with respect to $\phi$.

Also, $\phi N$ does not belong to $S(TM^{\perp})$, and since $S(TM^{\perp})$ is orthogonal to $S(TM)$, it implies that $\overline{g}(\phi N, \phi\xi)$ must be zero. But from (2.1), we have that

$$\overline{g}(\phi N, \phi\xi) = \overline{g}(N, \xi) - \eta(N)\eta(\xi) = g(N, \xi) = 1 \neq 0$$

for some $\xi \in \Gamma Rad\, TM$, which is again a contradiction and hence that $\phi ltr(TM)$ is a distribution on $M$.

Moreover, $\phi N$ does not belong to $Rad\ TM$. Indeed, if $\phi N \in \Gamma Rad\ TM$, we would have $\phi^2 N = -N + \eta(N)V = -N \in \Gamma\phi Rad\, TM$, which is not possible. Similarly, $\phi N$ does not belong to $\phi Rad\, TM$. Thus, we conclude that $\phi ltr(TM) \subset S(TM)$ and $\phi ltr(TM) \bigcap \phi Rad\, TM = \{0\}$. □

Next, we prove this lemma:

**Lemma 3.2** *Let $M$ be q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index 2q with structure vector field tangent to $M$. Suppose that $\phi Rad\ TM$ is a distribution on $M$ such that $Rad\ TM \bigcap \phi Rad\ TM = \{0\}$. Then any complementary distribution to $\phi ltr(TM) \bigoplus Rad\ TM$ in screen distribution $S(TM)$ is Riemannian.*

**Proof.** Let $D'$ be the complementary distribution to $\phi ltr(TM) \bigoplus \phi Rad\, TM$ in $S(TM)$ and $\dim(\overline{M}) = m + n$ and $\dim\,(M) = m$. We can choose a local quasi-orthonormal frame on $\overline{M}$ along $M$ as

$\{\xi_i, N_i, \phi\xi_i, \phi N_i, X_\alpha, V, W_a\}$, $i \in \{1, ..., q\}$, $\alpha \in \{3q+1, ..., m-1\}$, $a \in \{q+1, ..., n\}$, where $\{\xi_i\}$ and $\{N_i\}$ are lightlike bases of $Rad\, TM$ and $ltr(TM)$, respectively, and $\{\phi\xi_i, \phi N_i, X_\alpha, V\}$, is an orthonormal basis of $S(TM)$ and $\{W_a\}$ is an orthonormal basis of $S(TM^{\perp})$.

Now, we can construct the orthonormal basis $\{U_1, U_2, .., U_{2q}, V_1, V_2, ..., V_{2q}\}$ as

$$U_1 = \tfrac{1}{\sqrt{2}}\{\xi_1 + N_1\}, \qquad U_2 = \tfrac{1}{\sqrt{2}}\{\xi_1 - N_1\},$$
$$U_3 = \tfrac{1}{\sqrt{2}}\{\xi_2 + N_2\}, \qquad U_4 = \tfrac{1}{\sqrt{2}}\{\xi_2 - N_2\},$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$
$$U_{2q-1} = \tfrac{1}{\sqrt{2}}\{\xi_q + N_q\}, \qquad U_{2q} = \tfrac{1}{\sqrt{2}}\{\xi_q - N_q\},$$
$$V_1 = \tfrac{1}{\sqrt{2}}\{\phi\xi_1 + \phi N_1\}, \qquad V_2 = \tfrac{1}{\sqrt{2}}\{\phi\xi_1 - \phi N_1\},$$
$$V_3 = \tfrac{1}{\sqrt{2}}\{\phi\xi_2 + \phi N_2\}, \qquad V_2 = \tfrac{1}{\sqrt{2}}\{\phi\xi_2 - \phi N_2\},$$
$$\cdots \qquad \cdots$$
$$\cdots \qquad \cdots$$
$$V_{2q-1} = \tfrac{1}{\sqrt{2}}\{\phi\xi_q + \phi N_q\}, \qquad V_{2q} = \tfrac{1}{\sqrt{2}}\{\phi\xi_q - \phi N_q\}.$$

Hence, $\{\xi_i, N_i, \phi\xi_i, \phi N_i\}$ gives a non-degenerate space of constant index $2q$ which imply that $Rad\, TM \bigoplus ltr(TM) \bigoplus \phi Rad\, TM \bigoplus \phi ltr(TM)$ is non degenerate and of constant index $2q$ on $\overline{M}$. As $index(T\overline{M}) = index(Rad\, TM$

$\bigoplus ltr(TM)) + index(\phi Rad\, TM \bigoplus \phi ltr(TM)) + index(D' \perp S(TM^{\perp}))$, we have $2q = 2q + index(D' \perp S(TM^{\perp}))$, which implies that $index(D' \perp S(TM^{\perp})) = 0$. Hence $D'$ is Riemannian. □

As mentioned in the introduction, the purpose of this paper is to define slant lightlike submanifolds of indefinite Kenmotsu manifolds. To define this notion, one needs to consider angle between two vector fields. As we can see from Section 2, a lightlike submanifold has two distributions viz. radical and screen.

The radical distribution is totally lightlike and, therefore, it is not possible to define angle between two vector fields of radical distribution. The screen distribution is non-degenerate. Thus one way to define slant lightlike submanifolds is to choose a Riemannian screen distribution on lightlike submanifolds, for which we use Lemma 3.2.

**Definition 3.1** *Let $M$ be a q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index 2q with structure vector field tangent to $M$. Then we say that $M$ is a slant lightlike submanifold of $\overline{M}$ if the following conditions are satisfied:*

*(i) $Rad\, TM$ is a distribution on $M$ such that $Rad\, TM \bigcap \phi Rad\, TM = \{0\}$.*

*(ii) For all $x \in U \subset M$ and for each non zero vector field $X$ tangent to $\overline{D} = D \perp \{V\}$, if $X$ and $V$ are linearly independent, then the angle $\theta(X)$ between $\phi X$ and the vector space $\overline{D}_x$ is constant, where $D$ is complementary distribution to $\phi ltr(TM) \bigoplus \phi Rad\, TM$ in screen distribution $S(TM)$.*

The constant angle $\theta(X)$ is called the slant angle of $\overline{D}$. A slant lightlike submanifold $M$ is said to be proper if $D \neq \{0\}$, and $\theta \neq 0, \frac{\pi}{2}$.

The following result is an easy consequence of Definition 3.1.

**Proposition 3.1** *There exists no proper slant totally lightlike or isotropic submanifold $M$ in indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to $M$.*

In what follows, $(R_q^{2m+1}, \phi_0, V, \eta, g)$ will denote the manifold $R_q^{2m+1}$ with its usual Kenmotsu structure given by

$$\begin{cases} \eta = dz, \qquad V = \partial z, \\ \overline{g} = \eta \bigotimes \eta + e^{2z}(-\sum_{i=1}^{q/2} dx^i \bigotimes dx^i + dy^i \bigotimes dy^i + \sum_{i=q+1}^{m} dx^i \bigotimes dx^i + dy^i \bigotimes dy^i), \\ \phi_0(\sum_{i=1}^{m}(X_i \partial x^i + Y_i \partial y^i) + Z \partial z) = \sum_{i=1}^{m-1}(-X_{i+1}\partial x^i + X_i \partial x^{i+1} - Y_{i+1}\partial y^i + Y_i \partial y^{i+1}) \end{cases}$$

where $(x_i, y_i, z)$ are cartesian coordinates.

**Example 3.1** Let $\overline{M} = (R_2^9, \overline{g})$ be a semi-Euclidean space of signature (-, -, +, +, +, +, +, +, +) with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z\}$.

Consider a submanifold $M$ of $R_2^9$, defined by

$$X(u, v, \theta_1, \theta_2, s, t) = (u, v, \sin\theta_1, \cos\theta_1, -\theta_1 \sin\theta_2, -\theta_1 \cos\theta_2, u, s, t)$$

Then a local frame of $TM$ is given by

$$\begin{cases} Z_1 = e^{-z}(\partial x_1 + \partial y_3), \qquad Z_2 = e^{-z}\partial x_2, \\ Z_3 = e^{-z}(\cos\theta_1 \partial x_3 - \sin\theta_1 \partial x_4 - \sin\theta_2 \partial y_1 - \cos\theta_2 \partial y_2), \\ Z_4 = e^{-z}(-\theta_1 \cos\theta_2 \partial y_1 + \theta_1 \sin\theta_2 \partial y_2), \qquad Z_5 = e^{-z}\partial y_4, \\ Z_6 = V = \partial z. \end{cases}$$

Hence, $Rad\, TM = span\{Z_1\}$, $\phi_0 Rad\, TM = span\{Z_2 + Z_5\}$, and $Rad\, TM \bigcap \phi_0 Rad\, TM = \{0\}$. Next, $\overline{D} = D \perp \{V\} = \{Z_3, Z_4\} \perp \{V\}$ is Riemannian.

Then $M$ is slant lightlike with slant angle $\frac{\pi}{4}$. By direct calculations, we get

$$S(TM^{\perp}) = span \begin{cases} W_1 = e^{-z}(\cos\theta_1 \partial x_3 - \sin\theta_1 \partial x_4 + \sin\theta_2 \partial y_1 + \cos\theta_2 \partial y_2), \\ W_2 = e^{-z}(\sin\theta_1 \partial x_3 + \cos\theta_1 \partial x_4) \end{cases}$$

and $ltr(TM)$ is spanned by $N = \frac{e^{-z}}{2}(-\partial x_1 + \partial y_3)$, such that $\phi_0(N) = -Z_2 + Z_5 \in S(TM)$. It is easy to see that conditions (i) and (ii) of Definition 3.1 hold. Hence, $M$ is a proper slant lightlike submanifold of $R^9_2$.

**Proposition 3.3** *Slant lightlike submanifolds $M$ of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to $M$ do not include invariant and screen real lightlike submanifolds.*

**Proof.** Let $M$ be an invariant or screen real lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Since $\phi Rad\, TM = Rad\, TM$, the first condition of slant lightlike submanifold is not satisfied which proves our assertion. □

The following result gives a relation between slant lightlike and contact $CR$- lightlike submanifolds of an indefinite Kenmotsu manifold:

**Proposition 3.2** *Let $M$ be a q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index 2q with structure vector field tangent to $M$. Then any coisotropic CR-lightlike submanifold is a slant lightlike submanifold with $\theta = 0$. In particular, a lightlike real hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ of index 2 is a slant lightlike submanifold with $\theta = 0$. Moreover, any CR-lightlike submanifold of $\overline{M}$ with $D_0 = \{0\}$, is a slant lightlike submanifold with $\theta = \frac{\pi}{2}$.*

**Proof.** Let $M$ be a $q$-lightlike $CR$-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$. Then, $\phi Rad\ TM$ is a distribution on $M$ such that $Rad\ TM \bigcap \phi Rad\ TM = \{0\}$. If $M$ is coisotropic, then $S(TM^{\perp}) = \{0\}$. Then the complementary distribution to $\phi ltr(TM) \bigoplus \phi Rad\ TM$ in screen distribution $S(TM)$ is $\overline{D} = D_0 \perp \{V\}$ where $D_0$ is Riemannian by Lemma 3.2. Since $D_0$ is invariant with respect to $\phi$, it follows that $\theta = 0$. Our second assertion is obvious as a lightlike real hypersurface of $\overline{M}$ is coisotropic.

Now, if $M$ is CR-lightlike submanifold with $D_0 = \{0\}$, then the complementary distribution to $\phi ltr(TM) \bigoplus \phi Rad\, TM$ in screen distribution $S(TM)$ is $\overline{D} = D' \perp \{V\}$. Since $D'$ is anti-invariant with respect to $\phi$, it follows that $\theta = \frac{\pi}{2}$, whereby completing the proof. □

We know that for any $X \in \Gamma(TM)$ and $W \in \Gamma tr(TM)$, we have

$$\phi X = TX + FX, \qquad \phi W = BW + CW \tag{3.1}$$

where $TX$ and $FX$ are the tangential and transversal components of $\phi X$, respectively, and $BW$ and $CW$ are the tangential and transversal components of $\phi W$, respectively. Moreover, for a slant lightlike submanifold, we denote by $P_1, P_2, Q_1, Q_2$ and $\overline{Q}_2$ the projections on the distributions $Rad\ TM, \phi Rad\ TM, \phi ltr(TM), D$ and $\overline{D} = D \perp \{V\}$, respectively. Then for any $X \in \Gamma(TM)$, we can write

$$X = P_1 X + P_2 X + Q_1 X + \overline{Q}_2 X \tag{3.2}$$

where $\overline{Q}_2 X = Q_2 X + \eta(X)V.$

Using (3.1) in the above equation, we obtain

$$\phi X = \phi P_1 X + \phi P_2 X + TQ_2 X + FQ_1 X + FQ_2 X \tag{3.3}$$

for any $X \in \Gamma(TM)$. Then the tangential components are

$$TX = TP_1 X + TP_2 X + TQ_2 X. \tag{3.4}$$

We now prove two characterization theorems for slant lightlike submanifolds.

**Theorem 3.1** *Let $M$ be a q-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index 2q with structure vector field tangent to $M$. Then $M$ is slant lightlike submanifold if and only if the following conditions are satisfied:*

*(a) $\phi ltr(TM)$ is a distribution on $M$*

*(b) There exists a constant $\lambda \in [-1, 0]$ such that*

$$T^2 \overline{Q}_2 X = \lambda(\overline{Q}_2 X - \eta(\overline{Q}_2 X)V) \tag{3.5}$$

*$\forall X \in \Gamma(TM)$ linearly independent of structure vector field $V$. Moreover, in such a case, $\lambda = -cos^2\theta$, where $\theta$ is the slant angle of $M$.*

**Proof.** Let $M$ be a $q$-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$. If $M$ is a slant lightlike submanifold of $\overline{M}$, then $\phi Rad\ TM$ is a distribution on $S(TM)$, and hence from Lemma 3.1, it follows that $\phi ltr(TM)$ is also a distribution on $M$ and $\phi ltr(TM) \subset S(TM)$. Thus (a) is proved. □

For $X \in \Gamma(TM), Q_2 X \in \overline{D} - \{V\}$, we have

$$\cos\theta(Q_2 X) = \frac{\overline{g}(\phi Q_2 X, TQ_2 X)}{|\phi Q_2 X||TQ_2 X|} = -\frac{\overline{g}(Q_2 X, \phi TQ_2 X)}{|\phi Q_2 X||TQ_2 X|} = -\frac{\overline{g}(Q_2 X, T^2 Q_2 X)}{|Q_2 X||TQ_2 X|}. \tag{3.6}$$

On the other hand, we get

$$\cos\theta(Q_2 X) = \frac{|TQ_2 X|}{|\phi Q_2 X|}. \tag{3.7}$$

Thus, from (3.6) and (3.7), we find

$$\cos^2\theta(Q_2X) = -\frac{\overline{g}(Q_2X, T^2Q_2X)}{|Q_2X|^2}.$$

Since $\theta(Q_2X)$ is constant on $\overline{D}$, we conclude that

$$T^2(Q_2X) = \lambda Q_2X = \lambda(\overline{Q}_2X - \eta(\overline{Q}_2X)V), \lambda \in (-1, 0). \tag{3.8}$$

Moreover, in this case, $\lambda = -\cos^2\theta$. It is clear that equation (3.8) is valid for $\theta = 0$ and $\theta = \frac{\pi}{2}$. Hence, for $\overline{Q}_2X \in \overline{D}$, we find

$$T^2(\overline{Q}_2X) = \lambda(\overline{Q}_2X - \eta(\overline{Q}_2X)V), \lambda \in [-1, 0]. \tag{3.9}$$

Conversely, suppose that (a) and (b) are satisfied. Then (a) implies that $\phi Rad\, TM$ is a distribution on $M$. From Lemma 3.2, it follows that the complementary distribution to $\phi ltr(TM) \bigoplus \phi Rad\, TM$ is a Riemannian distribution. The rest of the proof is obvious.

Using (2.1), (3.1) and Theorem 3.1, we have the following corollary.

**Corollary 3.1** *Let $M$ be a slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to $M$. Then we have*

$$g(T\overline{Q}_2X, T\overline{Q}_2Y) = \cos^2\theta[g(\overline{Q}_2X, \overline{Q}_2Y) - \eta(\overline{Q}_2X)\eta(\overline{Q}_2Y)] \tag{3.10}$$

$$g(F\overline{Q}_2X, F\overline{Q}_2Y) = \sin^2\theta[g(\overline{Q}_2X, \overline{Q}_2Y) - \eta(\overline{Q}_2X)\eta(\overline{Q}_2Y)]. \tag{3.11}$$

*for $X, Y \in \Gamma(TM)$.*

**Theorem 3.2** *Let $M$ be a $q$-lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ of index $2q$ with structure vector field tangent to $M$. Then $M$ is slant lightlike submanifold if and only if the following conditions are satisfied:*

*(A) $\phi ltr(TM)$ is a distribution on $M$*

*(B) There exists a constant $\mu \in [-1, 0]$ such that*

$$BF\overline{Q}_2X = \mu(\overline{Q}_2X - \eta(\overline{Q}_2X)V), \forall X \in \Gamma(TM).$$

*Moreover, in such a case, $\mu = -sin^2\theta$ where $\theta$ is the slant angle of $M$.*

**Proof.** It is easy to see that $\phi Rad\, TM \bigcap \phi ltrTM = \{0\}$ and $\phi Rad\, TM$ is a subbundle of $S(TM)$. Moreover, the complementary distribution to $\phi ltr(TM) \bigoplus \phi Rad\, TM$ in $S(TM)$ is Riemannian. Furthermore, from the proof of Lemma 3.2, $S(TM^{\perp})$ is also Riemannian. Thus condition (i) in the Definition 3.1 of slant lightlike submanifold is satisfied. On the other hand, from (3.1) and (3.3), we obtain

$$-X = -P_1X - P_2X + T^2Q_2X + FTQ_2X + \phi FQ_1X + BFQ_2X + CFQ_2X.$$

Since $\phi FQ_1X = -Q_1X \in S(TM)$, taking the tangential parts, we have

$$-X + \eta(X)V = -P_1X - P_2X + T^2Q_2X - Q_1X + BFQ_2X.$$

From (3.2), we find

$$-Q_2X = -T^2Q_2X + BFQ_2X. \tag{3.12}$$

Now, if $M$ is slant lightlike then from Theorem 3.1, we have $T^2Q_2X = -\cos^2\theta Q_2X$, and hence we get $BFQ_2X = -\sin^2\theta Q_2X$. Since $FV = 0$ and $\overline{Q}_2X = Q_2X + \eta(X)V$, we have
$BF\overline{Q}_2X = -\sin^2\theta(\overline{Q}_2X - \eta(\overline{Q}_2X)V)$.

Conversely, suppose that $BFQ_2X = \mu Q_2X$. Then, from (3.12), we obtain

$$T^2Q_2X = -(1+\mu)Q_2X.$$

Thus, the proof follows from Theorem 3.1. □

## 4. Minimal slant lightlike submanifolds

In this section we study minimal slant lightlike submanifolds of indefinite Kenmotsu manifolds. We have the following.

**Example 4.1** Let $\overline{M} = (R^9_2, \overline{g})$ be a semi-Euclidean space of signature (-, -, +, +, +, +, +, +, +) with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial z)$.

Consider a submanifold $M$ of $R^9_2$ defind by

$$\begin{array}{rr} x_1 = u_1\cosh\theta, & x_2 = u_2\cosh\theta, \\ x_3 = -u_3 + u_1\sinh\theta, & x_4 = u_1 + u_3\sinh\theta, \\ y_1 = \cos u_4\cosh u_5, & y_2 = \cos u_4\sinh u_5, \\ y_3 = \sin u_4\sinh u_5, & y_4 = \sin u_4\cosh u_5, \\ & z = t, \end{array}$$

where $u_1 \in (0, \frac{\pi}{2})$.

Then a local frame of $TM$ is given by

$$\begin{cases} Z_1 = e^{-z}(\cosh\theta\partial x_1 + \sinh\theta\partial x_3 + \partial x_4), Z_2 = e^{-z}\cosh\theta\partial x_2, Z_3 = e^{-z}(-\partial x_3 + \sinh\theta\partial x_4), \\ Z_4 = e^{-z}(-\sin u_4\cosh u_5\partial y_1 - \sin u_4\sinh u_5\partial y_2 + \cos u_4\sinh u_5\partial y_3 + \cos u_4\cosh u_5\partial y_4), \\ Z_5 = e^{-z}(\cos u_4\sinh u_5\partial y_1 + \cos u_4\cosh u_5\partial y_2 + \sin u_4\cosh u_5\partial y_3 + \sin u_4\sinh u_5\partial y_4), \\ Z_6 = V = \partial z \end{cases}$$

We define an almost-contact structure $\phi_1$ as

$$\phi_1(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) = (-x_2, x_1, -x_4, x_3, -y_3\cos\alpha - y_2\sin\alpha, -y_4$$
$$\cos\alpha + y_1\sin\alpha, y_1\cos\alpha + y_4\sin\alpha, y_2\cos\alpha - y_3\sin\alpha, 0),$$

where $\alpha \in (0, \frac{\pi}{2})$. Hence, $Rad\ TM = span\{Z_1\}, \phi_1 Rad\ TM = span\{Z_2 + Z_3\}$ and $Rad\ TM \bigcap \phi_1 Rad\ TM = \{0\}$.

Next, $\overline{D} = D\perp\{V\} = \{Z_4 + Z_5\}\perp\{V\}$ is Riemannian. Then $M$ is slant lightlike with slant angle $\alpha$ with respect to $\phi_1$. By direct calculation, we get $S(TM^{\perp}) = span\{W_1 = e^{-z}(-\cosh u_5\partial y_1 + \sinh u_5\partial y_2 + \tan u_4 \sinh u_5\partial y_3 - \tan u_4 \cosh u_5\partial y_4), W_2 = e^{-z}(-\tan u_4 \sinh u_5\partial y_1 + \tan u_4 \cosh u_5\partial y_2 - \cosh u_5\partial y_3 + \sinh u_5\partial y_4)\}$ and $ltr(TM)$ is spanned by $N = e^{-z}(\tanh\theta \sinh\theta\partial x_1 + \sinh\theta\partial x_3 + \partial x_4)$ such that $\phi_1 N = \tanh^2\theta Z_2 + Z_3 \in S(TM)$. It is easy to see that condition, (i) and (ii) of Definition 3.1 hold. By direct calculation, and using the Gauss formula, we get

$$\begin{cases} h^s(X, Z_1) = h^s(X, \phi_1 Z_1) = 0 = h^s(X, \phi_1 N), h^l = 0, \forall X \in \Gamma(TM) \\ h^s(Z_4, Z_4) = \frac{e^{-z}\cos u_4}{(\cosh^2 u_5 + \sinh^2 u_5)} W_1, h^s(e_2, e_2) = -\frac{e^{-z}\cos u_4}{(\cosh^2 u_5 + \sinh^2 u_5)} W_1. \end{cases}$$

Thus $M$ is a minimal slant lightlike submanifold of $(R^9_2, \phi_1)$.

In what follows, we prove two characterization results for minimal slant lightlike submanifolds.

We have the following lemma:

**Lemma 4.1** *Let $M$ be a proper slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ such that $dim(D) = dim(S(TM^{\perp}))$. If $\{e_1, \ldots, e_m\}$ is a local orthonormal basis of $\Gamma(D)$, then $\{\csc\theta Fe_1, ..., \csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^{\perp})$.*

**Proof.** Since $\{e_1, ..., e_m\}$ is a local orthonormal basis of $D$ and $D$ is Riemannian, from Corollary 3.1, we find

$$\overline{g}\{\csc\theta Fe_i, \csc\theta Fe_j\} = \delta_{ij},$$

where $i, j = 1, 2, ..., m$. This proves the result. □

**Theorem 4.1** *Let $M$ be a proper slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to $M$. Then $M$ is minimal if and only if*

$$trace\ A_{W_j|S(TM)} = 0,\ trace\ A^*_{\xi_k|S(TM)} = 0,\ and\ \ \overline{g}(D^l(X, W), Y) = 0,$$

*for $X, Y \in \Gamma(Rad\ TM)$ and $W \in \Gamma(S(TM^{\perp}))$, where $\{\xi_k\}_{k=1}^r$ is a basis of $Rad\ (TM)$ and $\{W_j\}_{j=1}^r$ is a basis of $S(TM^{\perp})$.*

**Proof.** Since $\overline{\nabla}_V V = 0$, from (2.7), we get $h^l(V, V) = h^s(V, V) = 0$. Now, take an orthonormal frame $\{e_1, ..., e_m\}$ of $D$.

We know that $h^l = 0$ on $Rad\ (TM)(cf.[2])$, Proposition 3.1). Thus $M$ is minimal if and only if

$$\textstyle\sum_{k=1}^r h(\phi\xi_k, \phi\xi_k) + \sum_{k=1}^r h(\phi N_k, \phi N_k) + \sum_{i=1}^m h(e_i, e_i) = 0.$$

Using (2.10) and (2.14), we obtain

$$\sum_{k=1}^r h(\phi\xi_k, \phi\xi_k) = \sum_{k=1}^r \frac{1}{r} \sum_{a=1}^r g(A^*_{\xi_a}\phi\xi_k, \phi\xi_k)N_a + \frac{1}{m}\sum_{j=1}^m g(A_{W_j}\phi\xi_k, \phi\xi_k)W_j. \tag{4.1}$$

Similarly, we have

$$h(\phi N_k, \phi N_k) = \sum_{k=1}^{r} \frac{1}{r} \sum_{a=1}^{r} g(A^*_{\xi_a} N_k, \phi N_k) N_a + \frac{1}{m} \sum_{j=1}^{m} g(A_{W_j} \phi N_k, \phi N_k) W_j. \tag{4.2}$$

and

$$\sum_{i=1}^{m} h(e_i, e_i) = \sum_{i=1}^{m} \frac{1}{r} \sum_{a=1}^{r} g(A^*_{\xi_a} e_i, e_i) N_a + \frac{1}{m} \sum_{j=1}^{m} g(A_{W_j} e_i, e_i) W_j. \tag{4.3}$$

Thus our assertion follows from (4.1)$\sim$(4.3). □

**Theorem 4.2** *Let $M$ be a proper slant lightlike submanifold of an indefinite Kenmotsu manifold $\overline{M}$ with structure vector field tangent to $M$ such that $dim(D) = dim(S(TM^{\perp}))$. Then $M$ is minimal if and only if*

$$trace\ A_{Fe_j|S(TM)} = 0,\ trace\ A^*_{\xi_k|S(TM)} = 0,\ and\ \ \overline{g}(D^l(X, Fe_j), Y) = 0,$$

*for $X, Y \in \Gamma(Rad\ TM)$, where $\{\xi_k\}_{k=1}^{r}$ is a basis of $Rad\ TM$ and $\{e_j\}_{j=1}^{m}$ is a basis of $D$.*

**Proof.** Since $\overline{\nabla}_V V = 0$, from (2.7), we get $h^l(V, V) = h^s(V, V) = 0$. We know that $h^l = 0$ on $Rad\ (TM)(cf.[2]$, Proposition 3.1). Also, from Lemma 4.1, $\{\csc\theta Fe_1, \ldots, \csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^{\perp})$. Thus

$$h^s(X, X) = \sum_{i=1}^{m} \csc\theta g(A_{Fe_i} X, X) Fe_i$$

for $X \in \Gamma(\phi Rad\ TM \bigoplus \phi ltr TM \perp D)$. Thus the proof follows from Theorem 4.1. □

**Remarks**

**(a)** It is known that a proper slant submanifold of a Kenmotsu manifold is odd dimensional, but this is not true in case of our definition of slant lightlike submanifold. For instance, see the two examples given in this paper.

**(b)** We notice that the second fundamental forms and their shape operators of a non-degenerate submanifold are related by means of the metric tensor field. Contrary to this we see from (2.7)–(2.11) that in case of lightlike submanifolds there are interrelations between these geometric objects and those of its screen distributions. Thus, the geometry of lightlike submanifolds depends on the triplet $(S(TM),\ S(TM^{\perp}), ltr\ (TM))$. However, it is important to highlight that, as per Proposition 2.1 of this paper; our results are stable with respect to any change in the above triplet.

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