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Generalized derivations on Lie ideals in prime rings

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Abstract

Let R be a prime ring with characteristic different from two, U a nonzero Lie ideal of R and f be a generalized derivation associated with d. We prove the following results: (i) If $[u, f(u)] \in Z$, for all $u \in U$, then $U \subset Z$. (ii) (f, d) and (g, h) be two generalized derivations of R such that $f(u)v = ug(v)$, for all $u, v \in U$, then $U \subset Z$. (iii) $f([u, v]) = \pm[u, v]$, for all $u, v \in U$, then $U \subset Z$.

Key Words: Derivations, Lie ideals, generalized derivations, centralizing mappings, prime rings.

1. Introduction

Throughout R will represent an assosiative ring with center Z. Recall that a ring R is prime if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U, r \in R$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. Let S be a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$, for all $x \in S$. In [11], Posner showed that if a prime ring has a nontrivial derivation which is centralizing on the entire ring, then the ring must be commutative. In [3], Awtar considered centralizing derivations on Lie and Jordan ideals. For prime rings Awtar showed that a nontrivial derivation which is centralizing on Lie ideal implies that the ideal is contained in the center if the ring is not of characteristic two or three. In [10], Lee and Lee obtained the same result while removing the restriction of characteristic not three.

In the year 1991, Bresar [5], defined the following concept. An additive mapping $f : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$f(xy) = f(x)y + xd(y) \ \textit{for all} \ x, y \in R.$$

One may observe that the concept of generalized derivation includes the concept of derivations, also of left multipliers when $d = 0$. Hence it should be interesting to extend some results concerning these notions to generalized derivations. In [2], Argaç and Albaş extended a well known result of Posner for generalized

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derivations of prime rings. Our first objective in this paper is to prove corresponding results for generalized derivations on Lie ideals.

On the other hand, in [6] Daif and Bell showed that if a semiprime ring R has a derivation d satisfiying the following condition, then I is a central ideal; there exists a nonzero ideal I of R such that either $d([x,y]) = [x,y]$ for all $x,y \in I$ or $d([x,y]) = -[x,y]$ for all $x,y \in I$.

These results are extended for semiprime rings in [1]. Our second objective of this note is to show the same conditions imposed on Lie ideals of a prime ring with generalized derivation.

Throughout the present paper, R will denote a prime ring of characteristic not two and U will denote a nonzero Lie ideal of R. We make some extensive use of the basic commutator identities:

$[x,yz] = y[x,z] + [x,y]z$

$[xy,z] = [x,z]y + x[y,z]$

$[[x,y],z] = [[x,z],y] + [x,[y,z]]$.

We denote a generalized derivation $f : R \to R$ determined by derivation d of R by (f,d). If $d = 0$ then $f(xy) = f(x)y$ for all $x,y \in R$ and there exists $q \in Q_r(R_C)$ (a right Martindale ring of quotients) such that $f(x) = qx$, for all $x \in R$ by [9, Lemma 2]. So, we assume that $d \neq 0$.

2. Preliminaries

We shall require the following lemmas.

Lemma 2.1 [*10, Theorem 5*] *Let R be a prime ring with $charR \neq 2$, d be a nonzero derivation of R and U be a Lie ideal of R. If $[u,d(u)] \in Z$ for all $u \in U$, then $U \subset Z$.*

Lemma 2.2 [*4, Theorem 1*] *Let R be a prime ring with $charR \neq 2$, d be a nonzero derivation of R and U be Lie ideal of R. If $d^2(U) = 0$, then $U \subset Z$.*

Lemma 2.3 [*4, Lemma 6*] *Let R be a prime ring with $charR \neq 2$, d be a nonzero derivation of R and U be Lie ideal of R. If $d(U) \subseteq Z$, then $U \subset Z$.*

Lemma 2.4 [*4, Lemma 1*] *Let R be a prime ring with $charR \neq 2$. If $U \not\subseteq Z$ is a Lie ideal of R, then there exists an ideal M of R such that $[M,R] \subset U$, but $[M,R] \not\subseteq Z$.*

Lemma 2.5 [*8, Lemma 1*] *Let R be a semiprime 2-torsion free ring and U be a Lie ideal of R. Suppose that $[U,U] \subset Z$, then $U \subset Z$.*

Lemma 2.6 [*10, Theorem 2*] *Let R be a prime ring with $charR \neq 2$, d be a nonzero derivation of R, U be a Lie ideal of R and $a \in R$ such that $[a,d(U)] \subset Z$. Then either $a \in Z$ or $U \subset Z$.*

3. Results

Definition 3.1 *[7, Definition] Let R be a ring, d a derivation of R. An additive mapping $f : R \to R$ is said to be right generalized derivation of R associated with d if*

$$f(xy) = f(x)y + xd(y) \text{ for all } x, y \in R$$

and f is said to be left generalized derivation of R associated with d if

$$f(xy) = d(x)y + xf(y) \text{ for all } x, y \in R.$$

f is said to be a generalized derivation of R associated with d if it is both a left and right generalized derivation of R associated with d.

Remark 3.2 *For all $x, y \in R$,*

$$f([x, y]) = f(xy - yx) = f(x)y + xd(y) - d(y)x - yf(x) = [f(x), y] + [x, d(y)].$$

Theorem 3.3 *If $[u, f(u)] \in Z$ for all $u \in U$, then $U \subset Z$.*

Proof. Writing u by $u + v, v \in U$ in the hypothesis, we have

$$[u, f(v)] + [v, f(u)] \in Z, \text{ for all } u, v \in U.$$

Replacing v by $[u, r]$, $r \in R$ in this equation, we get

$$[u, [f(u), r]] + [u, [u, d(r)]] + [[u, r], f(u)] \in Z, \text{ for all } u \in U, r \in R.$$

Using Jacobi identity and the hypothesis in this equation, we obtain

$$[u, [u, d(r)]] \in Z, \text{ for all } u \in U, r \in R.$$

This yields that $[u, I_{d(r)}(u)] \in Z$, for all $u \in U$, where $I_{d(r)} : R \to R$, $I_{d(r)} = [x, d(r)]$ is an inner derivation of R. We have $d(R) \subset Z$ or $U \subset Z$ by Lemma 2.1. If $d(R) \subset Z$, then R is commutative and so, $U \subset Z$. □

Theorem 3.4 *Let (f, d) and (g, h) be two generalized derivations of R. If $f(u)v = ug(v)$ for all $u, v \in U$, then $U \subset Z$.*

Proof. Assume that $U \not\subseteq Z$. Then there exists a nonzero ideal M of R such that $[R, M] \not\subseteq Z$, but $[R, M] \subset U$ by Lemma 2.4. For any $x \in R$ and $m \in M$, $m[x, m] = [mx, m] \in U$. If we take $m[x, m]$ instead of u in the hypothesis, we have

$$f(m[x, m])v = m[x, m]g(v)$$
$$d(m)[x, m]v + mf([x, m])v = m[x, m]g(v).$$

Using the hypothesis in the above relation, we get

$$d(m)[x, m]v + m[x, m]g(v) = m[x, m]g(v)$$

and so

$$d(m)[x,m]v = 0, \ for\ all\ m \in M, v \in U, x \in R.$$

Replacing v by $[v,r]$, $r \in R$ in above equation and using this, we have

$$d(m)[x,m]rv = 0, \ for\ all\ m \in M, v \in U, x, r \in R.$$

and so

$$d(m)[x,m]RU = \{0\}, \ for\ all\ m \in M, x \in R.$$

Since R is prime ring and $U \neq \{0\}$, it follows that

$$d(m)[x,m] = 0, \ for\ all\ m \in M, x \in R.$$

Writing x by $xy, y \in R$ in the last equation and using this, we obtain that

$$d(m)R[y,m] = \{0\}, \ for\ all\ m \in M, y \in R.$$

Primeness of R yields that for a fixed $m \in M$,

$$m \in Z \ or\ d(m) = 0.$$

Let $L = \{m \in M \mid m \in Z \}$ and $K = \{m \in M \mid d(m) = 0\}$. Clearly each of L and K is additive subgroup of M such that $M = L \cup K$. But, a group can not be the set-theoretic union of its two proper subgroups. Hence $L = M$ or $K = M$. In the former case, $M \subset Z$, which forces R to be commutative. This is impposible because of $U \not\subseteq Z$. In the latter case, $d(M) = 0$. Since R is prime ring M a nonzero ideal of R, we get $d = 0$, which is a contradiction. This completes the proof. □

Corollary 3.5 *Let (f,d) and (g,h) be two generalized derivations of R. If $f(u)u = ug(u)$, for all $u \in U$, then $U \subset Z$.*

Theorem 3.6 *If (f,d) satisfies one of the following conditions then $U \subset Z$.*

(i) $f([u,v]) = [u,v]$, for all $u, v \in U$.

(ii) $f([u,v]) = -[u,v]$, for all $u, v \in U$.

(iii) For each $u, v \in U$, either $f([u,v]) = [u,v]$ or $f([u,v]) = -[u,v]$.

Proof. (i) For any $u, v \in U$, we have $f([u,v]) = [u,v]$, which gives

$$f([u,v]) = [f(u),v] + [u,d(v)] = [u,v].$$

Replacing u by $[u,w]$, $w \in U$, we get

$$[f([u,w]),v] + [[u,w],d(v)] = [[u,w],v].$$

Using the hypothesis, we obtain

$$[[u,w],v] + [[u,w],d(v)] = [[u,w],v]$$

and so

$$[[u,w],d(v)]=0,\ for\ all\ u,v,w\in U.$$

That is

$$[[U,U],d(U)]=0.$$

By Lemma 2.6, we have $[U,U]\subset Z$ or $U\subset Z$. If $[U,U]\subset Z$, then again $U\subset Z$ by Lemma 2.5. This completes the proof.

(ii) can be proved by using the same techniques.

(iii) For each $w\in U$, we put

$U_w=\{v\in U \mid f([w,v])=[w,v]\}$ and $U_w^*=\{v\in U \mid f([w,v])=-[w,v]\}$.

Then $(U,+)=U_w\cup U_w^*$, but a group cannot be the union of its two proper subgroups, hence $U=U_w$ or $U=U_w^*$. By the same method in (i) or (ii), we complete the proof. □

Corollary 3.7 *If (f,d) satisfies one of the following conditions then $U\subset Z$.*

(i) $f(uv)=uv$, for all $u,v\in U$.

(ii) $f(uv)=-uv$, for all $u,v\in U$.

(iii) For each $u,v\in U$, either $f(uv)=uv$ or $f(uv)=-uv$.

Proof. (i) Assume that $f(uv)=uv$ for all $u,v\in U$. Then we have

$$f(uv-vu)=f(uv)-f(vu)=uv-vu.$$

Hence $f([u,v])=[u,v]$, for all $u,v\in U$. By Theorem 3.6 (i), we obtain that $U\subset Z$.

(ii) can be proved similarly.

(iii) can be proved by using the similar arguments in Theorem 3.6 (iii). □

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