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Existence of three solutions to a non-homogeneous multi-point BVP of second order differential equations

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Abstract

This paper is concerned with a non-homogeneous multi-point boundary value problem of second order differential equation with one-dimensional p-Laplacian. Using multiple fixed point theorems, sufficient conditions to guarantee the existence of at least three solutions of this kind of BVP are established. Two examples are presented to illustrate the main results.

Key word and phrases: Second order differential equation with p-Laplacian, generalized Sturm-Liouville boundary value problem, fixed point theorem in cone.

1. Introduction

In recent years, the solvability of multi-point boundary-value problems (BVPs for short) for second order differential equations or higher order differential equations on finite intervals have been studied by different authors, see papers [1–29]. The methods used in above mentioned papers, are the Guo-Krasnoselskii fixed point theorem, the fixed-point theorem due to Avery and Peterson, the Leggett-Williams fixed point theorem, the five functional fixed point theorem, the monotone iterative techniques and Mawhin coincidence degree theory, et cetera.

Ma in [21, 22] studied the following more generalized BVP

$$\begin{cases} [p(t)x'(t)]' - q(t)x(t) + f(t, x(t)) = 0, \quad t \in (0, 1), \\ \alpha x(0) - \beta p(0)x'(0) = \sum_{i=1}^{m} a_i x(\xi_i), \\ \gamma x(1) + \delta p(1)x'(1) = \sum_{i=1}^{m} b_i x(\xi_i), \end{cases} \tag{1}$$

where $0 < \xi_1 < \cdots < \xi_m < 1$, $\alpha, \beta, \gamma, \delta \geq 0, a_i, b_i \geq 0$ with $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$. By using Green′s functions (which complicate the studies of BVP(1)) and Guo-Krasnoselskii fixed point theorem, the existence and multiplicity of positive solutions for BVP(1) were given.

There has been a large number of papers in which many exciting results concerned with the existence of

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multiple positive solutions of the following BVPs of second order differential equations with p-Laplacian

$$\begin{cases} [\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \\ x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i), \end{cases} \tag{2}$$

or

$$\begin{cases} [\phi(x'(t))]' + f(t, x(t)) = 0, \quad t \in (0,1), \\ x(0) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \\ x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i), \end{cases} \tag{3}$$

are obtained. To see these interesting results, one may see the text book [6].

In paper [8], the authors studied the existence of three positive solutions of the boundary value problem of the form

$$\begin{cases} [\phi(x'(t))]' + e(t)f(t, x(t)) = 0, \quad t \in (0,1), \\ x(0) - B_0(x'(0)) = 0, \\ x(1) + B_1(x'(1)) = 0, \end{cases} \tag{4}$$

where $B_0, B_1 : R \to R$ are continuous, increasing on R and $0 < xB_i(x)(i = 0,1)$ for all $x \neq 0$, e and f are continuous and nonnegative.

The boundary conditions in BVP(1), BVP(2), BVP(3) and BVP(4) are homogeneous cases. In many applications, BVPs consist of differential equations coupled with nonhomogeneous BCs, see [16, 17].

In papers [10–12], using lower and upper solutions methods, Kong and Kong established existence results for solutions and positive solutions of the following two problems

$$\begin{cases} x''(t)) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x'(0) - \sum_{i=1}^{m} \alpha_i x'(\xi_i) = \lambda_1, \\ x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{cases} \tag{5}$$

and

$$\begin{cases} x''(t)) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) - \sum_{i=1}^{m} \alpha_i x(\xi_i) = \lambda_1, \\ x(1) - \sum_{i=1}^{m} \beta_i x(\xi_i) = \lambda_2, \end{cases} \tag{6}$$

respectively. These papers may be the first papers concerned with the BVPs with two parameter multi-point non-homogeneous BCs. In Kong$'$s results, the existence of lower and upper solutions with certain relations are supposed.

A problem appears, under what conditions BVP(5) and BVP(6) have at least three solutions?

To address above problem, in recent paper [17], the author established existence results for three solutions of the following BVP,

$$\begin{cases} [\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) - \alpha x(0) = A, \\ x(1) - \sum_{i=1}^{m} b_i x(\xi_i) = B, \end{cases}$$

by using a three functionals fixed point theorem.

Motivated by paper [10–12, 17, 21, 22] and the problem, the purpose of this paper is to investigate the BVP

$$\begin{cases} [p(t)\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) - B_0(x'(0)) = \sum_{i=1}^{m} a_i x(\xi_i) + A, \\ x(1) + B_1(x'(1)) = \sum_{i=1}^{m} b_i x(\xi_i) + B, \end{cases} \tag{7}$$

where

- $B_0, B_1 : R \to R$ are continuous, increasing on R;
- $0 < \xi_1 < \cdots < \xi_m < 1$, $A, B \in R$, $a_i \geq 0, b_i \geq 0$ for all $i = 1, \cdots, m$;
- f defined on $[0,1] \times R \times R$ is continuous, p defined on $[0,1]$ continuous differentiable;
- $\phi : R \to R$ with $\phi \in C^1(R)$ and $\phi'(x) \geq 0$ for all $x \in R$, and there exists its inverse function denoted by ϕ^{-1}. It is easy to see that $p-$Laplacian function $\phi(x) = |x|^{p-2}x$ with $p > 1$ is such a function.

A function $x : [0,1] \to R$ is called a solution of BVP(7) if $x \in C^1[0,1]$, $p\phi(x') \in C^1[0,1]$ and all equations in (7) are satisfied.

Sufficient conditions for the existence of at least three solutions of BVP(7) are established by using the five functional fixed point theorem [1]. The Green$'s$ functions are not used in the proofs of the main results.

Applying the result to the special case

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0,1), \\ x(0) = \sum_{i=1}^{m} a_i x(\xi_i) + A, \\ x(1) = \sum_{i=1}^{m} b_i x(\xi_i) + B, \end{cases} \tag{8}$$

our result is different from those in [10–12] since we get three solutions of BVP(8). It is easy to see that BVP(7) is the precise nature of combinations of multi-points, p-Laplacian and non-homogeneity. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results and their proofs are presented in Section 3, and some examples are given in Section 4.

2. Preliminary results

To the reader$'$s convenience, some background definitions in Banach spaces and an important three fixed point theorem are presented.

As usual, let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ and $x + y \in P$ for all $x, y \in P$ and $a \geq 0$, and $x \in X$ and $-x \in X$ imply $x = 0$. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave (or convex) functional map provided ψ is nonnegative, continuous and satisfies

$$\psi(tx + (1-t)y) \geq \text{ (or } \leq \text{) } t\psi(x) + (1-t)\psi(y) \text{ for all } x, y \in P, t \in [0,1].$$

An operator $T : X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Let $c_1, c_2, c_3, c_4, c_5 > 0$ be positive constants, α_1, α_2 be two nonnegative continuous concave functionals on the cone P, $\beta_1, \beta_2, \beta_3$ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as

$$\begin{aligned}
&P_{c_5} = \{x \in P : ||x|| < c_5\},\\
&P(\beta_1, \alpha_1; c_2, c_5) = \{x \in P : \alpha_1(x) \geq c_2,\ \beta_1(x) \leq c_5\},\\
&P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5) = \{x \in P : \alpha_1(x) \geq c_2,\ \beta_3(x) \leq c_4,\ \beta_1(x) \leq c_5\},\\
&Q(\beta_1, \beta_2; , c_1, c_5) = \{x \in P : \beta_2(x) \leq c_1,\ \beta_1(x) \leq c_5\},\\
&Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5) = \{x \in P : \alpha_2(x) \geq c_3,\ \beta_2(x) \leq c_1,\ \beta_1(x) \leq c_5\}.
\end{aligned}$$

Theorem 2.1 [1] *Let X be a real Banach space, P a cone in X. α_1, α_2 be two nonnegative continuous concave functionals on the cone P, β_1, β_2, β be three nonnegative continuous convex functionals on the cone P. Then T has at least three fixed points y_1, y_2 and y_3 such that*

$$\beta_2(y_1) < c_1,\ \alpha_1(y_2) > c_2,\ \beta_2(y_3) > c_1,\ \alpha_1(y_3) < c_2$$

if

- **(A1)** *$T : X \to X$ is a completely continuous operator;*
- **(A2)** *there exist a constant $M > 0$ such that*

$$\alpha_1(x) \leq \beta_2(x),\ ||x|| \leq M\beta_1(x) \text{ for all } x \in P;$$

- **(A3)** *there exist positive numbers c_1, c_2, c_3, c_4, c_5 with $c_1 < c_2$ such that*
  - **(i)** $T\overline{P_{c_5}} \subset \overline{P_{c_5}}$;
  - **(ii)** $\{y \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5) | \alpha_1(x) > c_2\} \neq \emptyset$ *and*

$$\alpha_1(Tx) > c_2 \text{ for every } x \in P(\beta_1, \beta_3, \alpha_1; c_2, c_4, c_5);$$

  - **(iii)** $\{y \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5) | \beta_2(x) < c_1\} \neq \emptyset$ *and*

$$\beta_2(Tx) < c_1 \text{ for every } x \in Q(\beta_1, \beta_2, \alpha_2; c_3, c_1, c_5);$$

  - **(iv)** $\alpha_1(Ty) > c_2$ *for* $y \in P(\beta_1, \alpha_1; c_2, c_5)$ *with* $\beta_3(Ty) > c_4$;
  - **(v)** $\beta_2(Tx) < c_1$ *for each* $x \in Q(\beta_1, \beta_2; c_1, c_5)$ *with* $\alpha_2(Tx) < c_3$.

Lemma 2.1 *Suppose that $p : [0, 1] \to (0, +\infty)$ with $p \in C^1[0, 1]$, $x \in C^1[0, 1]$ with $[p\phi(x')]' \in C^0[0, 1]$, $x(t) \geq 0$ for all $t \in [0, 1]$ and $[p(t)\phi(x'(t))]' \leq 0$ on $[0, 1]$. Then x is concave and*

$$x(t) \geq \min\{t, 1 - t\} \max_{t\in[0,1]} x(t),\quad t \in [0, 1]. \tag{9}$$

Proof. Since $x \in C^1[0, 1]$, suppose $x(t_0) = \max_{t\in[0,1]} x(t)$. If $t_0 < 1$, for $t \in (t_0, 1)$, since $x'(t_0) \leq 0$, we have $p(t_0)\phi(x'(t_0)) \leq 0$. Then $p(t)\phi(x'(t)) \leq 0$ for all $t \in (t_0, 1]$. It follows that $x'(t) \leq 0$ for all $t \in [t_0, 1]$. Thus $x(t_0) \geq x(t) \geq x(1) \geq 0$. Let

$$\tau(t) = \frac{\int_{t_0}^{t} \phi^{-1}\left(\frac{1}{p(s)}\right) ds}{\int_{t_0}^{1} \phi^{-1}\left(\frac{1}{p(s)}\right) ds}.$$

Since

$$\frac{d\tau}{dt} = \frac{\phi^{-1}\left(\frac{1}{p(t)}\right)}{\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds} > 0,$$

we get $\tau \in C^1([t_0,1],[0,1])$ and is increasing on $[t_0,1]$ with $\tau(t_0)=0$ and $\tau(1)=1$. Thus

$$\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\frac{\phi^{-1}\left(\frac{1}{p(t)}\right)}{\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds}$$

implies that

$$p(t)\phi(x'(t)) = \phi\left(\frac{dx}{d\tau}\right)\frac{1}{\phi\left(\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds\right)}.$$

Hence

$$\phi'\left(\frac{dx}{d\tau}\right)\frac{\phi^{-1}\left(\frac{1}{p(t)}\right)}{\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds}\frac{d^2x}{d\tau^2} = \phi\left(\int_{t_0}^1 \phi^{-1}\left(\frac{1}{p(s)}\right) ds\right)\left[\,p(t)\phi(x'(t))\right]' \le 0 \text{ for all } t \in [t_0,1].$$

It follows $\phi'(x) \ge 0$ that $\frac{d^2x}{d\tau^2} \le 0$. Together with $x''(\tau) \le 0(\tau \in [0,1])$. It follows that x is concave on [0,1].

Now $x(t) \ge 0(t \in [t_0,1])$, we get that there exist $t_0 \le \eta \le t \le \xi \le 1$ such that

$$\begin{aligned}
\frac{x(t_0)-x(1)}{t_0-1} - \frac{x(t)-x(1)}{t-1} &= -\frac{(t-1)[x(t_0)-x(t)] + (t_0-t)[x(1)-x(t)]}{(t-1)(t_0-1)} \\
&= -\frac{(t-1)(t_0-t)x'(\eta) + (t_0-t)(1-t)tx'(\xi)}{(t-1)(t_0-1)} \\
&\le -\frac{(t-1)(t_0-t)x'(\xi) + (t_0-t)(1-t)tx'(\xi)}{(t-1)(t_0-1)} = 0.
\end{aligned}$$

It follows for $t \in (t_0,1)$ that

$$x(t) \ge x(1) + (t-1)\frac{x(t_0)-x(1)}{t_0-1} = x(1)\left(1 - \frac{1-t}{1-t_0}\right) + \frac{1-t}{1-t_0}x(t_0) \ge (1-t)x(t_0).$$

If $t_0 > 0$, for $t \in (0,t_0)$, similarly to above discussion, we can get that

$$x(t) \ge tx(t_0),\ t \in (0,t_0).$$

Then one gets that $x(t) \ge \min\{t,1-t\}\max_{t\in[0,1]} x(t)$ for all $t \in [0,1]$. The proof is complete. □

Suppose that

(B1) $1-\sum_{i=1}^m a_i \ne 0$, $f : [0,1] \times [h_1,+\infty) \times R \to [0,+\infty)$, where $h_1 = \frac{A}{1-\sum_{i=1}^m a_i}$, is continuous with $f(t,0,0) \not\equiv 0$ on each sub-interval of [0,1];

(B1)$'$ $1-\sum_{i=1}^m b_i \neq 0$, $f:[0,1]\times[h_2,+\infty)\times R\to[0,+\infty)$, where $h_2=\frac{B}{1-\sum_{i=1}^m b_i}$, is continuous with $f(t,0,0)\not\equiv 0$ on each sub-interval of [0,1];

(B2) $p:[0,1]\to(0,+\infty)$ with $p\in C^1[0,1]$;

(B3) $a_i\geq 0$, $b_i\geq 0$ for all $i=1,\cdots,m$;

(B4) $\sum_{i=1}^m a_i<1$, $\sum_{i=1}^m b_i<1$ and $\frac{A}{1-\sum_{i=1}^m a_i}\leq\frac{B}{1-\sum_{i=1}^m b_i}$;

(B4)$'$ $\sum_{i=1}^m a_i<1$, $\sum_{i=1}^m b_i<1$ and $\frac{A}{1-\sum_{i=1}^m a_i}\geq\frac{B}{1-\sum_{i=1}^m b_i}$;

(B5) $\sigma:[0,1]\to[0,+\infty)$ is a continuous function and $\sigma(t)\not\equiv 0$ on each subinterval of [0,1];

(B6) there exist nonnegative numbers $\beta,\beta',\delta,\delta'$ such that $\beta' x^2\leq xB_0(x)\leq\beta x^2$ and $\delta' x^2\leq xB_1(x)\leq\delta x^2$ for all $x\in R$.

Let $X=C^1[0,1]$ be with the norm $||x||=\max\left\{\max_{t\in[0,1]}|x(t)|,\ \max_{t\in[0,1]}|x'(t)|\right\}$. Then X is a Banach space. For $x,y\in X$, we call $x\leq y$ for $x,y\in X$ if $x(t)\leq y(t)$ for all $t\in[0,1]$. It is easy to see that X is a semi-ordered real Banach space.

Lemma 2.2 *Suppose that* $(B2)$*-*$(B6)$ *hold. If* y *is a solution of the BVP*

$$\begin{cases}[p(t)\phi(y'(t))]'+\sigma(t)=0,\ \ t\in(0,1),\\ y(0)-B_0(y'(0))=\sum_{i=1}^m a_i y(\xi_i),\\ y(1)+B_1(y'(1))=\sum_{i=1}^m b_i y(\xi_i)+\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right),\end{cases}\tag{10}$$

then y *is concave and positive on* $(0,1)$.

Proof. Suppose y satisfies (10). Then $(B5)$ implies that $[p(t)\phi(y'(t))]'\leq 0$ for all $t\in[0,1]$. Suppose that $y(t_0)=\max_{t\in[0,1]}y(t)$. Then either $t_0>0$ or $t_0<1$.

If $t_0>0$, for $t\in[0,t_0]$, let

$$\tau(t)=\frac{\int_0^t\phi^{-1}\left(\frac{1}{p(s)}\right)ds}{\int_0^{t_0}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}.$$

It is easy to see that $\tau\in C([0,t_0],[0,1])$ and

$$\frac{d\tau}{dt}=\frac{\phi^{-1}\left(\frac{1}{p(t)}\right)}{\int_0^{t_0}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}>0.$$

Thus

$$\frac{dy}{dt}=\frac{dy}{d\tau}\frac{d\tau}{dt}=\frac{dy}{d\tau}\frac{\phi^{-1}\left(\frac{1}{p(t)}\right)}{\int_0^{t_0}\phi^{-1}\left(\frac{1}{p(s)}\right)ds}.$$

It follows that

$$\phi\left(\int_0^{t_0}\phi^{-1}\left(\frac{1}{p(s)}\right)ds\right)\ p(t)\phi\left(\frac{dy}{dt}\right)=\phi\left(\frac{dy}{d\tau}\right).$$

Hence

$$\phi\left(\int_0^{t_0}\phi^{-1}\left(\frac{1}{p(s)}\right)ds\right)\ \left[p(t)\phi\left(\frac{dy}{dt}\right)\right]' = \phi'\left(\frac{dy}{d\tau}\right)\frac{d^2y}{d\tau^2}\frac{d\tau}{dt}.$$

Since $[p(t)\phi(y'(t))]' \le 0$, $\phi'(x) \ge 0$, we get that $\frac{d^2y}{d\tau^2} \le 0$. Then y is concave on [0,1].

If $t_0 < 1$, we can get that $\frac{d^2y}{d\tau^2} \le 0$ similarly. It follows that y is concave on $[0,1]$.

Now, we prove that $y'(0) \ge 0$. If $y'(0) < 0$, we get that $y'(t) < 0$ for all $t \in [0,1]$. (B6) implies that $B_0(y'(0)) < 0$ and $B_1(y'(1)) < 0$. Then the BCs in (10) and the assumptions $(B3), (B4)$ imply that

$$y(0) - B_0(y'(0)) = \sum_{i=1}^m a_i y(\xi_i) \le \sum_{i=1}^m a_i y(0)$$

and

$$y(1) + B_1(y'(1)) = \sum_{i=1}^m b_i y(\xi_i) + \left(B - \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right) \ge \sum_{i=1}^m b_i y(1).$$

It follows that

$$\left(1-\sum_{i=1}^m a_i\right) y(0) - B_0(y'(0)) \le 0,\quad \left(1-\sum_{i=1}^m b_i\right) y(1) + B_1(y'(1)) \ge 0.$$

Then (B3) and (B4) imply that $y(0) \le 0$ and $y(1) \ge 0$. It follows that $y(t) \equiv 0$ on [0,1]. We get that $\sigma(t) = [p(t)\phi(y'(t))]' \equiv 0$, a contradiction to $(B5)$.

It follows from above discussion that $y'(0) \ge 0$. Since $y(t)$ is concave on [0,1], we get that $y(t) \ge \min\{y(0), y(1)\}$ for $t \in [0,1]$. Then

$$y(0) - B_0(y'(0)) = \sum_{i=1}^m a_i y(\xi_i) \ge \sum_{i=1}^m a_i \min\{y(0), y(1)\}$$

and

$$y(1) + B_1(y'(1)) = \sum_{i=1}^m b_i y(\xi_i) + \left(B - \frac{\gamma-\sum_{i=1}^m b_i}{\alpha-\sum_{i=1}^m a_i}A\right) \ge \sum_{i=1}^m b_i \min\{y(0), y(1)\}.$$

Case 1. If $\min\{y(0), y(1)\} = y(0)$, then $(1-\sum_{i=1}^m a_i)\, y(0) - B_0(y'(0)) \ge 0$. We get $y(0) \ge 0$ since $y'(0) \ge 0$ and $xB_0(x) > 0$ for all $x \ne 0$. Then $\min\{y(0), y(1)\} \ge 0$. Hence $y(t) \ge \min\{y(0), y(1)\} \ge 0$ for $t \in [0,1]$.

Case 2. If $\min\{y(0), y(1)\} = y(1)$, then $(1-\sum_{i=1}^m b_i)\, y(1) + B_1(y'(1)) \ge 0$.

If $y'(1) > 0$, then

$$\frac{d^2y}{d\tau^2} \le 0$$

implies that $y'(t) > 0$ for all $t \in [0,1]$. It follows that $\min\{y(0), y(1)\} = y(0)$, a contradiction. Then $y'(1) \le 0$, we get from $(1-\sum_{i=1}^m b_i)\, y(1) + B_1(y'(1)) \ge 0$, that $y(1) \ge 0$. Hence $y(t) \ge \min\{y(0), y(1)\} \ge 0$ for $t \in [0,1]$. From cases 1 and 2, we get that $y(t) > 0$ for $t \in (0,1)$. The proof is complete. □

Define

$$\begin{aligned}
l_1 &= \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(\beta+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds\right),\\
l_2 &= \left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds,\\
l_3 &= \delta\phi^{-1}\left(\frac{1}{p(1)}p(0)\right)+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds,\\
a &= \frac{B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A}{l_1+l_2+l_3}
\end{aligned}$$

and

$$\begin{aligned}
l_1' &= \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(\beta'+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds\right),\\
l_3' &= \delta'\phi^{-1}\left(\frac{1}{p(1)}p(0)\right)+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds,\\
a' &= \frac{B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A}{l_1'+l_2+l_3'}.
\end{aligned}$$

Lemma 2.3 *Suppose that (B2)-(B6) hold. If y is a solution of BVP(10), then there exists an unique constant A_σ such that*

$$y(t)=B_\sigma+\int_0^t\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds,\quad t\in[0,1],$$

where $A_\sigma\in\left[a,\phi^{-1}\left(\frac{\phi(a')}{p(0)}+\frac{1}{p(0)}\int_0^1\sigma(u)du\right)\right]$ satisfies

$$\begin{aligned}
&\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(B_0(A_\sigma)+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\right)\\
=\;& -\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\\
&-B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(A_\sigma)-\frac{1}{p(1)}\int_0^1\sigma(u)du\right)\right)\\
&+\sum_{i=1}^m b_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\\
&+\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right),
\end{aligned}$$

and B_σ *satisfies*

$$B_\sigma = \frac{1}{1-\sum_{i=1}^m a_i}\left(B_0(A_\sigma)+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\right).$$

Proof. Since y is solution of (10), we get

$$p(t)\phi(y'(t)) = p(0)\phi(y'(0)) - \int_0^t \sigma(u)du, \quad t\in[0,1].$$

Then

$$y(t) = y(0)+\int_0^t\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds.$$

The BCs in (10) imply that

$$y(0)-B_0(y'(0)) = y(0)\sum_{i=1}^m a_i+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds$$

and

$$\begin{aligned}
& y(0)+\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds \\
& +B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(y'(0))-\frac{1}{p(1)}\int_0^1\sigma(u)du\right)\right) \\
= \; & y(0)\sum_{i=1}^m b_i+\sum_{i=1}^m b_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds \\
& +\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
& \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(B_0(y'(0))+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\right) \\
= \; & -\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds \\
& -B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(y'(0))-\frac{1}{p(1)}\int_0^1\sigma(u)du\right)\right) \\
& +\sum_{i=1}^m b_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(y'(0))-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds \\
& +\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right).
\end{aligned}$$

Let

$$
\begin{aligned}
G(c) \;=\;& \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(B_0(c)+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(c)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\right)\\
&+\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(c)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\\
&+B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(c)-\frac{1}{p(1)}\int_0^1\sigma(u)du\right)\right)\\
&-\sum_{i=1}^m b_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(c)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\\
&-\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right).
\end{aligned}
$$

It is easy to see that $G(c)$ is increasing on $(-\infty,+\infty)$. Using $(B6)$, one gets

$$
\begin{aligned}
G(a) \;=\;& \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(B_0(a)+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(a)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\right)\\
&+\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(a)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\\
&+B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(a)-\frac{1}{p(1)}\int_0^1\sigma(u)du\right)\right)\\
&+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(a)-\frac{1}{p(s)}\int_0^s\sigma(u)du\right)ds\\
&-\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right)\\
\le\;& \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(\beta a+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(a)\right)ds\right)\\
&+\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(a)\right)ds+\delta\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(a)\right)\right)\\
&+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(a)\right)ds-\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right)\\
=\;& a\left[\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(\beta+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds\right)\right.
\end{aligned}
$$

$$+\left(1-\sum_{i=1}^{m} b_i\right)\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds+\delta\phi^{-1}\left(\frac{1}{p(1)}p(0)\right)$$
$$+\sum_{i=1}^{m} b_i\int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds\Bigg]-\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right)$$
$$= a(l_1+l_2+l_3)-\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right)=0,$$

and

$$\begin{aligned}
& G\left(\phi^{-1}\left(\frac{\phi(a')}{p(0)}+\frac{1}{p(0)}\int_0^1\sigma(u)du\right)\right)\\
= & \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\Bigg[B_0\left(\phi^{-1}\left(\frac{\phi(a')}{p(0)}+\frac{1}{p(0)}\int_0^1\sigma(u)du\right)\right)\\
& +\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{\phi(a')}{p(s)}+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds\Bigg]\\
& +B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi\left(\phi^{-1}\left(\frac{\phi(a')}{p(0)}+\frac{1}{p(0)}\int_0^1\sigma(u)du\right)\right)-\frac{1}{p(1)}\int_0^1\sigma(u)du\right)\right)\\
& +\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{\phi(a')}{p(s)}+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds\\
& +\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{\phi(a')}{p(s)}+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds-\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right)\\
\geq & \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left[B_0\left(\phi^{-1}\left(\frac{\phi(a')}{p(0)}\right)\right)+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{\phi(a')}{p(s)}\right)ds\right]\\
& +B_1\left(\phi^{-1}\left(\frac{\phi(a')}{p(1)}\right)\right)\\
& +\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{\phi(a')}{p(s)}\right)ds+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{\phi(a')}{p(s)}\right)ds\\
& -\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right)\\
= & a'\Bigg[\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(\beta'+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds\right)+\delta'\phi^{-1}\left(\frac{1}{p(1)}p(0)\right)\\
& +\left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds\Bigg]\\
& -\left(B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right)
\end{aligned}$$

$$= \quad a'(l_1'+l_2+l_3') - \left(B - \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right) = 0.$$

Hence we get that there exists unique constant $A_\sigma \in \left[a, \phi^{-1}\left(\frac{\phi(a')}{p(0)} + \frac{1}{p(0)}\int_0^1 \sigma(u)du\right)\right]$ such that

$$\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(B_0(A_\sigma) + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma) - \frac{1}{p(s)}\int_0^s \sigma(u)du\right)ds\right)$$
$$+\left(1-\sum_{i=1}^m b_i\right)\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma) - \frac{1}{p(s)}\int_0^s \sigma(u)du\right)ds$$
$$+B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(A_\sigma) - \frac{1}{p(1)}\int_0^1 \sigma(u)du\right)\right)$$
$$+\sum_{i=1}^m b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma) - \frac{1}{p(s)}\int_0^s \sigma(u)du\right)ds = B - \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A.$$

Then $A_\sigma = y'(0)$ and

$$B_\sigma = \frac{1}{1-\sum_{i=1}^m a_i}\left(B_0(A_\sigma) + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_\sigma) - \frac{1}{p(s)}\int_0^s \sigma(u)du\right)ds\right).$$

The proof is completed. □

Note $h_1 = \frac{A}{1-\sum_{i=1}^m a_i}$. Let $x(t) = y(t) + h_1$. Then BVP(7) is transformed into the following BVP

$$\begin{cases} [p(t)\phi(y'(t))]' + f(t, y(t)+h_1, y'(t)) = 0, \quad t \in (0,1), \\ y(0) - B_0(y'(0)) = \sum_{i=1}^m a_i y(\xi_i), \\ y(1) + B_1(y'(1)) = \sum_{i=1}^m b_i y(\xi_i) + \left(B - \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A\right). \end{cases} \tag{11}$$

Let

$$P_1 = \left\{y \in X : \ y(t) \ge 0 \text{ for all } t \in [0,1],\ y'(t) \text{ is decreasing on } [0,1],\ \right\}.$$

Then P_1 is a nonempty cone in X since $x(t) = t^2 \in P_1$ for example. Choose $\sigma_0 \in (0, 1/2)$. Define the functionals on $P_1 \to R$ by

$$\begin{aligned} \beta_1(y) &= \max_{t\in[0,1]} |y'(t)|,\ y \in P_1, \\ \beta_2(y) &= \max_{t\in[0,1]} |y(t)|,\ y \in P_1, \\ \beta_3(y) &= \max_{t\in[\sigma_0, 1-\sigma_0]} |y(t)|,\ y \in P_1, \\ \alpha_1(y) &= \min_{t\in[\sigma_0, 1-\sigma_0]} |y(t)|,\ y \in P_1, \\ \alpha_2(y) &= \min_{t\in[\sigma_0, 1-\sigma_0]} |y(t)|,\ y \in P_1. \end{aligned}$$

Define the nonlinear operator $T_1 : P_1 \to X$ by

$$(T_1y)(t) = B_y + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u, y(u) + h_1, y'(u))du\right) ds, \quad y \in P_1,$$

where $A_y \in \left[a, \phi^{-1}\left(\frac{\phi(a')}{p(0)} + \frac{1}{p(0)}\int_0^1 f(u, y(u) + h_1, y'(u))du\right)\right]$ such that

$$\begin{aligned}
&\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\Bigg[B_0(A_y) \\
&\qquad + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u, y(u) + h_1, y'(u))du\right) ds\Bigg] \\
&+ \left(1 - \sum_{i=1}^m b_i\right) \int_0^1 \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u, y(u) + h_1, y'(u))du\right) ds \\
&+ B_1\left(\phi^{-1}\left(\frac{1}{p(1)}p(0)\phi(A_y) - \frac{1}{p(1)}\int_0^1 f(u, y(u) + h_1, y'(u))du\right)\right) \\
&+ \sum_{i=1}^m b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u, y(u) + h_1, y'(u))du\right) ds \\
= \; & B - \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i} A.
\end{aligned}$$

and B_y satisfies

$$B_y = \frac{B_0(A_y) + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u, y(u) + h_1, y'(u))du\right) ds}{1-\sum_{i=1}^m a_i}.$$

Then

$$\begin{aligned}
(T_1y)(t) = \; & \frac{B_0(A_y) + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{p(0)}{p(s)}\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u, y(u) + h_1, y'(u))du\right) ds}{1-\sum_{i=1}^m a_i} \\
& + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u, y(u) + h_1, y'(u))du\right) ds, \quad y \in P.
\end{aligned}$$

Remark 2.1 The operator T_1 defined relies on the constant A_y which changes with each y, Lemma 2.3 implies that A_y is unique, so T_1 is well defined.

Lemma 2.4 *Suppose that (B1)-(B4),(B6) hold. It is easy to show that*

(i) T_1y *satisfies the equalities*

$$\begin{cases}
[p(t)\phi((T_1y)')]' + f(t, y(t) + h_1, y'(t)) = 0, \quad t \in (0,1), \\
(T_1y)(0) - B_0((T_1y)'(0)) = \sum_{i=1}^m a_i (T_1y)(\xi_i), \\
(T_1y)(1) + B_1((T_1y)'(1)) = \sum_{i=1}^m b_i (T_1y)(\xi_i) + \left(B - \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i} A\right);
\end{cases}$$

(ii) *$T_1 y \in P_1$ for each $y \in P_1$;*

(iii) *x is a solution of BVP(7) if and only if $x = y + h_1$ and y is a solution of the operator equation $y = T_1 y$ in P_1;*

(iv) *T_1 is completely continuous.*

Proof. The proofs of (i), (ii) and (iii) are simple. To prove (iv), it suffices to prove that T_1 is continuous on P and T_1 is relative compact. We divide the proof into two steps:

Step 1. For each bounded subset $D \subset P$, and each $x_0 \in D$, since $f(t,u,v)$ is continuous in u, v, we can prove that T_1 is continuous at $y(t)$.

Step 2. For each bounded subset $D \subset X$, prove that T_1 is relatively compact on D.

It is similar to that of the proof of Lemmas in [16, 23] and are omitted. □

Lemma 2.5 *Suppose that $(B2), (B3), (B4)'$, $(B5)$ and $(B6)$ hold. If y is a solution of the BVP*

$$\begin{cases} [p(t)\phi(y'(t))]' + \sigma(t) = 0, \ \ t \in (0,1), \\ y(0) - B_0(y'(0)) = \sum_{i=1}^m a_i y(\xi_i) + \left(A - \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i} B\right), \\ y(1) + B_1(y'(1)) = \sum_{i=1}^m b_i y(\xi_i). \end{cases} \tag{12}$$

then y is concave and positive on $(0,1)$.

Proof. The proof is similar to that of the proof of Lemma 2.3 and is omitted. □

Define

$$\begin{aligned} m_1 &= \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}\left(\delta + \sum_{i=1}^m b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(1)\right) ds\right), \\ m_2 &= \left(1 - \sum_{i=1}^m a_i\right) \int_0^1 \phi^{-1}\left(\frac{1}{p(s)}p(1)\right) ds, \\ m_3 &= \beta\phi^{-1}\left(\frac{1}{p(0)}p(1)\right) + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(1)\right) ds, \\ b &= \frac{A - \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}B}{m_1+m_2+m_3} \end{aligned}$$

and

$$\begin{aligned} m_1' &= \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}\left(\delta' + \sum_{i=1}^m b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(1)\right) ds\right), \\ m_3' &= \beta'\phi^{-1}\left(\frac{1}{p(0)}p(1)\right) + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(1)\right) ds, \\ b' &= \frac{A - \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}B}{m_1'+m_2+m_3'}. \end{aligned}$$

Lemma 2.6 *Suppose that* $(B2),(B3),(B4)'$, $(B5)$ *and* $(B6)$ *hold. If* y *is a solution of BVP(12), then*

$$y(t)=B_\sigma-\int_t^1\phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_\sigma)+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds,\ \ t\in[0,1],$$

where $A_\sigma\in\left[-\phi^{-1}\left(\frac{\phi(b')}{p(1)}+\frac{1}{p(1)}\int_0^1\sigma(u)du\right),-b\right]$ *satisfies*

$$\begin{aligned}&-\frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}\left(B_1(A_\sigma)+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_\sigma)+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds\right)\\ =&\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_\sigma)+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds\\ &+B_0\left(\phi^{-1}\left(\frac{1}{p(0)}p(1)\phi(A_\sigma)+\frac{1}{p(0)}\int_0^1\sigma(u)du\right)\right)\\ &-\sum_{i=1}^m a_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_\sigma)+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds\\ &+\left(A-\frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}B\right).\end{aligned}$$

B_σ *satisfies*

$$B_\sigma=-\frac{1}{1-\sum_{i=1}^m b_i}\left(B_1(A_\sigma)+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_\sigma)+\frac{1}{p(s)}\int_s^1\sigma(u)du\right)ds\right).$$

Proof. The proof is similar to that of the proof of Lemma 2.4. □

Note $h_2=\frac{B}{1-\sum_{i=1}^m b_i}$. Let $x(t)=y(t)+h_2$. Then BVP(7) is transformed into the BVP

$$\left\{\begin{array}{l}[p(t)\phi(y'(t))]'+f\left(t,y(t)+h_2,y'(t)\right)=0,\ \ t\in(0,1),\\ y(0)-B_0(y'(0))=\sum_{i=1}^m a_iy(\xi_i)+\left(A-\frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}B\right),\\ y(1)+B_1(y'(1))=\sum_{i=1}^m b_iy(\xi_i).\end{array}\right. \tag{13}$$

Let

$$P_2=\left\{y\in X:\ \ y(t)\geq 0 \text{ for all } t\in[0,1],\ y'(t) \text{ is decreasing on } [0,1],\ \right\}.$$

Then P_2 is a nonempty cone in X. Choose $\sigma_0\in(0,1/2)$. Define the functionals on $P_2\to R$ by

$$\begin{aligned}\beta_1(y)&=\max_{t\in[0,1]}|y'(t)|,\ y\in P_2,\\ \beta_2(y)&=\max_{t\in[0,1]}|y(t)|,\ y\in P_2,\\ \beta_3(y)&=\max_{t\in[\sigma_0,1-\sigma_0]}|y(t)|,\ y\in P_2,\\ \alpha_1(y)&=\min_{t\in[\sigma_0,1-\sigma_0]}|y(t)|,\ y\in P_2,\\ \alpha_2(y)&=\min_{t\in[\sigma_0,1-\sigma_0]}|y(t)|,\ y\in P_2.\end{aligned}$$

Define the nonlinear operator $T_2 : P_2 \to X$ by

$$(T_2y)(t) = B_y - \int_t^1 \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(A_y) + \frac{1}{p(s)}\int_s^1 f(u, y(u) + h_2, y'(u))du\right)ds, \quad y \in P,$$

where

$$A_y \in \left[-\phi^{-1}\left(\frac{\phi(b')}{p(1)} + \frac{1}{p(1)}\int_0^1 f(u, y(u) + h_2, y'(u))du\right), -b\right]$$

satisfies

$$\begin{aligned}
&-\frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}\Big(B_1(A_y) \\
&\qquad + \sum_{i=1}^m b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_y) + \frac{1}{p(s)}\int_s^1 f(u, y(u) + h_2, y'(u))du\right)ds\Big) \\
= &\left(1 - \sum_{i=1}^m a_i\right)\int_0^1 \phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_y) + \frac{1}{p(s)}\int_s^1 f(u, y(u) + h_2, y'(u))du\right)ds \\
&+B_0\left(\phi^{-1}\left(\frac{1}{p(0)}p(1)\phi(A_y) + \frac{1}{p(0)}\int_0^1 f(u, y(u) + h_2, y'(u))du\right)\right) \\
&+\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_y) + \frac{1}{p(s)}\int_s^1 f(u, y(u) + h_2, y'(u))du\right)ds \\
&+\left(A - \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}B\right).
\end{aligned}$$

B_y satisfies

$$B_y = -\frac{B_1(A_y) + \sum_{i=1}^m b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_y) + \frac{1}{p(s)}\int_s^1 f(u, y(u) + h_2, y'(u))du\right)ds}{\gamma - \sum_{i=1}^m b_i}.$$

Then

$$\begin{aligned}
(T_2y)(t) = &-\frac{1}{1-\sum_{i=1}^m b_i}\Big(B_1(A_y) \\
&+\sum_{i=1}^m b_i \int_{\xi_i}^1 \phi^{-1}\left(\frac{1}{p(s)}p(1)\phi(A_y) + \frac{1}{p(s)}\int_s^1 f(u, y(u) + h_2, y'(u))du\right)ds\Big) \\
&-\int_t^1 \phi^{-1}\left(\frac{p(1)}{p(s)}\phi(A_y) + \frac{1}{p(s)}\int_s^1 f(u, y(u) + h_2, y'(u))du\right)ds, \quad y \in P_2.
\end{aligned}$$

Lemma 2.7 *Suppose that $(B1)', (B2), (B3)$ and $(B4)'$ and $(B6)$ hold. It is easy to show that*

(i) *T_2y satisfies the following equalities:*

$$\begin{cases}
[p(t)(\phi(T_2y)')]'(t) + f(t, y(t) + h_2, y'(t)) = 0, \quad t \in (0,1), \\
(T_2y)(0) - B_0((T_2y)'(0)) = \sum_{i=1}^m a_i (T_2y)(\xi_i) + \left(A - \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}B\right), \\
(T_2y)(1) + B_1((T_2y)'(1)) = \sum_{i=1}^m b_i (T_2y)(\xi_i);
\end{cases}$$

(ii) $T_2 y \in P_2$ *for each* $y \in P_2$*;*

(iii) x *is a solution of BVP(7) if and only if* $x = y + h_2$ *and* y *is a solution of the operator equation* $y = T_2 y$ *in* P_2*;*

(iv) $T_2 : P_2 \to P_2$ *is completely continuous.*

Proof. The proof is similar to that of Lemma 2.4 and is omitted. □

3. Main results

In this section, we given the main results and their proofs. Let β is defined in (B6) and

$$\begin{aligned} M &= 1 + \frac{1}{1 - \sum_{i=1}^m a_i}\left(\beta + \sum_{i=1}^m a_i \xi_i\right), \\ L &= \frac{2\beta}{1 - \sum_{i=1}^m a_i}\phi^{-1}\left(\frac{1}{p(0)}\right) + \int_0^1 \phi^{-1}\left(\frac{2-s}{p(s)}\right) ds \\ &\quad + \frac{1}{1 - \sum_{i=1}^m a_i}\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{2-s}{p(s)}\right) ds. \end{aligned}$$

Theorem 3.1 *Choose* $\sigma_0 \in (0, 1/2)$*. Suppose that (B1)-(B4) and* (B6) *hold. Let* e_1, e_2, c *be positive numbers and* Q, W *and* E *given by*

$$\begin{aligned} Q &= \min\left\{\phi\left(\frac{c}{L}\right), \quad \frac{\phi(c)p(0)}{2}, \quad \phi(c)p(1)\right\}; \\ W &= \phi\left(\frac{e_2}{\sigma_0 \min\left\{\int_{\sigma_0}^{\frac{1}{2}} \phi^{-1}\left(\frac{\frac{1}{2}-s}{p(s)}\right) ds, \ \int_{\frac{1}{2}}^{1-\sigma_0} \phi^{-1}\left(\frac{s-\frac{1}{2}}{p(s)}\right) ds\right\}}\right); \\ E &= \min\left\{\phi\left(\frac{e_1}{L}\right), \ \frac{\phi(e_1)p(0)}{2}, \ \phi(e_1)p(1)\right\}. \end{aligned}$$

Let a' *be defined in Section 2. If*

$$\min\{c, e_1\} \geq \max\left\{\frac{e_2}{\sigma_0}, \ La, \ \phi^{-1}\left(\frac{2}{p(0)}\right) a, \ \phi^{-1}\left(\frac{1}{p(1)}\right) a\right\}$$

$$e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$$

and

(B7) $f(t, u, v) \leq Q$ *for all* $t \in [0, 1], u \in [h_1, Mc + h_1], v \in [-c, c]$*;*

(B8) $f(t, u, v) \geq W$ *for all* $t \in [\sigma_0, 1 - \sigma_0], u \in [e_2 + h_1, e_2/\sigma_0 + h_1], v \in [-c, c]$*;*

(B9) $f(t, u, v) \leq E$ *for all* $t \in [0, 1], u \in [h_1, e_1 + h_1], v \in [-c, c]$*;*

then BVP(7) has at least three solutions x_1, x_2, x_3 *such that*

$$\max_{t\in[0,1]} x_1(t) < e_1 + h_1, \quad \min_{t\in[\sigma_0,1-\sigma_0]} x_2(t) > e_2 + h_1,$$

and

$$\max_{t\in[0,1]} x_3(t) > e_1 + h_1, \quad \min_{t\in[\sigma_0,1-\sigma_0]} x_3(t) < e_2 + h_1.$$

Proof. To apply Theorem 2.1, we prove that all conditions in Theorem 2.1 are satisfied.

By the definitions, it is easy to see that α_1, α_2 are nonnegative continuous concave functional on the cone P, $\beta_1, \beta_2, \beta_3$ nonnegative continuous convex functional on the cone P, and $\alpha_1(x) \le \beta_2(x)$ for all $x \in P$. Lemma 2.4 implies that $x = x(t)$ is a positive solution of BVP(7) if and only if $x(t) = y(t) + h_1$ and $y(t)$ is a solution of the operator equation $y = T_1 y$ in P_1. T_1 is completely continuous.

Since $y \in P_1$ implies that

$$y(0) - \beta y'(0) - \sum_{i=1}^{m} a_i y(\xi_i) \le 0, \; y(0) - \beta' y'(0) - \sum_{i=1}^{m} a_i y(\xi_i) \ge 0,$$

we get that

$$\begin{aligned} y(0) &= \frac{y(0) - y(0)\sum_{i=1}^m a_i}{1 - \sum_{i=1}^m a_i} \\ &\le \frac{\beta y'(0) + \sum_{i=1}^m a_i y(\xi_i) - y(0)\sum_{i=1}^m a_i}{1 - \sum_{i=1}^m a_i} \\ &= \frac{1}{1 - \sum_{i=1}^m a_i}\left(\beta y'(0) + \sum_{i=1}^{m} a_i \xi_i y'(\eta_i)\right) \text{ where } \eta_i \in [0, \xi_i] \\ &\le \frac{1}{1 - \sum_{i=1}^m a_i}\left(\beta + \sum_{i=1}^{m} a_i \xi_i\right) \max_{t\in[0,1]} |y'(t)| \end{aligned}$$

and

$$\begin{aligned} y(0) &= \frac{y(0) - y(0)\sum_{i=1}^m a_i}{1 - \sum_{i=1}^m a_i} \\ &\ge \frac{\beta' y'(0) + \sum_{i=1}^m a_i y(\xi_i) - y(0)\sum_{i=1}^m a_i}{1 - \sum_{i=1}^m a_i} \\ &= \frac{1}{1 - \sum_{i=1}^m a_i}\left(\beta' y'(0) + \sum_{i=1}^{m} a_i \xi_i y'(\eta_i)\right) \text{ where } \eta_i \in [0, \xi_i] \\ &\le -\frac{1}{1 - \sum_{i=1}^m a_i}\left(\beta' + \sum_{i=1}^{m} a_i \xi_i\right) \max_{t\in[0,1]} |y'(t)|. \end{aligned}$$

It follows that

$$|y(0)| \le \frac{1}{1 - \sum_{i=1}^m a_i}\left(\beta + \sum_{i=1}^{m} a_i \xi_i\right) \max_{t\in[0,1]} |y'(t)|.$$

Then

$$|y(t)| \le |y(t)-y(0)|+|y(0)| \le \left(1+\frac{1}{1-\sum_{i=1}^m a_i}\left(\beta+\sum_{i=1}^m a_i\xi_i\right)\right)\max_{t\in[0,1]}|y'(t)|.$$

Then

$$\max_{t\in[0,1]}|y(t)| \le \left(1+\frac{1}{1-\sum_{i=1}^m a_i}\left(\beta+\sum_{i=1}^m a_i\xi_i\right)\right)\max_{t\in[0,1]}|y'(t)| = M\max_{t\in[0,1]}|y'(t)|.$$

It follows that there exist constants $M_1>0$ such that $||y|| \le M_1\beta_1(y)$ for all $y\in P_1$. Hence $(A1)$ and $(A2)$ of Theorem 2.1 are satisfied. Now we prove that $(A3)$ holds.

Corresponding to Theorem 2.1,

$$c_1=e_1,\ \ c_2=e_2,\ \ c_3=\sigma_0 e_1,\ \ c_4=\frac{e_2}{\sigma_0},\ \ c_5=c.$$

Now, we prove that $(A3)$ of Theorem 2.1 are satisfied. One sees that $c_1<c_2$ since $e_1<e_2$. The remainder is divided into five steps.

Step 1. Prove that $T_1\overline{P_{c_5}}\subset\overline{P_{c_5}}$;

For $y\in\overline{P_{c_5}}$, we have $||y||\le c$. Then

$$0\le y(t)\le c,\ \ -c\le y'(t)\le c \text{ for all } t\in[0,1].$$

So (B7) implies that

$$f(t,y(t)+h_1,y'(t))<Q,\ t\in[0,1].$$

Then Lemma 2.3 implies

$$A_y\in\left[a,\phi^{-1}\left(\frac{\phi(a')}{p(0)}+\frac{1}{p(0)}\int_0^1 f(u,y(u)+h_1,y'(u))du\right)\right].$$

Since

$$c\ge\max\left\{\frac{e_2}{\sigma_0},\ La',\ \phi^{-1}\left(\frac{2}{p(0)}\right)a',\ \phi^{-1}\left(\frac{1}{p(1)}\right)a'\right\},$$

we get that

$$\phi(a')\le\min\left\{\phi\left(\frac{c}{L}\right),\ \frac{\phi(c)p(0)}{2},\ \phi(c)p(1)\right\}=Q.$$

Hence

$$
\begin{aligned}
(T_1y)(t) &= B_y + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds \\
&= \frac{1}{1-\sum_{i=1}^m a_i}\Bigg(B_0(A_y) \\
&\qquad\qquad + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds\Bigg) \\
&\quad + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds \\
&\le \frac{1}{1-\sum_{i=1}^m a_i}\Bigg(\beta A_y \\
&\qquad\qquad + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds\Bigg) \\
&\quad + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds \\
&\le \frac{\beta}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\frac{\phi(a')}{p(0)} + \frac{1}{p(0)}\int_0^1 f(u,y(u)+h_1,y'(u))du\right) \\
&\quad + \frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{1}{p(s)}\int_s^1 f(u,y(u)+h_1,y'(u))du\right)ds\right) \\
&\quad + \int_0^t \phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{1}{p(s)}\int_s^1 f(u,y(u)+h_1,y'(u))du\right)ds \\
&\le \frac{\beta}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\frac{\phi(a')}{p(0)} + \frac{Q}{p(0)}\right) \\
&\quad + \frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{Q(1-s)}{p(s)}\right)ds\right) \\
&\quad + \int_0^1 \phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{Q(1-s)}{p(s)}\right)ds \\
&\le 2\phi^{-1}(Q)\frac{\beta}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\frac{1}{p(0)}\right) + \frac{\phi^{-1}(Q)}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m a_i \int_0^{\xi_i}\phi^{-1}\left(\frac{2-s}{p(s)}\right)ds\right) \\
&\quad + \phi^{-1}(Q)\int_0^1 \phi^{-1}\left(\frac{2-s}{p(s)}\right)ds \\
&\le c.
\end{aligned}
$$

On the other hand, similarly to above discussion, since $(T_1y)'(t)$ is decreasing and $(T_1y)'(0) \ge 0$, $(T_1y)'(1) \le 0$, we have from Lemma 2.3 that

$$\begin{aligned}
(T_1y)'(t) &= \phi^{-1}\left(\frac{1}{p(t)}p(0)\phi(A_y)-\frac{1}{p(t)}\int_0^t f(u,y(u)+h_1,y'(u))du\right),\\
\max_{t\in[0,1]}|(T_1y)'(t)| &= \max\{|(Ty)'(0)|,\ |(Ty)'(1)|\}\\
&= \max\left\{A_y,\ \phi^{-1}\left(-\frac{1}{p(1)}p(0)\phi(A_y)+\frac{1}{p(1)}\int_0^1 f(u,y(u)+h_1,y'(u))du\right)\right\}\\
&\le \max\left\{\phi^{-1}\left(\frac{\phi(a')}{p(0)}+\frac{1}{p(0)}\int_0^1 f(u,y(u)+h_1,y'(u))du\right),\right.\\
&\qquad \left.\phi^{-1}\left(-\frac{1}{p(1)}p(0)\phi(A_y)+\frac{1}{p(1)}\int_0^1 f(u,y(u)+h_1,y'(u))du\right)\right\}\\
&\le \max\left\{\phi^{-1}\left(\frac{\phi(a')}{p(0)}+\frac{1}{p(0)}Q\right),\ \ \phi^{-1}\left(-\frac{1}{p(1)}p(0)\phi(A_y)+\frac{1}{p(1)}Q\right)\right\}\\
&\le \max\left\{\phi^{-1}\left(\frac{2Q}{p(0)}\right),\ \ \phi^{-1}\left(\frac{Q}{p(1)}\right)\right\}\\
&\le c.
\end{aligned}$$

It follows that

$$||T_1y|| = \max\left\{\max_{t\in[0,1]}|(T_1y)(t)|,\max_{t\in[0,1]}|(T_1y)'(t)|\right\}\le c.$$

Then $T(\overline{P_{c_5}})\subseteq\overline{P_{c_5}}$. This completes the proof of (A3)(i) of Theorem 2.1.

Step 2. Prove that $\{y\in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)|\alpha_1(x)>c_2\}\neq\emptyset$ and

$$\alpha_1(T_1x)>c_2 \text{ for every } x\in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5);$$

It is easy to give a function y such that $y\in P$ and

$$\alpha(y)=\frac{e_2}{2\sigma_0}>e_2,\ \theta(y)=\frac{e_2}{2\sigma_0}\le\frac{e_2}{\sigma_0},\ \gamma(y)=0<c.$$

It follows that $\{y\in P_1(\gamma,\theta,\alpha;a,b,c)|\alpha(y)>a\}\neq\emptyset$.

For $y\in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)$, one has that

$$\alpha_1(y)=\min_{t\in[\sigma_0,1-\sigma_0]}y(t)\ge e_2,\ \beta_3(y)=\max_{t\in[\sigma_0,1-\sigma_0]}y(t)\le\frac{e_2}{\sigma_0},\ \beta_1(y)=\max_{t\in[0,1]}|y'(t)|\le c.$$

Then

$$e_2\le y(t)\le\frac{e_2}{\sigma_0},\ t\in[\sigma_0,1-\sigma_0],\ \ |y'(t)|\le c.$$

Thus $(B8)$ implies that

$$f(t,y(t)+h_1,y'(t))\ge W,\ \ n\in[\sigma_0,1-\sigma_0].$$

Since $T_1y \in P$, we get

$$\alpha_1(T_1y) = \min_{t\in[\sigma_0,1-\sigma_0]}(T_1y)(t) \geq \sigma_0 \max_{t\in[0,1]}(T_1y)(t).$$

It follows from Lemma 2.2 that $(T_1y)'(0) \geq 0$ and $(T_1y)'(1) \leq 0$. Then there exists $\xi \in [0,1]$ such that $(T_1y)'(\xi) = 0$. Then the definition of T_1 implies that

$$\phi((T_1y)'(t)) = \frac{1}{p(t)} \begin{cases} -\int_\xi^t f(u, y(u) + h_1, y'(u))du, & t \geq \xi, \\ \int_t^\xi f(u, y(u) + h_1, y'(u))du, & t \leq \xi. \end{cases}$$

Then

$$(T_1y)(t) = \begin{cases} (T_1y)(1) + \int_t^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(u, y(u) + h_1, y'(u))du\right)ds, & t \geq \xi, \\ (T_1y)(0) + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(u, y(u) + h_1, y'(u))du\right)ds, & t \leq \xi. \end{cases}$$

Hence

$$\begin{aligned} \max_{t\in[0,1]}(T_1y)(t) &= (T_1y)(1) + \int_\xi^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(u, y(u) + h_1, y'(u))du\right)ds \\ &= ((T_1y)(0) + \int_0^\xi \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(u, y(u) + h_1, y'(u))du\right)ds. \end{aligned}$$

Then $(Ty)(0) \geq 0$ and $(Ty)(1) \geq 0$ imply that

$$\begin{aligned} \alpha_1(T_1y) &\geq \sigma_0 \max_{t\in[0,1]}(T_1y)(t) \\ &\geq \sigma_0 \max\left\{\int_\xi^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(u, y(u) + h_1, y'(u))du\right)ds,\right. \\ &\qquad \left.\int_0^\xi \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(u, y(u) + h_1, y'(u))du\right)ds\right\}. \end{aligned}$$

It is easy to see that

$$\int_\xi^1 \phi^{-1}\left(\frac{1}{p(s)}\int_\xi^s f(u, y(u) + h_1, y'(u))du\right)ds \geq \int_{\frac{1}{2}}^{1-\sigma_0} \phi^{-1}\left(\frac{1}{p(s)}\int_{\frac{1}{2}}^s f(u, y(u) + h_1, y'(u))du\right)ds$$

if $\xi \leq \frac{1}{2}$ and

$$\int_0^\xi \phi^{-1}\left(\frac{1}{p(s)}\int_s^\xi f(u, y(u) + h_1, y'(u))du\right)ds \geq \int_{\sigma_0}^{\frac{1}{2}} \phi^{-1}\left(\frac{1}{p(s)}\int_s^{\frac{1}{2}} f(u, y(u) + h_1, y'(u))du\right)ds$$

if $\xi \geq \frac{1}{2}$. We get that

$$\begin{aligned} \alpha_1(T_1y) &\geq \sigma_0 \min\left\{\int_{\frac{1}{2}}^{1-\sigma_0} \phi^{-1}\left(\frac{1}{p(s)}\int_{\frac{1}{2}}^{s} f(u,y(u)+h_1,y'(u))du\right)ds,\right. \\ &\qquad \left.\int_{\sigma_0}^{\frac{1}{2}} \phi^{-1}\left(\frac{1}{p(s)}\int_{s}^{\frac{1}{2}} f(u,y(u)+h_1,y'(u))du\right)ds\right\} \\ &\geq \sigma_0 \min\left\{\int_{\sigma_0}^{\frac{1}{2}} \phi^{-1}\left(W\frac{\frac{1}{2}-s}{p(s)}\right)ds,\ \int_{\frac{1}{2}}^{1-\sigma_0} \phi^{-1}\left(W\frac{s-\frac{1}{2}}{p(s)}\right)ds\right\} \\ &= e_2. \end{aligned}$$

It follows that $\alpha_1(T_1x) > c_2$ for every $x \in P(\beta_1,\beta_3,\alpha_1;c_2,c_4,c_5)$. This completes the proof of (A3)(ii) of Theorem 2.1.

Step 3. Prove that $\{y \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5)|\beta_2(x) < c_1\} \neq \emptyset$ and

$$\beta_2(T_1x) < c_1 \text{ for every } x \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5);$$

It is easy to give a function y such that $y \in P$ and

$$\alpha_2(y) \geq c_3,\ \beta_2(y) < c_1,\ \beta_1(y) \leq c_5.$$

It follows that $\{x \in P(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5) : \alpha_2(x) < c_1\} \neq \emptyset$.

For $y \in Q(\beta_1,\beta_2,\alpha_2;c_3,c_1,c_5)$, one has that

$$\alpha_2(y) = \min_{n\in[\sigma_0,1-\sigma_0]} y(t) \geq c_3,\ \beta_2(y) = \max_{t\in[0,1]} y(t) \leq c_1\ \beta_1(y) = \max_{t\in[0,1]} |y'(t)| \leq c_5.$$

Then

$$0 \leq y(t) \leq e_1, \quad |y'(t)| \leq c,\ \ t \in [0,1].$$

Thus (B9) implies that

$$f(t,y(t)+h_1,y'(t)) \leq E,\ \ t \in [0,1].$$

Since

$$e_1 \geq \max\left\{\frac{e_2}{\sigma_0},\ La',\ \phi^{-1}\left(\frac{2}{p(0)}\right)a',\ \phi^{-1}\left(\frac{1}{p(1)}\right)a'\right\},$$

we get

$$\phi(a') \leq \min\left\{\phi\left(\frac{e_1}{L}\right),\ \frac{\phi(e_1)p(0)}{2},\ \phi(e_1)p(1)\right\} = E.$$

So

$$
\begin{aligned}
(T_1y)(t) &= B_y + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds \\
&= \frac{1}{1-\sum_{i=1}^m a_i}\Big(B_0(A_y) \\
&\qquad + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds\Big) \\
&\quad + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds \\
&\le \frac{1}{1-\sum_{i=1}^m a_i}\Big(\beta A_y \\
&\qquad + \sum_{i=1}^m a_i \int_0^{\xi_i} \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds\Big) \\
&\quad + \int_0^t \phi^{-1}\left(\frac{1}{p(s)}p(0)\phi(A_y) - \frac{1}{p(s)}\int_0^s f(u,y(u)+h_1,y'(u))du\right)ds \\
&\le \frac{\beta}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\frac{\phi(a')}{p(0)} + \frac{1}{p(0)}\int_0^1 f(u,y(u)+h_1,y'(u))du\right) \\
&\quad + \frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m a_i \int_0^{\xi_i}\phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{1}{p(s)}\int_s^1 f(u,y(u)+h_1,y'(u))du\right)ds\right) \\
&\quad + \int_0^t \phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{1}{p(s)}\int_s^1 f(u,y(u)+h_1,y'(u))du\right)ds \\
&\le \frac{\beta}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\frac{\phi(a')}{p(0)} + \frac{E}{p(0)}\right) \\
&\quad + \frac{1}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m a_i \int_0^{\xi_i}\phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{E(1-s)}{p(s)}\right)ds\right) \\
&\quad + \int_0^1 \phi^{-1}\left(\frac{\phi(a')}{p(s)} + \frac{E(1-s)}{p(s)}\right)ds \\
&\le 2\phi^{-1}(E)\frac{\beta}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\frac{1}{p(0)}\right) + \frac{\phi^{-1}(E)}{1-\sum_{i=1}^m a_i}\left(\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{2-s}{p(s)}\right)ds\right) \\
&\quad + \phi^{-1}(E)\int_0^1 \phi^{-1}\left(\frac{2-s}{p(s)}\right)ds \\
&\le e_1 = c_1.
\end{aligned}
$$

Hence $\max_{t\in[0,1]}(T_1y)(t) \le c_1$. it follows that $\beta_2(T_1y) < c_1$. This completes the proof of (A3)(iii) of Theorem 2.1.

Step 4. Prove that $\alpha_1(T_1y) > c_2$ for $y \in P(\beta_1, \alpha_1; c_2, c_5)$ with $\beta_3(T_1y) > c_4$;

For $y \in P(\beta_1, \alpha_1; c_2, c_5)$ with $\beta_3(T_1y) > c_4$, we have that

$$\alpha_1(y) = \min_{t\in[\sigma_0,1-\sigma_0]} y(t) \geq c_2 = e_2$$

$$\beta_1(y) = \max_{t\in[0,1]} |y'(t)| \leq c_5$$

and

$$\beta_3(T_1y) = \max_{t\in[\sigma_0,1-\sigma_0]} (T_1y)(t) > \frac{e_2}{\sigma_0} = c_4.$$

Then

$$\alpha_1(T_1y) = \min_{t\in[\sigma_0,1-\sigma_0]} (T_1y)(t) \geq \sigma_0\beta_2(T_1y) > \sigma_0\frac{e_2}{\sigma_0} = e_2 = c_2.$$

This completes the proof of (A3)(iv) of Theorem 2.1.

Step 5. Prove that $\beta_2(T_1x) < c_1$ for each $x \in Q(\beta_1, \beta_2; c_1, c_5)$ with $\alpha_2(T_1x) < c_3$;

For $y \in Q(\beta_1, \beta_2; c_1, c_5)$ with $\alpha_2(T_1y) < c_3$, we have that

$$\beta_2(y) = \min_{t\in[\sigma_0,1-\sigma_0]} y(t) \leq e_1$$

and

$$\beta_1(y) = \max_{t\in[0,1]} |y'(t)| \leq c_5$$

and

$$\alpha_2(T_1y) = \min_{t\in[\sigma_0,1-\sigma_0]} (T_1y)(t) < c_3 = \sigma_0 e_1.$$

Using (B) and the methods in Step 3, we get

$$\beta_2(T_1y) = \max_{t\in[0,1]} (T_1y)(t) < c_1.$$

This completes the proof of (A3)(v) of Theorem 2.1.

Then Theorem 2.1 implies that T_1 has at least three fixed points y_1, y_2 and y_3 such that

$$\beta_2(y_1) < e_1,\ \alpha_1(y_2) > e_2,\ \beta_2(y_3) > e_1,\ \alpha_1(y_3) < e_2.$$

Hence BVP(7) has three decreasing positive solutions x_1, x_2 and x_3 such that

$$\max_{t\in[0,1]} y_1(t) < e_1 + h_1,\quad \min_{t\in[\sigma_0,1-\sigma_0]} y_2(t) > e_2 + h_1,$$

and

$$\max_{t\in[0,1]} y_3(t) > e_1 + h_1,\quad \min_{t\in[\sigma_0,1-\sigma_0]} y_3(t) < e_2 + h_1.$$

The proof of Theorem 3.1 is completed. □

Let δ is defined in (B6) and

$$\begin{aligned} M' &= 1+\frac{1}{1-\sum_{i=1}^m b_i}\left(\delta+\sum_{i=1}^m b_i(1-\xi_i)\right), \\ L &= \frac{2\delta}{1-\sum_{i=1}^m b_i}\phi^{-1}\left(\frac{1}{p(1)}\right)+\int_0^1 \phi^{-1}\left(\frac{1+s}{p(s)}\right)ds \\ &\quad +\frac{1}{1-\sum_{i=1}^m b_i}\sum_{i=1}^m b_i\int_{\xi_i}^1 \phi\left(\frac{1+s}{p(s)}\right)ds. \end{aligned}$$

Theorem 3.2 *Choose $\sigma_0 \in (0,1/2)$. Suppose that $(B1)', (B2), (B3), (B4)'$ and $(B6)$ hold. Let e_1, e_2, c be positive numbers and Q, W and E given by*

$$\begin{aligned} Q &= \min\left\{\phi\left(\frac{c}{L}\right), \quad \frac{\phi(c)p(0)}{2}, \quad \phi(c)p(1)\right\}; \\ W &= \phi\left(\frac{e_2}{\sigma_0\min\left\{\int_{\sigma_0}^{\frac{1}{2}}\phi^{-1}\left(\frac{\frac{1}{2}-s}{p(s)}\right)ds, \quad \int_{\frac{1}{2}}^{1-\sigma_0}\phi^{-1}\left(\frac{s-\frac{1}{2}}{p(s)}\right)ds\right\}}\right); \\ E &= \phi\left(\frac{e_1}{L}\right). \end{aligned}$$

Let b be defined in Section 2. If

$$M'c > c \geq \max\left\{\frac{e_2}{\sigma_0},\ Lb,\ \phi^{-1}\left(\frac{2}{p(1)}\right)b,\ \phi^{-1}\left(\frac{1}{p(0)}\right)b\right\}$$

$$e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$$

and

(B10) $f(t,u,v) < Q$ *for all* $t \in [0,1], u \in [h_2, M'c+h_2], v \in [-c,c]$*;*

(B11) $f(t,u,v) > W$ *for all* $t \in [\sigma_0, 1-\sigma_0], u \in [e_2+h_2, e_2/\sigma_0+h_2], v \in [-c,c]$*;*

(B12) $f(t,u,v) \leq E$ *for all* $t \in [0,1], u \in [h_2, e_1/\sigma_0+h_2], v \in [-c,c]$*;*

then BVP(7) has at least three solutions x_1, x_2, x_3 such that

$$\max_{t\in[0,1]} x_1(t) < e_1+h_2, \quad \min_{t\in[\sigma_0,1-\sigma_0]} x_2(t) > e_2+h_2,$$

and

$$\max_{t\in[0,1]} x_3(t) > e_1+h_2, \quad \min_{t\in[\sigma_0,1-\sigma_0]} x_3(t) < e_2+h_2.$$

Proof. The proof is similar to that of the proof of Theorem 3.1 and is omitted. □

Remark 3.1 Consider the corresponding difference equation related to BVP(7),

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + f(n, x(n), \Delta x(n)) = 0, \quad n \in [0, N]), \\ x(0) - B_0(\Delta x(0)) = \sum_{i=1}^m a_i x(n_i) + A, \\ x(N+2) + B_1(\Delta x(N+1)) = \sum_{i=1}^m b_i x(n_i) + B, \end{cases}$$

where $A, B \in R$, $0 < n_1 < n_2 < \cdots < n_m$ are integers, $a_i, b_i \in R$ for $i = 1, 2, \cdots, m$, $B_0, B_1 : R \to R$ are continuous functions, $f : [0, N] \times R^2 \to R$ is continuous. It is interesting to work on this kind of BVP by using these types of techniques.

Remark 3.2 It is interesting to apply our methods to establish three solutions of the following BVPs

$$\begin{cases} [p(t)\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \\ x(0) - B_0(x'(0)) = \sum_{i=1}^{m} a_i x'(\xi_i) + A, \\ x(1) + B_1(x'(1)) = \sum_{i=1}^{m} b_i x(\xi_i) + B, \end{cases}$$

$$\begin{cases} [p(t)\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \\ x(0) - B_0(x'(0)) = \sum_{i=1}^{m} a_i x'(\xi_i) + A, \\ x(1) + B_1(x'(1)) = \sum_{i=1}^{m} b_i x'(\xi_i) + B, \end{cases}$$

and

$$\begin{cases} [p(t)\phi(x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \\ x(0) - B_0(x'(0)) = \sum_{i=1}^{m} a_i x(\xi_i) + A, \\ x(1) + B_1(x'(1)) = \sum_{i=1}^{m} b_i x'(\xi_i) + B. \end{cases}$$

Since the methods are similar, we omit the details. The readers should try it.

4. Examples

Now, we present two examples to illustrate the main results.

Example 4.1 *Consider the following BVP of second order differential equation with nonlinear boundary conditions*

$$\begin{cases} x'' + f(t, x(t), x'(t)) = 0, \ \ t \in (0, 1), \\ x(0) - B_0(x'(0)) = \frac{1}{2}x(1/2) + 2, \\ x(1) + B_1(x'(1)) = \frac{1}{4}x(1/4) + \frac{1}{4}x(1/2) + 8, \end{cases} \tag{14}$$

where $B_1(x) = 2B_0(x)$,

$$B_0(x) = \begin{cases} \frac{x}{1+e^{-x}} \ \textit{for}\ x \geq 0, \\ \frac{x}{1+e^{x}} \ \textit{for}\ x \leq 0 \end{cases}$$

$$f(t, u, v) = f_0(u) + \frac{t|v|}{25500000}$$

and

$$f_0(u) = \begin{cases} \frac{2}{51}u, & u \in [0, 4], \\ \frac{8}{51}, & u \in [4, 12], \\ 564000 + \frac{564000 - \frac{8}{51}}{1004 - 12}(u - 1004), & u \in [12, 1004], \\ 564000, & u \in [1004, 4004], \\ 564000, & u \in [4004, 7000004], \\ 564000e^{u - 7000004}, & u \geq 7000004. \end{cases}$$

Corresponding to BVP(7), one sees that $\phi(x) = x$, $p(t) \equiv 1$, $\xi_1 = 1/4, \xi_2 = 1/2$, $a_1 = 0, a_2 = 1/2$, $b_1 = 1/4, b_2 = 1/4$, $A = 2, B = 8$. *It is easy to see that* $\frac{2}{1-1/2} < \frac{8}{1-1/4-1/4}$.

Use Theorem 3.1 *It is easy to see that*

$$xB_0(x) \le x^2,\ xB_1(x) \le 2x^2,\ \ x \in R.$$

One sees that (B1)-(B4) hold,(B6) hold with $\beta' = 0, \delta' = 0$, $\beta = 1$ *and* $\delta = 2$, $h_1 = \frac{2}{1-1/2} = 4$. *Choose constants* $\sigma_0 = \frac{1}{4}$, $e_1 = 2, e_2 = 1000, c = 2000000$, *then*

$$\begin{aligned}
l_1 &= \frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}\left(\beta+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds\right) = \frac{5}{4},\\
l_2 &= \left(1-\sum_{i=1}^m b_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds = \frac{1}{2},\\
l_3 &= \delta\phi^{-1}\left(\frac{1}{p(1)}p(0)\right)+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(0)\right)ds = \frac{37}{16},\\
a &= \frac{B-\frac{1-\sum_{i=1}^m b_i}{1-\sum_{i=1}^m a_i}A}{l_1+l_2+l_3} = \frac{96}{65},\\
L &= \frac{2\beta}{1-\sum_{i=1}^m a_i}\phi^{-1}\left(\frac{1}{p(0)}\right)+\int_0^1\phi^{-1}\left(\frac{2-s}{p(s)}\right)ds\\
&\quad +\frac{1}{1-\sum_{i=1}^m a_i}\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{2-s}{p(s)}\right)ds = \frac{51}{8},\\
M &= 1+\frac{1}{1-\sum_{i=1}^m a_i}\left(\beta+\sum_{i=1}^m a_i\xi_i\right) = \frac{7}{2},\\
Q &= \min\left\{\phi\left(\frac{c}{L}\right),\ \ \frac{\phi(c)p(0)}{2},\ \ \phi(c)p(1)\right\} = 1000000;\\
W &= \phi\left(\frac{e_2}{\sigma_0\min\left\{\int_{\sigma_0}^{\frac{1}{2}}\phi^{-1}\left(\frac{\frac{1}{2}-s}{p(s)}\right)ds,\ \ \int_{\frac{1}{2}}^{1-\sigma_0}\phi^{-1}\left(\frac{s-\frac{1}{2}}{p(s)}\right)ds\right\}}\right) = 128000;\\
E &= \phi\left(\frac{e_1}{L}\right) = \frac{16}{51}.
\end{aligned}$$

It is easy to see that

$$c \ge \max\left\{\frac{e_2}{\sigma_0},\ La,\ \phi^{-1}\left(\frac{2}{p(0)}\right)a,\ \phi^{-1}\left(\frac{1}{p(1)}\right)a\right\}$$

$$e_2 > \frac{e_1}{\sigma_0} > e_1 > 0$$

and

$f(t,u,v) \le 1000000$ *for all* $t \in [0,1], u \in [4, 7000004], v \in [-2000000, 2000000]$;

$f(t,u,v) \geq 128000$ for all $t \in [1/4, 3/4], u \in [1004, 4004], v \in [-2000000, 2000000]$;

$f(t,u,v) \leq \frac{16}{51}$ for all $t \in [0,1], u \in [4, 12], v \in [-2000000, 2000000]$;

Hence $(B7), (B8), (B9)$ hold. Then applying Theorem 3.1 BVP(14) has at least three solutions x_1, x_2, x_3 such that

$$\max_{t\in[0,1]} x_1(t) < 6, \quad \min_{t\in[1/4,3/4]} x_2(t) > 1004,$$

and

$$\max_{t\in[0,1]} x_3(t) > 6, \quad \min_{t\in[1/4,3/4]} x_3(t) < 1004.$$

Example 4.2 *Consider the following BVP of second order differential equation with nonlinear boundary conditions*

$$\begin{cases} [|x'(t)|^2 x'(t)]' + f(t, x(t), x'(t)) = 0, & t \in (0,1), \\ x(0) - B_0(x'(0)) = 4, \\ x(1) + B_1(x'(1)) = 2, \end{cases} \tag{15}$$

where $B_1(x) = 2B_0(x)$,

$$B_0(x) = \begin{cases} \frac{x}{1+e^{-x}} & \text{for } x \geq 0, \\ \frac{x}{1+e^{x}} & \text{for } x \leq 0 \end{cases}$$

$$f(t,u,v) = f_0(u) + \frac{t|v|}{255 \times 10^8}$$

and

$$f_0(u) = \begin{cases} \frac{128}{31^3} u, & u \in [0,2], \\ \frac{256}{31^3}, & u \in [2,10], \\ 4 \times 10^{18} + \frac{4\times 10^{18} - \frac{256}{31^3}}{1002-10}(u - 1002), & u \in [10, 1002], \\ 4 \times 10^{18}, & u \in [1002, 40022], \\ 4 \times 10^{18}, & u \in [4002, 6000002], \\ 4 \times 10^{18} e^{u-60000042}, & u \geq 6000002. \end{cases}$$

Corresponding to BVP(7), one sees that $\phi(x) = |x|^2 x$ with $p = 4$ is a one-dimensional $p-$laplacian, $p(t) \equiv 1$, $m = 0$ or $a_i = 0, b_i = 0$ for all i, $A = 4, B = 2$. It is easy to see that $A > B$, (B2), (B3) and (B4)$'$ hold and

(B1)$'$ *$1 - \sum_{i=1}^m b_i = 1 \neq 0$, $h_2 = \frac{B}{1-\sum_{i=1}^m b_i} = 2$, $f : [0,1] \times [2, +\infty) \times R \to [0, +\infty)$ is continuous with $f(t,0,0) \not\equiv 0$ on each sub-interval of [0,1];*

(B6) *there exist nonnegative numbers $\beta = 1, \beta' = \frac{1}{2}, \delta = 2, \delta' = 1$ such that $\beta' x^2 \leq x B_0(x) \leq \beta x^2$ and $\delta' x^2 \leq x B_1(x) \leq \delta x^2$ for all $x \in R$.*

To use Theorem 3.2, one sees

$$
\begin{aligned}
M' &= 1+\frac{1}{1-\sum_{i=1}^m b_i}\left(\delta+\sum_{i=1}^m b_i(1-\xi_i)\right)=3,\\
L &= \frac{2\delta}{1-\sum_{i=1}^m b_i}\phi^{-1}\left(\frac{1}{p(1)}\right)+\int_0^1\phi^{-1}\left(\frac{1+s}{p(s)}\right)ds\\
&\quad +\frac{1}{1-\sum_{i=1}^m b_i}\sum_{i=1}^m b_i\int_{\xi_i}^1\phi\left(\frac{1+s}{p(s)}\right)ds=\frac{31}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
m_1 &= \frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}\left(\delta+\sum_{i=1}^m b_i\int_{\xi_i}^1\phi^{-1}\left(\frac{1}{p(s)}p(1)\right)ds\right)=2,\\
m_2 &= \left(1-\sum_{i=1}^m a_i\right)\int_0^1\phi^{-1}\left(\frac{1}{p(s)}p(1)\right)ds=\frac{3}{2},\\
m_3 &= \beta\phi^{-1}\left(\frac{1}{p(0)}p(1)\right)+\sum_{i=1}^m a_i\int_0^{\xi_i}\phi^{-1}\left(\frac{1}{p(s)}p(1)\right)ds=1,\\
b &= \frac{A-\frac{1-\sum_{i=1}^m a_i}{1-\sum_{i=1}^m b_i}B}{m_1+m_2+m_3}=\frac{4}{9}.
\end{aligned}
$$

Choose $e_1=2, e_2=1000, c=2000000$ *and* $\sigma_0=\frac{1}{4}$ *and* Q, W *and* E *given by*

$$
\begin{aligned}
Q &= \min\left\{\phi\left(\frac{c}{L}\right),\quad \frac{\phi(c)p(0)}{2},\quad \phi(c)p(1)\right\}=8\times 10^{18};\\
W &= \phi\left(\frac{e_2}{\sigma_0\min\left\{\int_{\sigma_0}^{\frac12}\phi^{-1}\left(\frac{\frac12-s}{p(s)}\right)ds,\ \int_{\frac12}^{1-\sigma_0}\phi^{-1}\left(\frac{s-\frac12}{p(s)}\right)ds\right\}}\right)=\frac{64000\sqrt[3]{4}}{3};\\
E &= \phi\left(\frac{e_1}{L}\right)=\frac{8^3}{31^3}.
\end{aligned}
$$

Let b *be defined in Section 2. If*

$$
M'c>c\ge\max\left\{\frac{e_2}{\sigma_0},\ Lb,\ \phi^{-1}\left(\frac{2}{p(1)}\right)b,\ \phi^{-1}\left(\frac{1}{p(0)}\right)b\right\}
$$

$$
e_2>\frac{e_1}{\sigma_0}>e_1>0
$$

and

(B10) $f(t,u,v)<8\times 10^{18}$ *for all* $t\in[0,1], u\in[2,6000002], v\in[-2000000,2000000]$*;*

(B11) $f(t,u,v)>\frac{64000\sqrt[3]{4}}{3}$ *for all* $t\in\left[\frac14,\frac34\right], u\in[1002,4002], v\in[-2000000,2000000]$*;*

(B12) $f(t,u,v) \le \frac{8^3}{31^3}$ *for all* $t \in [0,1], u \in [2,10], v \in [-2000000, 2000000]$*;*

By Theorem 3.2, BVP(15) has at least three solutions x_1, x_2, x_3 *such that*

$$\max_{t\in[0,1]} x_1(t) < 4, \quad \min_{t\in[\frac{1}{4},\frac{3}{4}]} x_2(t) > 1002,$$

and

$$\max_{t\in[0,1]} x_3(t) > 4, \quad \min_{t\in[\frac{1}{4},\frac{3}{4}]} x_3(t) < 1002.$$

Remark 4.1 One can not get three solutions of BVPs in Examples 4.1–4.2 by using the theorems obtained in papers [1–6, 8, 10-13, 18, 16, 24–29].

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