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Blow-up time for a semilinear parabolic equation with variable reaction

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Abstract

In this paper, we address the solution of a semilinear heat equation with variable reaction subject to Dirichlet boundary conditions and nonnegative initial datum. Under some assumptions, we show that the solution of the above problem blows up in a finite time, and its blow-up time goes to that of the solution of a certain differential equation. Finally, we give some numerical results to illustrate our analysis.

Key Words: Semilinear parabolic equation, blow-up, numerical blow-up time.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Consider the initial-boundary value problem for a semilinear parabolic equation with variable reaction subject to Dirichlet boundary conditions of the form

$$u_t(x,t) = \varepsilon\Delta u(x,t) + e^{p(x)u(x,t)} \quad \text{in} \quad \Omega\times(0,T), \tag{1.1}$$

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \tag{1.3}$$

where $p \in \mathrm{C}^1(\overline{\Omega})$, $\sup_{x\in\overline{\Omega}} p(x) = p_0 > 0$, Δ is the Laplacian and ε a positive parameter. The initial datum $u_0 \in C^1(\overline{\Omega})$ and $u_0(x)$ is nonnegative in Ω. Here, $(0,T)$ is the maximal time interval on which the solution u exists. The time T may be finite or infinite. When T is infinite, then we say that the solution u exists globally. When T is finite, then the solution u develops a singularity in a finite time, namely,

$$\lim_{t\to T} \|u(\cdot,t)\|_\infty = \infty,$$

where $\|u(\cdot,t)\|_\infty = \sup_{x\in\Omega} |u(x,t)|$. In this last case, we say that the solution u blows up in a finite time, and the time T is called the blow-up time of the solution u.

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Throughout this paper, we suppose that there exists $a \in \Omega$ such that

$$M = \sup_{x \in \overline{\Omega}} u_0(x) = u_0(a) \text{ and } p_0 = \sup_{x \in \overline{\Omega}} p(x) = p(a).$$

For our problem described in (1.1)–(1.3), it is well known that the local in time existence and uniqueness of a classical solution have been proved (see [9], [13], [19]). Solutions of semilinear parabolic equations which blow up in a finite time have been the subject of investigations by many authors (see, [4], [6]–[8], [10]–[12], [14], [18], [19], and the references cited therein). In particular, in the case where $p(x) = q$, q being a positive constant, the phenomenon of blow-up has been studied for the problem considered in (1.1)–(1.3) (see, [6]–[8], [14]). The particularity of our problem is that the potential $p(x)$ can take negative values. Due to this fact, to the best of our knowledge, the phenomenon of blow-up regarding small diffusions and large initial datum for the above problem has not been treated. Our work was motivated by the paper of Friedman and Lacey in [8], where they considered the initial-boundary value problem

$$u_t = \varepsilon \Delta u + f(u) \quad \text{in} \quad \Omega \times (0, T),$$

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0, T),$$

$$u(x, 0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega.$$

Here, $f(s)$ is a positive, increasing, convex function for nonnegative values of s, $\int_0^\infty \frac{ds}{f(s)} < \infty$, ε is a positive parameter, and the initial datum $u_0(x)$ is a continuous function in Ω. Under some additional conditions on the initial datum, they proved that the solution u of the above problem blows up in a finite time, and its blow-up time goes to that of the solution of the differential equation

$$\alpha^{'}(t) = f(\alpha(t)), \quad \alpha(0) = M, \tag{1.4}$$

as ε goes to zero. In the same way, in [16], Nabongo and Boni obtained a result as the one found by Friedman and Lacey for solutions which quench in a finite time (we say that a solution quenches in a finite time if it reaches a singular value in a finite time). Let us notice that for this kind of problems, other parameters have been taken such that the L^∞ norm of the initial datum (see for instance [10]). One may also consult the paper of Boni and Diby in [5] for an analogous problem where the L^∞ norm of the initial datum has been taken as parameter within the framework of the phenomenon of quenching. In the present paper, we take either the L^∞ norm of the initial datum or ε as parameter, and obtain analogous results using both a modification of Kaplan's method (see, [12]) and a method based on the construction of upper solutions.

The remainder of the paper is organized as follows. In the next section, under some assumptions, we show that when $\varepsilon = 1$, the solution u of (1.1)–(1.3) blows up in a finite time and its blow-up time tends to that of the solution of a certain differential equation when the L^∞ norm of the initial datum is large enough. We then obtain an analogous result for small diffusions. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Blow-up solutions

In this section, under some assumptions, we show that the solution u of (1.1)–(1.3) blows up in a finite time, and its blow-up time tends to that of the solution of a certain differential equation. Our first result concerns the case where $\varepsilon = 1$ and the L^∞ norm of the initial datum is large enough, and our second result handles the case where the diffusion is small enough. In the introduction of the paper, we mentioned that there exists $a \in \Omega$ such that

$$M = \sup_{x\in\overline{\Omega}} u_0(x) = u_0(a) \text{ and } p_0 = \sup_{x\in\overline{\Omega}} p(x) = p(a).$$

Consider the following eigenvalue problem

$$-\Delta\varphi = \lambda_\delta\varphi \quad \text{in} \quad B(a,\delta), \tag{2.1}$$

$$\varphi = 0 \quad \text{on} \quad \partial B(a,\delta), \tag{2.2}$$

$$\varphi > 0 \quad \text{in} \quad B(a,\delta), \tag{2.3}$$

where $\delta > 0$, such that, $B(a,\delta) = \{x \in \mathbb{R}^N; \|x-a\| < \delta\} \subset \Omega$. It is well known that the above eigenvalue problem admits a solution $(\varphi, \lambda_\delta)$ such that $0 < \lambda_\delta = \frac{D}{\delta^2}$, where D is a positive constant which depends only on the dimension N, and we can normalize φ so that $\int_{B(a,\delta)} \varphi(x)dx = 1$.

Now, we are in a position to state the main result of this paper in the case where $\varepsilon = 1$ and the L^∞ norm of the initial datum is large enough.

Theorem 2.1 *Let K be an upper bound of the first derivatives of u_0 and p. Assume that*

$$M > \max\{(p_0/2)^{-1/2}, 2^{1/3}, A, (Kdist(a,\partial\Omega))^{-1/2}\},$$

where $A = \frac{4DK^2 8^5 5!}{p_0^6}$. Then the solution u of (1.1)–(1.3) blows up in a finite time, and its blow-up time T satisfies the following estimates

$$0 \le T - T_M \le \big(1 + p_0 + A)M^{-1}T_M + o(M^{-1}T_M),$$

where $T_M = \frac{e^{-p_0 M}}{p_0}$ is the blow-up time of the solution $\alpha(t)$ of the differential equation defined as

$$\alpha^{'}(t) = e^{p_0\alpha(t)}, \quad t > 0, \quad \alpha(0) = M. \tag{2.4}$$

Proof. Since $(0,T)$ is the maximal time interval of existence of the solution u, our aim is to show that T is finite and satisfies the above inequalities. Due to the fact that the initial datum u_0 is nonnegative in Ω, owing to the maximum principle, we see that u is also nonnegative in $\Omega \times (0,T)$. Applying the mean value theorem and the triangle inequality, we find that

$$u_0(x) \ \ge \ M - M^{-2} \quad \text{for} \quad x \in B(a,\delta), \tag{2.5}$$

$$p(x) \ \ge \ p_0 - M^{-2} \quad \text{for} \quad x \in B(a,\delta), \tag{2.6}$$

where $\delta = \frac{M^{-2}}{K}$. Introduce the function $v(t)$ defined as follows

$$v(t) = \int_{B(a,\delta)} u(x,t)\varphi(x)dx \quad \text{for} \quad t \in [0,T).$$

Take the derivative of v in t and use (1.1) to obtain

$$v'(t) = \int_{B(a,\delta)} \Delta u(x,t)\varphi(x)dx + \int_{B(a,\delta)} e^{p(x)u(x,t)}\varphi(x)dx \quad \text{for} \quad t \in (0,T).$$

Applying Green's formula, we arrive at

$$\begin{aligned} v'(t) &= \int_{B(a,\delta)} u(x,t)\Delta\varphi(x)dx + \int_{\partial B(a,\delta)} \varphi(x)\frac{\partial u(x,t)}{\partial \nu}ds \\ &- \int_{\partial B(a,\delta)} u(x,t)\frac{\partial \varphi(x)}{\partial \nu}ds + \int_{B(a,\delta)} e^{p(x)u(x,t)}\varphi(x)dx \quad \text{for} \quad t \in (0,T), \end{aligned}$$

ν being the exterior normal unit vector to $\partial\Omega$. It is well known that $\frac{\partial\varphi(x)}{\partial\nu} < 0$ for $x \in \partial B(a,\delta)$. In view of the above inequality, and using (2.1)–(2.2), we deduce that

$$v'(t) \geq -\lambda_\delta v(t) + \int_{B(a,\delta)} e^{p(x)u(x,t)}\varphi(x)dx \quad \text{for} \quad t \in (0,T). \tag{2.7}$$

Since the function $x \longmapsto e^x$ is nondecreasing, then taking into account (2.6) and (2.7), we find that

$$v'(t) \geq -\lambda_\delta v(t) + \int_{B(a,\delta)} e^{(p_0-M^{-2})u(x,t)}\varphi(x)dx \quad \text{for} \quad t \in (0,T). \tag{2.8}$$

Apply Jensen's inequality to arrive at

$$v'(t) \geq -\lambda_\delta v(t) + e^{(p_0-M^{-2})v(t)} \quad \text{for} \quad t \in (0,T), \tag{2.9}$$

which implies that

$$v'(t) \geq e^{(p_0-M^{-2})v(t)}\left(1 - DK^2M^4v(t)e^{(-p_0+M^{-2})v(t)}\right) \quad \text{for} \quad t \in (0,T), \tag{2.10}$$

because $\lambda_\delta = \frac{D}{\delta^2} = DK^2M^4$. It is easy to see that $0 \leq v(t) \leq \frac{4}{p_0}e^{(p_0/4)v(t)}$ for $t \in (0,T)$, and $e^{(-p_0+M^{-2})v(t)} \leq e^{-(p_0/2)v(t)}$ for $t \in (0,T)$. It follows from (2.10) that

$$v'(t) \geq e^{(p_0-M^{-2})v(t)}\left(1 - \frac{4DK^2M^4}{p_0}e^{-(p_0/4)v(t)}\right) \quad \text{for} \quad t \in (0,T). \tag{2.11}$$

We observe that $e^{\frac{p_0M}{8}} \geq \frac{p_0^5M^5}{8^55!}$, which implies that $\frac{4DK^2M^4}{p_0}e^{-p_0M/8} \leq AM^{-1}$. In view of the above inequality, and using the fact that $v(0) \geq M/2$, we observe that $v'(0) > 0$, and we claim that $v'(t) > 0$ for $t \in (0,T)$.

To prove the claim, we argue by contradiction. Let $t_0 \in (0,T)$ be the first t such that $v'(t) > 0$ for $t \in [0,t_0)$ but $v'(t_0) = 0$. This implies that $v(t_0) \geq v(0) \geq M/2$, and

$$0 = v'(t_0) \geq e^{(p_0 - M^{-2})v(0)}(1 - AM^{-1}) > 0,$$

which is a contradiction. Consequently, we have $v(t) \geq v(0) \geq M/2$ for $t \in (0,T)$, which leads us to $v'(t) \geq e^{(p_0 - M^{-2})v(t)}\big(1 - AM^{-1}\big)$ for $t \in (0,T)$, or equivalently, $e^{(-p_0 + M^{-2})v}dv \geq \big(1 - AM^{-1}\big)dt$ for $t \in (0,T)$. Integrate the above inequality over $(0,T)$ to obtain $\frac{e^{(-p_0+M^{-2})v(0)}}{p_0 - M^{-2}} \geq (1 - AM^{-1})T$, which implies that

$$T \leq \frac{e^{-p_0 M}}{p_0} \frac{e^{(1+p_0)M^{-1}}}{(1 - AM^{-1})(1 - \frac{M^{-2}}{p_0})}. \tag{2.12}$$

We deduce that the solution u blows up in a finite time because the quantity on the right hand side of the above inequality is finite. Apply Taylor's expansion to obtain

$$\frac{e^{(1+p_0)M^{-1}}}{(1 - AM^{-1})(1 - \frac{M^{-2}}{p_0})} = 1 + (1 + p_0 + A)M^{-1} + o(M^{-1}).$$

Using (2.12) and the above relation, we find that

$$T \leq T_M + (1 + p_0 + A)M^{-1}T_M + o(M^{-1}T_M),$$

and the second estimate of the theorem is shown. In order to prove the first one, we proceed in the following manner. We recall that $\alpha(t)$ is the solution of the Cauchy problem below

$$\alpha'(t) = e^{p_0 \alpha(t)}, \quad t > 0, \quad \alpha(0) = M.$$

Since $p(x) \leq p_0$ in Ω, we see that

$$u_t(x,t) \leq \Delta u(x,t) + e^{p_0 u(x,t)} \quad \text{in} \quad \Omega \times (0,T).$$

Also, it is not hard to check that $\alpha(0) \geq u(x,0)$ in Ω. We deduce from the comparison theorem that

$$\alpha(t) \geq u(x,t) \quad \text{in} \quad \Omega \times (0,T_*), \tag{2.13}$$

where $T_* = \min\{T, T_M\}$. We claim that

$$T \geq T_M. \tag{2.14}$$

To prove the claim, we argue by contradiction. Suppose that $T < T_M$. Taking into account (2.13), we discover that $\|u(\cdot,T)\|_\infty \leq \alpha(T) < \infty$, which contradicts the fact that $(0,T)$ is the maximal time interval of existence of the solution u. This ends the proof. □

Remark 2.1 *It is important to point out that the estimates obtained in Theorem 2.1 about the blow-up time can be rewritten as*

$$0 \leq \frac{T}{T_M} - 1 \leq (1 + p_0 + A)M^{-1} + o(M^{-1}).$$

This implies that $\lim_{M\to\infty} \frac{T}{T_M} = 1$.

To end this section, let us give our second result which concerns the case of small diffusions. It is stated in the following theorem.

Theorem 2.2 *Let K be an upper bound of the first derivatives of u_0 and p. Assume that*

$$\varepsilon < \min\{(p_0/2)^3, A^{-3}, (Kdist(a, \partial\Omega))^3\},$$

where $A = \frac{2DK^2}{p_0}$. Then the solution u of (1.1)–(1.3) blows up in a finite time, and its blow-up time T satisfies the following estimates

$$0 \leq T - T_M \leq \big(M + p_0 + \frac{1}{p_0} + A)T_M\varepsilon^{1/3} + o(T_M\varepsilon^{1/3}),$$

where $T_M = \frac{e^{-p_0 M}}{p_0}$ is the blow-up time of the solution $\alpha(t)$ of the differential equation defined as

$$\alpha^{'}(t) = e^{p_0\alpha(t)}, \quad t > 0, \quad \alpha(0) = M. \tag{2.15}$$

Proof. Since $(0,T)$ is the maximal time interval of existence of the solution u, our aim is to show that T is finite and satisfies the above inequality. Due to the fact that the initial datum u_0 is nonnegative in Ω, owing to the maximum principle, we see that u is also nonnegative in $\Omega\times(0,T)$. Applying the mean value theorem and the triangle inequality, we find that

$$u_0(x) \geq M - \varepsilon^{1/3} \quad \text{for} \quad x \in B(a,\delta), \tag{2.16}$$

$$p(x) \geq p_0 - \varepsilon^{1/3} \quad \text{for} \quad x \in B(a,\delta), \tag{2.17}$$

where $\delta = \frac{\varepsilon^{1/3}}{K}$. Introduce the function $v(t)$ defined as

$$v(t) = \int_{B(a,\delta)} u(x,t)\varphi(x)dx \quad \text{for} \quad t \in [0,T). \tag{2.18}$$

Reasoning as in the proof of Theorem 2.1, we find that

$$v'(t) \geq e^{(p_0-\varepsilon^{1/3})v(t)}\left(1 - DK^2\varepsilon^{1/3}v(t)e^{(-p_0+\varepsilon^{1/3})v(t)}\right) \quad \text{for} \quad t \in (0,T), \tag{2.19}$$

because $\lambda_\delta = \frac{D}{\delta^2} = \frac{DK^2}{\varepsilon^{2/3}}$. It is easy to see that $e^{(-p_0+\varepsilon^{1/3})v(t)} \leq e^{(-p_0/2)v(t)}$ for $t \in (0,T)$. Consequently, in view of (2.19), we arrive at

$$v'(t) \geq e^{(p_0-\varepsilon^{1/3})v(t)}\left(1 - DK^2\varepsilon^{1/3}v(t)e^{-(p_0/2)v(t)}\right) \quad \text{for} \quad t \in (0,T). \tag{2.20}$$

We observe that $\frac{2}{p_0} = \int_0^\infty e^{-(p_0/2)\sigma}d\sigma \geq \sup_{t\geq 0}\int_0^t e^{-(p_0/2)\sigma}d\sigma \geq \sup_{t\geq 0} te^{-(p_0/2)t}$. In view of the above inequalities and making use of (2.20), we deduce that $v'(t) \geq e^{(p_0-\varepsilon^{1/3})v(t)}(1-A\varepsilon^{1/3})$ for $t\in(0,T)$, or equivalently

$$e^{(-p_0+\varepsilon^{1/3})v}dv \geq (1-A\varepsilon^{1/3})dt \quad \text{for} \quad t\in(0,T).$$

Integrating the above inequality over $(0,T)$ to obtain

$$\frac{e^{(-p_0+\varepsilon^{1/3})v(0)}}{p_0-\varepsilon^{1/3}} \geq (1-A\varepsilon^{1/3})T.$$

Since $v(0)\geq M-\varepsilon^{1/3}$, we deduce that $T\leq \frac{e^{(-p_0+\varepsilon^{1/3})(M-\varepsilon^{1/3})}}{(p_0-\varepsilon^{1/3})(1-A\varepsilon^{1/3})}$, which implies that

$$T \leq \frac{e^{-p_0M}}{p_0}\frac{e^{(p_0+M)\varepsilon^{1/3}}}{(1-\frac{\varepsilon^{1/3}}{p_0})(1-A\varepsilon^{1/3})}. \tag{2.21}$$

We conclude that the solution u blows up in a finite time because the quantity on the right hand side of the above inequality is finite. Apply Taylor's expansion to obtain

$$\frac{e^{(p_0+M)\varepsilon^{1/3}}}{(1-\frac{\varepsilon^{1/3}}{p_0})(1-A\varepsilon^{1/3})} = 1+(p_0+M+\frac{1}{p_0}+A)\varepsilon^{1/3}+o(\varepsilon^{1/3}).$$

It follows from (2.21) that

$$T\leq T_M+(p_0+M+\frac{1}{p_0}+A)T_M\varepsilon^{1/3}+o(\varepsilon^{1/3})T_M,$$

and the second estimate of the theorem is proved. In order to demonstrate the first one, it suffices to argue as in the proof of Theorem 2.1. This finishes the proof. □

Remark 2.2 *Let us notice that the estimates obtained in Theorem 2.2 may be rewritten in the form*

$$0\leq \frac{T}{T_M}-1\leq (M+p_0+\frac{1}{p_0}+A)\varepsilon^{1/3}+o(\varepsilon^{1/3}).$$

We deduce that $\lim_{\varepsilon\longrightarrow 0}\frac{T}{T_M}=1$.

3. Numerical results

In this section, we give some computational results to illustrate the theory developed in the earlier section. For this fact, we consider the radial symmetric solution of the following initial-boundary value problem

$$\begin{aligned} u_t(x,t) &= \varepsilon\Delta u(x,t)+e^{p(x)u(x,t)} \quad \text{in} \quad B\times(0,T),\\ u(x,t) &= 0 \quad \text{on} \quad S\times(0,T),\\ u(x,0) &= u_0(x) \quad \text{in} \quad B,\end{aligned}$$

where $B=\{x\in\mathbb{R}^N;\|x\|<1\}, S=\{x\in\mathbb{R}^N;\|x\|=1\}$, $u_0(x)=M\cos(\frac{\pi}{2}(1-2\|x\|))$ with $M\in(0,\infty)$, and $p(x)=\frac{16\|x\|(1-\|x\|)-1}{3}$. Thus, the above problem may be rewritten in the form

$$\begin{aligned} u_t(r,t) &= \varepsilon\left(u_{rr}(r,t)+\frac{N-1}{r}u_r(r,t)\right)+e^{\psi(r)u(r,t)}, \qquad (3.1)\\ & \quad r\in(0,1), \quad t\in(0,T),\\ u_r(0,t) &= 0, \quad u(1,t)=0, \quad t\in(0,T), \qquad (3.2)\\ u(r,0) &= \varphi(r), \quad r\in(0,1), \qquad (3.3)\end{aligned}$$

where $\varphi(r)=M\cos(\frac{\pi}{2}(1-2r))$ and $\psi(r)=\frac{16r(1-r)-1}{3}$. We start by the construction of some adaptive schemes as follows.

Let I be a positive integer and let $h=1/I$. Define the grid $x_i=ih, 0\le i\le I$, and approximate the solution u of (3.1)–(3.3) by the solution $U_h^{(n)}=(U_0^{(n)},...,U_I^{(n)})^T$ of the following explicit scheme

$$\begin{aligned}\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n} &= \varepsilon N\frac{2U_1^{(n)}-2U_0^{(n)}}{h^2}+e^{\psi_0 u_0^{(n)}},\\ \frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n} &= \varepsilon\left(\frac{U_{i+1}^{(n)}-2U_i^{(n)}+U_{i-1}^{(n)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n)}-U_{i-1}^{(n)}}{2h}\right)\\ &\quad +e^{\psi_i u_i^{(n)}}, \quad 1\le i\le I-1,\\ U_I^{(n)} &= 0,\\ U_i^{(0)} &= \varphi_i, \quad 0\le i\le I,\end{aligned}$$

where $n\ge 0$, $\varphi_i=M\cos(\frac{\pi}{2}(1-2ih))$ and $\psi_i=\frac{16ih(1-ih)-1}{3}$. In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the blow-up time T, we need to adapt the size of the time step so that we take

$$\Delta t_n=\min\{\frac{h^2}{2N},h^2e^{(-\|\psi_h\|_\infty\|U_h^{(n)}\|_\infty)}\},$$

where $\|\psi_h\|_\infty=\sup_{0\le i\le I}|\psi_i|$ and $\|U_h^{(n)}\|_\infty=\sup_{0\le i\le I}|U_i^{(n)}|$. We also approximate the solution u of (3.1)–(3.3) by the solution $U_h^{(n)}$ of the implicit scheme below

$$\begin{aligned}\frac{U_0^{(n+1)}-U_0^{(n)}}{\Delta t_n} &= \varepsilon N\frac{2U_1^{(n+1)}-2U_0^{(n+1)}}{h^2}+e^{\psi_0 u_0^{(n)}},\\ \frac{U_i^{(n+1)}-U_i^{(n)}}{\Delta t_n} &= \varepsilon\left(\frac{U_{i+1}^{(n+1)}-2U_i^{(n+1)}+U_{i-1}^{(n+1)}}{h^2}+\frac{(N-1)}{ih}\frac{U_{i+1}^{(n+1)}-U_{i-1}^{(n+1)}}{2h}\right)\\ &\quad +e^{\psi_i u_i^{(n)}}, \quad 1\le i\le I-1,\\ U_I^{(n+1)} &= 0,\\ U_i^{(0)} &= \varphi_i, \quad 0\le i\le I,\end{aligned}$$

where $n \geq 0$. As in the case of the explicit scheme, here, we also choose

$$\Delta t_n = h^2 e^{(-\|\psi_h\|_\infty \|U_h^{(n)}\|_\infty)}$$

for the implicit scheme. We remark that $\lim_{r \longrightarrow 0} \frac{u_r(r,t)}{r} = u_{rr}(0,t)$. Hence, if $r \to 0$, then we have

$$u_t(0,t) = \varepsilon N u_{rr}(0,t) + e^{\psi(0)u(0,t)} \quad \text{for} \quad t \in (0,T).$$

This remark has been used in the construction of our schemes at the first node. Let us notice that in the explicit scheme, the restriction on the time step ensures the nonnegativity of the discrete solution. For the implicit scheme, existence and nonnegativity of the discrete solution are also guaranteed using standard methods (see, for instance [3]).

We need the following definition.

Definition 3.1 *We say that the discrete solution $U_h^{(n)}$ of the explicit scheme or the implicit scheme blows up in a finite time if $\lim_{n\to\infty} \|U_h^{(n)}\| = \infty$, and the series $\sum_{n=0}^{\infty} \Delta t_n$ converges. The quantity $\sum_{n=0}^{\infty} \Delta t_n$ is called the numerical blow-up time of the solution $U_h^{(n)}$.*

In the following tables we present some numerical results for meshes 16, 32, 64 and 128.

(1) In Tables 1 and 2, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 0$ and $\varepsilon = 1/10$.

(2) In Tables 3 and 4, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 0$ and $\varepsilon = 1/100$.

(3) In Tables 5 and 6, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 0$ and $\varepsilon = 1/1000$.

(4) In Tables 7 and 8, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 1$ and $\varepsilon = 1/10$.

(5) In Tables 9 and 10, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 1$ and $\varepsilon = 1/100$.

(6) In Tables 11 and 12, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 1$ and $\varepsilon = 1/1000$.

(7) In Tables 13 and 14, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 10$ and $\varepsilon = 1$.

(8) In Tables 15 and 16, we present the numerical blow-up times, the numbers of iterations, the CPU times (seconds) and the orders of the approximations obtained with the explicit and implicit Euler method, respectively, for $M = 20$ and $\varepsilon = 1$.

We take for the numerical blow-up time $t_n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |t_{n+1} - t_n| \leq 10^{-16}.$$

The order (s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $M = 0,\ N = 2$.

First case $\varepsilon = 1/10$

Table 1

I	t_n	n	CPU time	s
16	1.137646	8843	17	-
32	1.143852	36770	132	-
64	1.146566	141198	1041	1.19
128	1.147750	528086	9273	1.19

Table 2

I	t_n	n	CPU time	s
16	1.269595	11914	34	-
32	1.268552	47172	245	
64	1.268531	179015	2842	1.19
128	1.268519	557321	26361	1.17

Second case: $\varepsilon = 1/100$

Table 3

I	t_n	n	CPU time	s
16	1.023413	8477	16	-
32	1.023681	32583	115	-
64	1.023808	124963	867	1.07
128	1.023767	480019	7677	1.06

Table 4

I	t_n	n	CPU time	s
16	1.023608	8478	31	-
32	1.023749	32584	168	
64	1.023846	124965	2260	0.93
128	1.024011	480021	21205	0.76

Third case: $\varepsilon = 1/1000$

Table 5

I	t_n	n	CPU time	s
16	1.003518	8408	17	-
32	1.002820	32222	116	-
64	1.002668	123260	925	2.19
128	1.002636	470541	6736	2.24

Table 6

I	t_n	n	CPU time	s
16	1.003531	8408	29	-
32	1.002824	32222	155	-
64	1.002669	123260	1930	2.18
128	1.002637	470541	16130	2.27

Numerical experiments for $M = 1,\ N = 2$.

First case $\varepsilon = 1/10$

Table 7

I	t_n	n	CPU time	s
16	0.561495	11872	27	-
32	0.556983	35451	143	-
64	0.555652	136056	961	1.76
128	0.555096	520521	7412	1.26

Table 8

I	t_n	n	CPU time	s
16	0.593093	12023	35	-
32	0.584167	45568	240	-
64	0.580952	150793	2807	1.47
128	0.580396	535258	23051	1.53

Second case: $\varepsilon = 1/100$

Table 9

I	t_n	n	CPU time	s
16	0.398081	7917	16	-
32	0.392664	30312	121	-
64	0.391162	118042	987	1.85
128	0.389661	205772	7915	1.79

Table 10

I	t_n	n	CPU time	s
16	0.398167	7918	28	-
32	0.392686	30312	151	-
64	0.391168	118042	1926	1.85
128	0.389650	205772	16126	1.83

Third case: $\varepsilon = 1/1000$

Table 11

I	t_n	n	CPU time	s
16	0.377945	7843	15	-
32	0.372177	29936	120	-
64	0.370758	114093	931	2.02
128	0.370402	433853	6470	1.99

Table 12

I	t_n	n	CPU time	s
16	0.377955	7843	27	-
32	0.372180	29936	167	-
64	0.370761	113742	1979	2.02
128	0.369342	433805	16475	2.00

Numerical experiments for $\varepsilon = 1,\ M = 10,\ N = 2.$

Table 13

I	t_n	n	CPU time	s
16	5.541897 e-5	5583	10	-
32	4.795143 e-5	20536	68	-
64	4.625920 e-5	76145	579	2.14
128	4.585121 e-5	281694	3900	2.05

Table 14

I	t_n	n	CPU time	s
16	5.541991 e-5	5583	20	-
32	4.795175 e-5	20536	96	-
64	4.625929 e-5	76145	1280	2.14
128	4.585130 e-5	281694	8975	2.05

Numerical experiments for $\varepsilon = 1,\ M = 20,\ N = 2.$

Table 15

I	t_n	n	CPU time	s
16	3.031389 e-9	3179	6	-
32	2.270457 e-9	10408	35	-
64	2.111256 e-9	35166	495	2.25
128	2.072032 e-9	117190	1509	2.02

Table 16

I	t_n	n	CPU time	s
16	3.031389 e-9	3179	9	-
32	2.270457 e-9	10408	52	-
64	2.111256 e-9	35166	547	2.25
128	2.072032 e-9	117190	1638	2.02

Remark 3.1 *If we consider the problem (3.1)–(3.3) in the case where the initial datum is null and the potential* $\psi(r) = \frac{16r(1-r)-1}{3}$, *then it is not hard to see that the blow-up time* T_M *of the solution of the differential*

equation defined in Theorem 2.2 equals one. We observe from Tables 1–6 that when ε diminishes, then the numerical blow-up time of the discrete solution tends to one. In the same way, if one replaces the initial datum by $\varphi(r) = \cos(\frac{\pi}{2}(1-2r))$, then we notice that the blow-up time of the differential equation takes the value $T_M = 0.367879$. When we look at Tables 7–12, we see that the numerical blow-up time of the discrete solution goes to T_M provided the parameter ε tends to zero. These results confirm those found in Theorem 2.2. To finish the remark, we also observe that when $M = 10$ or $M = 20$, then the blow-up time of the differential equation $T_M = 4.539992\,e-5$ or $T_M = 2.061153\,e-9$ and we see, from Tables 13–16, that the numerical blow-up time of the discrete solution is approximately equal T_M when $M = 10$ or $M = 20$. These results confirm those found in Theorem 2.1.

In the following, we also give some plots to illustrate our analysis.

(1) In Figure 1, we plot the evolution of the discrete solution for $M = 0$, $I = 16$ and $\varepsilon = 1/10$.

(2) In Figure 2, we plot the evolution of the discrete solution for $M = 1$, $I = 16$ and $\varepsilon = 1/10$.

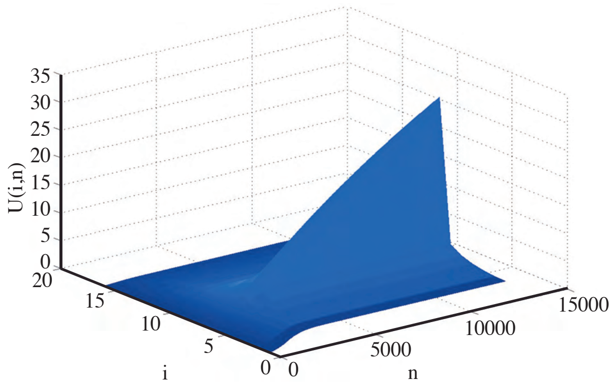

Figure 1

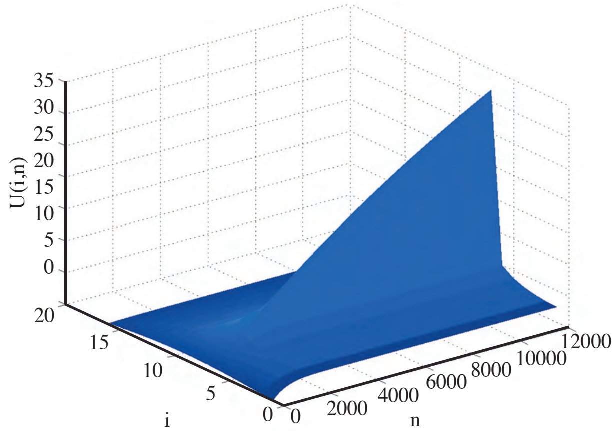

Figure 2

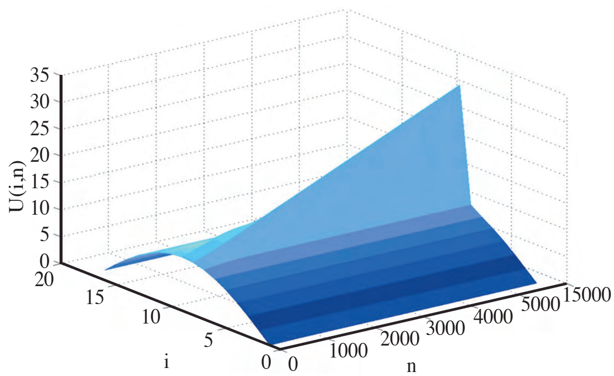

Figure 3

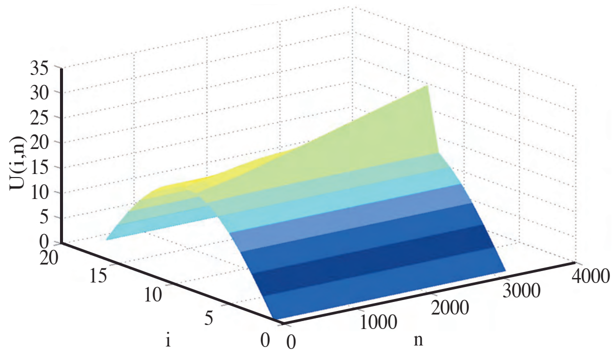

Figure 4

(3) In Figure 3, we plot the evolution of the discrete solution for $M = 10$, $I = 16$ and $\varepsilon = 1$.

(4) In Figure 4, we plot the evolution of the discrete solution for $M = 20$, $I = 16$ and $\varepsilon = 1$.

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