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Hypersurfaces with constant mean curvature in a real space form

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Abstract

Let M^n be an n $(n \geq 3)$-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)(c \geq 0)$ with constant mean curvature H and with two distinct principal curvatures, one of which is simple. We show that (1) if $c = 1$ and the squared norm of the second fundamental form of M^n satisfies a rigidity condition (1.3), then M^n is isometric to the Riemannian product $S^1(\sqrt{1-a^2}) \times S^{n-1}(a)$; (2) if $c = 0$, $H \neq 0$ and the squared norm of the second fundamental form of M^n satisfies $S \geq n^2H^2/(n-1)$, then M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$ or $S^1(a) \times \mathbf{R^{n-1}}$.

Key Words: Hypersurface, scalar curvature, mean curvature, principal curvature.

1. Introduction

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature c. According to $c > 0$ or $c = 0$, it is called sphere space or Euclidean space, respectively, and it is denoted by $S^{n+1}(c)$, $\mathbf{R}^{n+1}$. Let M^n be an n-dimensional hypersurface in $S^{n+1}(1)$ or $\mathbf{R}^{n+1}$. As it is well known there are many rigidity results for hypersurfaces with constant mean curvature or constant scalar curvature $n\ (n-1)r$ in $S^{n+1}(1)$ or $\mathbf{R}^{n+1}$; for example, see [1], [2], [4], [5], [7] and the author of [3] and [6]. In [7], Wei proved the following theorem.

Theorem 1.1 ([7]) *Let M^n be an n $(n \geq 3)$-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures, one of which is simple. If*

$$S \geq n + \frac{n^3H^2}{2(n-1)} + \frac{n\ (n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}, \tag{1.1}$$

then M^n is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{1}{2n(1+H^2)}[2 + nH^2 - \sqrt{n^2H^4 + 4(n-1)H^2}]$, and S denotes the squared norm of the second fundamental form of M^n.

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Theorem 1.2 **([7])** *Let M^n be an n $(n \geq 3)$-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures, one of which is simple. If*

$$S \leq n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}, \tag{1.2}$$

then M^n is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{1}{2n(1+H^2)}[2 + nH^2 + \sqrt{n^2H^4 + 4(n-1)H^2}]$, and S denotes the squared norm of the second fundamental form of M^n.

On the other hand, if M^n is an n-dimensional complete oriented hypersurface in $\mathbf{R}^{n+1}$ with constant scalar curvature $n\,(n-1)r$, Cheng [2] proved the following.

Theorem 1.3 **([2])** *Let M^n be an n $(n \geq 3)$-dimensional complete connected and oriented hypersurface in $\mathbf{R}^{n+1}$ with constant scalar curvature $n\,(n-1)r$ and with two distinct principal curvatures, one of which is simple. Then M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$ or $S^1(a) \times \mathbf{R^{n-1}}$, if $S \geq \frac{n(n-1)r}{n-2}$.*

In this paper, we shall also investigate n-dimensional hypersurfaces with constant mean curvature H in $S^{n+1}(c)$ or $\mathbf{R}^{n+1}$ and obtain the following result:

Theorem 1.4 *Let M^n be an n $(n \geq 3)$-dimensional complete connected and oriented hypersurface in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures, one of which is simple. If*

$$\begin{aligned} &n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2} \\ &\leq S \leq n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}, \end{aligned} \tag{1.3}$$

then M^n is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{1}{2n(1+H^2)}[2 + nH^2 \pm \sqrt{n^2H^4 + 4(n-1)H^2}]$.

Theorem 1.5 *Let M^n be an n $(n \geq 3)$-dimensional complete connected and oriented hypersurface in $\mathbf{R}^{n+1}$ with non-zero constant mean curvature H and with two distinct principal curvatures, one of which is simple. If*

$$S \geq \frac{n^2H^2}{n-1}, \tag{1.4}$$

then M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$ or $S^1(a) \times \mathbf{R^{n-1}}$.

2. Preliminaries

Let $M^{n+1}(c)$ be an $(n+1)$-dimensional connected Riemannian manifold with constant sectional curvature $c(\geq 0)$. Let M^n be an n-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)$. We choose a local orthonormal frame $e_1, \cdots, e_{n+1}$ in $M^{n+1}(c)$ such that $e_1, \cdots, e_n$ are tangent to M^n. Let $\omega_1, \cdots, \omega_{n+1}$ be the dual coframe. We use the following convention on the range of indices:

$$1 \leq A, B, C, \cdots \leq n+1; \quad 1 \leq i, j, k, \cdots \leq n.$$

The structure equations of $M^{n+1}(c)$ are given by

$$d\omega_A=\sum_B \omega_{AB}\wedge\omega_B, \quad \omega_{AB}+\omega_{BA}=0, \tag{2.1}$$

$$d\omega_{AB}=\sum_C \omega_{AC}\wedge\omega_{CB}+\Omega_{AB}, \tag{2.2}$$

where

$$\Omega_{AB}=-\frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C\wedge\omega_D, \tag{2.3}$$

$$K_{ABCD}=c(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC}). \tag{2.4}$$

Restricting to M^n such that

$$\omega_{n+1}=0, \tag{2.5}$$

$$\omega_{n+1i}=\sum_j h_{ij}\omega_j, \quad h_{ij}=h_{ji}, \tag{2.6}$$

the structure equations of M^n are

$$d\omega_i=\sum_j \omega_{ij}\wedge\omega_j, \qquad \omega_{ij}+\omega_{ji}=0, \tag{2.7}$$

$$d\omega_{ij}=\sum_k \omega_{ik}\wedge\omega_{kj}-\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \tag{2.8}$$

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+(h_{ik}h_{jl}-h_{il}h_{jk}), \tag{2.9}$$

$$R_{ij}=(n-1)c\delta_{ij}+nHh_{ij}-\sum_k h_{ik}h_{kj}, \tag{2.10}$$

$$n(n-1)(r-c)=n^2H^2-S, \tag{2.11}$$

where $n\ (n-1)r$ is the scalar curvature, H is the mean curvature and S is the squared norm of the second fundamental form of M^n.

Let M^n be an n $(n\geq 3)$-dimensional complete connected and oriented hypersurface in $M^{n+1}(c)$ with constant mean curvature and with two distinct principal curvatures, one of which is simple. Without loss of generality, we may assume

$$\lambda_1=\lambda_2=\cdots=\lambda_{n-1}=\lambda, \qquad \lambda_n=\mu, \tag{2.12}$$

where λ_i for $i=1,2,\cdots,n$ are the principal curvatures of M^n. We have

$$(n-1)\lambda+\mu=nH, \quad S=(n-1)\lambda^2+\mu^2. \tag{2.13}$$

From (2.13) and (2.11), we have, for $c=1$, that

$$\lambda\mu=(n-1)(r-1)-(n-2)H^2+(n-2)H\sqrt{H^2-(r-1)}, \tag{2.14}$$

on M^n, or

$$\lambda\mu = (n-1)(r-1) - (n-2)H^2 - (n-2)H\sqrt{H^2-(r-1)}, \tag{2.15}$$

on M^n.

On the other hand, from (2.13) and (2.11), we have, for $c=0$, that

$$\lambda\mu = (n-1)r - (n-2)H^2 + (n-2)H\sqrt{H^2-r}, \tag{2.16}$$

on M^n, or

$$\lambda\mu = (n-1)r - (n-2)H^2 - (n-2)H\sqrt{H^2-r}, \tag{2.17}$$

on M^n.

Example 2.1 Let $M_{1,n-1} := S^1(a)\times S^{n-1}(\sqrt{1-a^2})$. Then $M_{1,n-1}$ has two distinct constant principal curvatures $-\frac{a}{\sqrt{1-a^2}}$ and $\frac{\sqrt{1-a^2}}{a}$ with multiplicities $n-1$ and 1, respectively. It is easily seen that $a^2 = \frac{1}{2n(1+H^2)}[2+nH^2 \pm \sqrt{n^2H^4+4(n-1)H^2}]$ and $S = n + \frac{n^3H^2}{2(n-1)} \mp \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4+4(n-1)H^2}$.

Example 2.2 Let $M_{k,n-k} := S^{n-k}(a)\times \mathbf{R}^k$. Then $M_{k,n-k}$ has two distinct constant principal curvatures 0 and $\sqrt{a}$ with multiplicities k and $n-k$, respectively. It is easily seen that $S=\frac{n^2H^2}{n-k}$. Therefore, we know that for $S^{n-1}(a)\times\mathbf{R}$, $S=\frac{n^2H^2}{n-1}$ and for $S^1(a)\times\mathbf{R}^{\mathbf{n-1}}$, $S=n^2H^2$, where we denote $\mathbf{R}=\mathbf{R}^1$.

3. Proof of theorems

In order to prove Theorem 1.4, we need the following propositions due to [7].

Proposition 3.1 ([7]) *Let M^n be an n $(n\geq 3)$-dimensional connected hypersurface with constant mean curvature H and with two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. Then M^n is a locus of moving $(n-1)$-dimensional submanifold $M_1^{n-1}(s)$ along which the principal curvature λ of multiplicity $n-1$ is constant and which is locally isometric to an $(n-1)$-dimensional sphere $S^{n-1}(a(s)) = E^n(s)\cap S^{n+1}(1)$ of constant curvature and $\varpi = |\lambda-H|^{-\frac{1}{n}}$ satisfies the ordinary differential equation of order 2*

$$\frac{d^2\varpi}{ds^2} + \varpi[1+H^2+(2-n)H\varpi^{-n}+(1-n)\varpi^{-2n}] = 0, \tag{3.1}$$

for $\lambda - H > 0$ or

$$\frac{d^2\varpi}{ds^2} + \varpi[1+H^2+(n-2)H\varpi^{-n}+(1-n)\varpi^{-2n}] = 0, \tag{3.2}$$

for $\lambda-H<0$, where $E^n(s)$ is an n-dimensional linear subspace in the Euclidean space R^{n+2} which is parallel to a fixed $E^n(s_0)$.

Lemma 3.1 ([7]) *Equation (3.1) or (3.2) is equivalent to its first order integral*

$$(\frac{d\varpi}{ds})^2 + (1+H^2)\varpi^2 + 2H\varpi^{2-n} + \varpi^{2-2n} = C, \tag{3.3}$$

for $\lambda - H > 0$ *or*

$$(\frac{d\varpi}{ds})^2 + (1+H^2)\varpi^2 - 2H\varpi^{2-n} + \varpi^{2-2n} = C, \tag{3.4}$$

for $\lambda - H < 0$*, where* C *is a constant. Moreover, the constant solution of (3.1) or (3.2) corresponds to the Riemannian product* $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$*.*

By the same method in [7], we can prove the following proposition.

Proposition 3.2 *Let* M^n *be an* n $(n \geq 3)$*-dimensional complete connected hypersurface in* $S^{n+1}(1)$ *with constant mean curvature* H *and with two distinct principal curvatures* λ *and* μ *with multiplicities* $(n-1)$ *and* 1*, respectively. If* $\lambda\mu + 1 \geq 0$*, then* M^n *is isometric to the Riemannian product* $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$*.*

Proof. Let λ and μ be the two distinct principal curvatures of M^n with multiplicities $(n-1)$ and 1, respectively. Then, from $nH = (n-1)\lambda + \mu$, we have $\lambda\mu = nH\lambda - (n-1)\lambda^2$. Let $\varpi = |\lambda - H|^{-\frac{1}{n}}$. Then we have $\lambda = H + \varpi^{-n}$ for $\lambda - H > 0$ and $\lambda = H - \varpi^{-n}$ for $\lambda - H < 0$. If $\lambda - H > 0$, we have

$$\lambda\mu + 1 = 1 + H^2 + (2-n)H\varpi^{-n} + (1-n)\varpi^{-2n},$$

and if $\lambda - H < 0$, we have

$$\lambda\mu + 1 = 1 + H^2 + (n-2)H\varpi^{-n} + (1-n)\varpi^{-2n}.$$

Therefore, if $\lambda\mu + 1 \geq 0$, we obtain

$$1 + H^2 + (2-n)H\varpi^{-n} + (1-n)\varpi^{-2n} \geq 0,$$

for $\lambda - H > 0$ and

$$1 + H^2 + (n-2)H\varpi^{-n} + (1-n)\varpi^{-2n} \geq 0,$$

for $\lambda - H < 0$. From (3.1) and (3.2), we have $\frac{d^2\varpi}{ds^2} \leq 0$. Thus $\frac{d\varpi}{ds}$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, $\varpi(s)$ must be monotonic when s tends to infinity. From (3.3) and (3.4), we know that the positive function $\varpi(s)$ is bounded from above. Since $\varpi(s)$ is bounded and is monotonic when s tends infinity, we find that both $\lim_{s\to-\infty}\varpi(s)$ and $\lim_{s\to+\infty}\varpi(s)$ exist and then we have

$$\lim_{s\to-\infty}\frac{d\varpi(s)}{ds} = \lim_{s\to+\infty}\frac{d\varpi(s)}{ds} = 0. \tag{3.5}$$

By the monotonicity of $\frac{d\varpi}{ds}$, we see that $\frac{d\varpi}{ds} \equiv 0$ and $\varpi(s)$ is a constant. Then, by Lemma 3.1, it is easily see that M^n is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$. This completes the proof of Proposition 3.2. □

On the other hand, if $\lambda\mu + 1 \leq 0$, from above, we can obtain $\frac{d^2\varpi}{ds^2} \geq 0$. Combining $\frac{d^2\varpi}{ds^2} \geq 0$ with the boundedness of $\varpi(s)$, similar to the proof of Proposition 3.2, we know that $\varpi(s)$ is constant. Then, by Lemma 3.1, it is easily see that M^n is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$. Therefore, we have the following proposition.

Proposition 3.3 *Let M^n be an n $(n \geq 3)$-dimensional complete connected hypersurface in $S^{n+1}(1)$ with constant mean curvature H and with two distinct principal curvatures λ and μ with multiplicities $(n-1)$ and 1, respectively. If $\lambda\mu + 1 \leq 0$, then M^n is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$.*

Proof of theorem 1.4 Since M^n has two distinct principal curvatures λ and μ, if $H = 0$ on M^n, from (1.3) we have $S = n$, then M^n is isometric to a Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$. Therefore, we next only consider $H \neq 0$ on M^n. Since M^n is oriented and the mean curvature H is constant, we can choose an orientation for M^n such that $H > 0$. From (2.11), we know that (1.3) is equivalent to

$$\begin{aligned}&\frac{n(n-2)}{2(n-1)}[nH^2 - \sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)]\\ &\leq n(n-1)r \leq \frac{n(n-2)}{2(n-1)}[nH^2 + \sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)],\end{aligned}$$

that is

$$\begin{aligned}&\frac{1}{2(n-1)^2}[n^2H^2 - n\sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)]\\ &\leq \frac{n(r-1)+2}{n-2} \leq \frac{1}{2(n-1)^2}[n^2H^2 + n\sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)],\end{aligned} \tag{3.6}$$

where $n(n-1)r$ is the scalar curvature of M^n.

We define the function

$$f(x) = (n-1)^2x^2 - [n^2H^2 + 2(n-1)]x + 1. \tag{3.7}$$

Since $f(0) = 1$, we know that function (3.7) has two positive real roots

$$x_{1,2} = \frac{1}{2(n-1)^2}[n^2H^2 \pm n\sqrt{n^2H^4 + 4(n-1)H^2} + 2(n-1)]. \tag{3.8}$$

It can be easily checked that $x_1 \leq x_2$ and if $x_1 \leq x \leq x_2$, then $f(x) \leq 0$.

Now we set $x = \frac{n(r-1)+2}{n-2}$, from (3.6), we have

$$f(\frac{n(r-1)+2}{n-2}) \leq 0. \tag{3.9}$$

If there exists a point p on M^n such that (2.14) and (2.15) hold at p, that is, we have $H = 0$ or $H^2 = r - 1$ at p. If $H = 0$ at p, we have a contradiction to $H \neq 0$ on M^n. If $H^2 = r - 1$ at p, from (2.11) we have $S = nH^2$ at p, that is, p is a umbilical point on M^n, this is a contradiction to M^n has no umbilical points. Therefore, we only consider two cases:

Case (1) If (2.14) holds on M^n, next we shall prove that $\lambda\mu + 1 \geq 0$ on M^n. We consider three subcases:

(i) If $1 + (n-1)(r-1) - (n-2)H^2 \geq 0$ on M^n, then from (2.14), it is obvious that $\lambda\mu + 1 \geq 0$ on M^n.

(ii) If $1+(n-1)(r-1)-(n-2)H^2<0$ on M^n, suppose $\lambda\mu+1<0$ on M^n, from (2.14), we have

$$(n-2)H\sqrt{H^2-(r-1)}<-[1+(n-1)(r-1)-(n-2)H^2].$$

Therefore, we have

$$(n-2)^2H^2[H^2-(r-1)]<[1+(n-1)(r-1)-(n-2)H^2]^2,$$

that is, $f(\frac{n(r-1)+2}{n-2})>0$. This is a contradiction to (3.9); we deduce that $\lambda\mu+1\geq 0$ on M^n.

(iii) If $1+(n-1)(r-1)-(n-2)H^2\geq 0$ at a point p of M^n and $1+(n-1)(r-1)-(n-2)H^2<0$ at other points of M^n, in this case, from (i) and (ii), we have at point p, $\lambda\mu+1\geq 0$ and at other points of M^n, also $\lambda\mu+1\geq 0$. Therefore, we obtain $\lambda\mu+1\geq 0$ on M^n.

Therefore, we know that if (2.14) holds on M^n, then $\lambda\mu+1\geq 0$ on M^n. By Proposition 3.2, we obtain that M is isometric to the Riemannian product $S^1(a)\times S^{n-1}(\sqrt{1-a^2})$. From Example 2.1, we have $a^2=\frac{2+nH^2\pm\sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$.

Case (2) If (2.15) holds on M^n, we consider three subcases:

(i) If $1+(n-1)(r-1)-(n-2)H^2\leq 0$ on M^n, then from (2.15), it is obvious that $\lambda\mu+1\leq 0$ on M^n.

(ii) If $1+(n-1)(r-1)-(n-2)H^2>0$ on M^n, suppose $\lambda\mu+1>0$ on M^n, from (2.15), we have

$$1+(n-1)(r-1)-(n-2)H^2>(n-2)H\sqrt{H^2-(r-1)}.$$

Therefore, we have

$$[1+(n-1)(r-1)-(n-2)H^2]^2>(n-2)^2H^2[H^2-(r-1)],$$

that is $f(\frac{n(r-1)+2}{n-2})>0$. This is a contradiction to (3.9), we deduce that $\lambda\mu+1\leq 0$ on M^n.

(iii) If $1+(n-1)(r-1)-(n-2)H^2\leq 0$ at a point p of M^n and $1+(n-1)(r-1)-(n-2)H^2>0$ at other points of M^n , in this case, from (i) and (ii), we have at point p, $\lambda\mu+1\leq 0$ and at other points of M^n, also $\lambda\mu+1\leq 0$. Therefore, we obtain $\lambda\mu+1\leq 0$ on M^n.

Therefore, we know that if (2.15) holds on M^n, then $\lambda\mu+1\leq 0$ on M^n. By Proposition 3.3, we obtain that M is isometric to the Riemannian product $S^1(a)\times S^{n-1}(\sqrt{1-a^2})$. From Example 2.1, we have $a^2=\frac{2+nH^2\pm\sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$. This completes the proof of Theorem 1.4. □

In order to prove Theorem 1.5, we need the following Proposition 3.4, which can be proved by the same method due to Otsuki [5], also see Cheng [2].

Proposition 3.4 *Let M^n be an n $(n\geq 3)$-dimensional complete oriented hypersurface in $\mathbf{R}^{n+1}$ with constant mean curvature H and with two distinct principal curvatures, one of which is simple. Then M^n is isometric to one of the following hypersurfaces:*

(1) $S^1(a)\times\mathbf{R^{n-1}}$,

(2) a complete non-compact hypersurface of revolution $S^{n-1}(a(s))\times M^1$, where $S^{n-1}(a(s))$ is of constant curvature $\{\frac{d\{\log|\lambda-H|^{\frac{1}{n}}\}}{ds}\}^2+\lambda^2$ and M^1 is a plane curve and $\varpi=|\lambda-H|^{-\frac{1}{n}}$ satisfies the following ordinary

differential equation of order 2

$$\frac{d^2\varpi}{ds^2}+\varpi[H^2+(2-n)H\varpi^{-n}+(1-n)\varpi^{-2n}]=0, \tag{3.10}$$

for $\lambda-H>0$ *or*

$$\frac{d^2\varpi}{ds^2}+\varpi[H^2+(n-2)H\varpi^{-n}+(1-n)\varpi^{-2n}]=0, \tag{3.11}$$

for $\lambda-H<0$.

By a similar method in [7], we can prove the following lemma.

Lemma 3.2 *Equation (3.10) or (3.11) is equivalent to its first order integral*

$$(\frac{d\varpi}{ds})^2+H^2\varpi^2+2H\varpi^{2-n}+\varpi^{2-2n}=C, \tag{3.12}$$

for $\lambda-H>0$ *or*

$$(\frac{d\varpi}{ds})^2+H^2\varpi^2-2H\varpi^{2-n}+\varpi^{2-2n}=C, \tag{3.13}$$

for $\lambda-H<0$, *where* C *is a constant. Moreover, the constant solution of (3.10) or (3.11) corresponds to the Riemannian product* $S^{n-1}(a)\times\mathbf{R}$ *or* $S^1(a)\times\mathbf{R^{n-1}}$.

By the similar method in the proof of Proposition 3.2 and Proposition 3.3, we can also prove the following:

Proposition 3.5 *Let* M^n *be an* n $(n\geq 3)$*-dimensional complete connected and oriented hypersurface in* $\mathbf{R}^{n+1}$ *with constant mean curvature* H *and with two distinct principal curvatures, one of which is simple. If* $\lambda\mu\geq 0$, *then* M^n *is isometric to the Riemannian product* $S^{n-1}(a)\times\mathbf{R}$ *or* $S^1(a)\times\mathbf{R^{n-1}}$.

Proposition 3.6 *Let* M^n *be an* n $(n\geq 3)$*-dimensional complete connected and oriented hypersurface in* $\mathbf{R}^{n+1}$ *with constant mean curvature* H *and with two distinct principal curvatures, one of which is simple. If* $\lambda\mu\leq 0$, *then* M^n *is isometric to the Riemannian product* $S^{n-1}(a)\times\mathbf{R}$ *or* $S^1(a)\times\mathbf{R^{n-1}}$.

Proof of theorem 1.5 From (2.11), we know that $S\geq\frac{n^2H^2}{n-1}$ is equivalent to

$$n^2H^2\geq\frac{n(n-1)^2r}{n-2}. \tag{3.14}$$

If there exists a point p on M^n such that (2.16) and (2.17) hold at p, that is, we have $H=0$ or $H^2=r$ at p. If $H=0$ at p, this is a contradiction because of the assumption $H\neq 0$. If $H^2=r$ at p, from (2.11) we have $S=nH^2$ at p, that is, p is a umbilical point on M^n, this is a contradiction to M^n has no umbilical points. Therefore, we only consider two cases.

Case (1) If (2.16) holds on M^n, next we shall prove that $\lambda\mu\geq 0$ on M^n. We consider three subcases:

(i) If $(n-1)r-(n-2)H^2\geq 0$ on M^n, then from (2.16), it is obvious that $\lambda\mu\geq 0$ on M^n.

(ii) If $(n-1)r-(n-2)H^2<0$ on M^n, suppose $\lambda\mu<0$ on M^n, from (2.16), we have

$$(n-2)H\sqrt{H^2-r}<-[(n-1)r-(n-2)H^2].$$

Therefore, we have

$$(n-2)^2H^2(H^2-r) < [(n-1)r-(n-2)H^2]^2,$$

that is, $n^2H^2 < \frac{n(n-1)^2r}{n-2}$. This is a contradiction to (3.14), we deduce that $\lambda\mu \geq 0$ on M^n.

(iii) If $(n-1)r-(n-2)H^2 \geq 0$ at a point p of M^n and $(n-1)r-(n-2)H^2 < 0$ at other points of M^n, in this case, from (i) and (ii), we have at point p, $\lambda\mu \geq 0$ and at other points of M^n, also $\lambda\mu \geq 0$. Therefore, we obtain $\lambda\mu \geq 0$ on M^n.

Therefore, we know that if (2.16) holds on M^n, then $\lambda\mu \geq 0$ on M^n. By Proposition 3.5, we obtain that M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$ or $S^1(a) \times \mathbf{R^{n-1}}$.

Case (2) If (2.17) holds on M^n, we consider three subcases:

(i) If $(n-1)r-(n-2)H^2 \leq 0$ on M^n, then from (2.17), it is obvious that $\lambda\mu \leq 0$ on M^n.

(ii) If $(n-1)r-(n-2)H^2 > 0$ on M^n, suppose $\lambda\mu > 0$ on M^n, from (2.17), we have

$$(n-1)r-(n-2)H^2 > (n-2)H\sqrt{H^2-r}.$$

Therefore, we have

$$[(n-1)r-(n-2)H^2]^2 > (n-2)^2H^2(H^2-r),$$

that is $n^2H^2 < \frac{n(n-1)^2r}{n-2}$. This is a contradiction to (3.14), we deduce that $\lambda\mu \leq 0$ on M^n.

(iii) If $(n-1)r-(n-2)H^2 \leq 0$ at a point p of M^n and $(n-1)r-(n-2)H^2 > 0$ at other points of M^n, in this case, from (i) and (ii), we have at point p, $\lambda\mu \leq 0$ and at other points of M^n, also $\lambda\mu \leq 0$. Therefore, we obtain $\lambda\mu \leq 0$ on M^n.

Therefore, we know that if (2.17) holds on M^n, then $\lambda\mu \leq 0$ on M^n. By Proposition 3.6, we obtain that M^n is isometric to the Riemannian product $S^{n-1}(a) \times \mathbf{R}$ or $S^1(a) \times \mathbf{R^{n-1}}$. This completes the proof of Theorem 1.5. □

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