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Some products involving the fourth Greek letter family element $\tilde{\delta}_s$ in the Adams spectral sequence*

Xiugui Liu and He Wang

Abstract

Let p be an odd prime and A be the mod p Steenrod algebra. For computing the stable homotopy groups of spheres with the classical Adams spectral sequence, we must compute the E_2-term of the Adams spectral sequence, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$. In this paper we prove that in the cohomology of A, the product $k_0 h_n \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+7,t(s,n)+s}(\mathbb{Z}_p, \mathbb{Z}_p)$, is nontrivial for $n \geq 5$, and trivial for $n = 3, 4$, where $\tilde{\delta}_{s+4}$ is actually $\tilde{\alpha}_{s+4}^{(4)}$ described by Wang and Zheng, $p \geq 11$, $0 \leq s < p-4$ and $t(s,n) = 2(p-1)[(s+2)+(s+4)p+(s+3)p^2+(s+4)p^3+p^n]$.

Key word and phrases: Steenrod algebra, cohomology, Adams spectral sequence, May spectral sequence.

1. Introduction and statement of results

Let S be the sphere spectrum localized at an odd prime p and A be the mod p Steenrod algebra. To determine the stable homotopy of sphere $\pi_*(S)$ is one of the central problems in homotopy theory. So far, several methods have been found to determine the stable homotopy groups of spheres. For example we have the classical Adams spectral sequence (ASS)

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s}(S)$$

(cf. [1]) based on the Eilenberg-MacLane spectrum $K\mathbb{Z}_p$, where the differential is

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

We also have the Adams-Novikov spectral sequence (ANSS) (cf. [7]) based on the Brown-Peterson spectrum BP.

Throughout this paper, we fix $q = 2(p-1)$. Consider the spectra $V(k)$ given in [8] such that the $\mathbb{Z}_p$-cohomology

$$H^*V(k) \cong E(Q_0, Q_1, \cdots, Q_k),$$

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the exterior algebra generated by Milnor basis elements Q_0, Q_1, $\cdots$, Q_k in A. The existence of $V(k)$ is assured [8, Theorem 1.1] for $k=1$, $p\geq 3$, for $k=2$, $p\geq 5$ and for $k=3$, $p\geq 7$. Meanwhile, from [8] we also have the following four cofibrations:

$$S \xrightarrow{p} S \xrightarrow{i_0} V(0) \xrightarrow{j_0} \Sigma S, \quad p\geq 2, \tag{1.1}$$

$$\Sigma^q V(0) \xrightarrow{\alpha} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{q+1} V(0), \quad p\geq 3, \tag{1.2}$$

$$\Sigma^{(p+1)q} V(1) \xrightarrow{\beta} V(1) \xrightarrow{i_2} V(2) \xrightarrow{j_2} \Sigma^{(p+1)q+1} V(1), \quad p\geq 5, \tag{1.3}$$

$$\Sigma^{(p^2+p+1)q} V(2) \xrightarrow{\gamma} V(2) \xrightarrow{i_3} V(3) \xrightarrow{j_3} \Sigma^{(p^2+p+1)q+1} V(2), \quad p\geq 7, \tag{1.4}$$

where α, β, γ are so-called the Adams mapping, the v_2-mapping and the v_3-mapping respectively.

For computing the stable homotopy groups of spheres with the classical ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$. The known results on $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ are as follows. $\mathrm{Ext}_A^{0,*}(\mathbb{Z}_p,\mathbb{Z}_p)=\mathbb{Z}_p$ by its definition. From [6], we have $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of $a_0\in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p,\mathbb{Z}_p)$ and $h_i\in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p,\mathbb{Z}_p)$ for all $i\geq 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of α_2, a_0^2, $a_0h_i(i>0)$, $g_i(i\geq 0)$, $k_i(i\geq 0)$, $b_i(i\geq 0)$, and $h_ih_j(j\geq i+2, i\geq 0)$ whose internal degrees are $2q+1$, 2, $p^iq+1, p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq respectively. In 1980, Aikawa [2] determined $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ by λ-algebra.

In 1998, Wang and Zheng [9] proved the following theorem.

Theorem 1.1[9]. *For $p\geq 11$ and $0\leq s<p-4$, there exists the fourth Greek letter family element $\tilde{\delta}_{s+4}\neq 0\in \mathrm{Ext}_A^{s+4,t_1(s)+s}(\mathbb{Z}_p,\mathbb{Z}_p)$, where $t_1(s)=2(p-1)[(s+1)+(s+2)p+(s+3)p^2+(s+4)p^3]$.*

Note that we write $\tilde{\delta}_{s+4}$ for $\tilde{\alpha}^{(4)}_{s+4}$ which is described in [9].

In this note, our main result can be stated as follows.

Theorem 1.2. *Let $p\geq 11$ and $0\leq s<p-4$. Then in the cohomology of the mod p Steenrod algebra A in $\mathrm{Ext}_A^{s+7,t(s,n)+s}(\mathbb{Z}_p,\mathbb{Z}_p)$,*

(1) *the product $k_0h_n\tilde{\delta}_{s+4}$ is nontrivial for $n\geq 5$.*

(2) *the product $k_0h_n\tilde{\delta}_{s+4}$ is trivial for $n=3,4$.*

Here, $\tilde{\delta}_{s+4}$ are given in [9] and $t(s,n)=q[(s+1)+(s+3)p+(s+3)p^2+(s+4)p^3+p^n]$.

The paper is arranged as follows: after recalling some knowledge on the May spectral sequence (MSS) in Section 2, we introduce a method of detecting generators of the E_1-term $E_1^{*,*,*}$ of the MSS in Section 3. Section 4 is devoted to showing Theorem 1.2.

2. Some knowledge on the May spectral sequence

As we know, the most successful method to compute $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p,\mathbb{Z}_p)$ is the MSS. From [7], there is a May spectral sequence(MSS) $\{E_r^{s,t,*},d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p,\mathbb{Z}_p)$ with E_1-term

$$E_1^{*,*,*}=E(h_{m,i}|m>0,i\geq 0)\otimes P(b_{m,i}|m>0,i\geq 0)\otimes P(a_n|n\geq 0), \tag{2.1}$$

where $E(\ )$ is the exterior algebra, $P(\ )$ is the polynomial algebra, and

$$h_{m,i} \in E_1^{1,2(p^m-1)p^i,2m-1}, b_{m,i} \in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)}, a_n \in E_1^{1,2p^n-1,2n+1}.$$

One has

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r} \tag{2.2}$$

and if $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y). \tag{2.3}$$

In particular, the first May differential d_1 is given by

$$d_1(h_{i,j}) = \sum_{0<k<i} h_{i-k,k+j} h_{k,j},\ d_1(a_i) = \sum_{0\le k<i} h_{i-k,k} a_k,\ d_1(b_{i,j}) = 0. \tag{2.4}$$

There also exists a graded commutativity in the MSS: $x \cdot y = (-1)^{ss'+tt'} y \cdot x$ for $x, y = h_{m,i}, b_{m,i}$ or a_n.

For each element $x \in E_1^{s,t,u}$, we define *dim* $x = s$, *deg* $x = t$, *M(x)=u*. Then we have that

$$\begin{cases} \dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2, \\ \deg h_{i,j} = q(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = q(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = q(p^{i-1} + \cdots + 1) + 1, \deg a_0 = 1, \\ M(h_{i,j}) = M(a_{i-1}) = 2i-1, M(b_{i,j}) = (2i-1)p, \end{cases} \tag{2.5}$$

where $i \ge 1$, $j \ge 0$.

Note that by the knowledge on the p-adic expression in number theory, for each integer $m \ge 0$, it can be expressed uniquely as

$$m = q(c_n p^n + c_{n-1} p^{n-1} + \cdots + c_1 p + c_0) + e,$$

where $0 \le c_i < p$ $(0 \le i < n)$, $p > c_n > 0$, $0 \le e < q$.

3. The method of determining generators of $E_1^{*,*,*}$ in the MSS

In this section, we give a method used to compute generators of the E_1-term of MSS. Our method here originates from [3]. The method can also be found in [4].

We denote a_i, $h_{i,j}$ and $b_{i,j}$ by x, y and z, respectively. By the graded commutativity of $E_1^{*,*,*}$, we can suppose a generator $g = (x_1 \cdots x_u)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}$, where $t = (\bar{c}_0 + \bar{c}_1 + \cdots + \bar{c}_n p^n)q$ with $0 \le \bar{c}_i < p$ $(0 \le i < n)$, $0 < \bar{c}_n < p$ and $0 \le b < q$.

Assertion u must equal b if $s < b + q$. Otherwise, by the characteristics of deg a_i, deg $b_{i,j}$, deg $h_{i,j}$ and deg g, we would get $u = b + wq$ for some integer $w > 0$. It follows that dim $g \ge b + wq > s =$ dim g, which is a contradiction. The assertion is proved.

So we have $g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}$. By (2.5), the degrees of x_i, y_i and z_i can be expressed uniquely as

$$\begin{cases} \deg\, x_i = (x_{i,0} + x_{i,1}p + \cdots + x_{i,n}p^n)q + 1, \\ \deg\, y_i = (y_{i,0} + y_{i,1}p + \cdots + y_{i,n}p^n)q, \\ \deg\, z_i = (0 + z_{i,1}p + \cdots + z_{i,n}p^n)q, \end{cases}$$

and

(a) $(x_{i,0}, x_{i,1}, \cdots, x_{i,n})$ is of the form $(1, \cdots, 1, 0, \cdots, 0)$;

(b) $(y_{i,0}, y_{i,1}, \cdots, y_{i,n})$ is of the form $(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$;

(c) $(0, z_{i,1}, \cdots, z_{i,n})$ is of the form $(0, \cdots, 0, 1, \cdots, 1, 0, \cdots, 0)$.

By the graded commutativity of $E_1^{*,*,*}$, $g = (x_1 \cdots x_b)(y_1 \cdots y_v)(z_1 \cdots z_l) \in E_1^{s,t+b,*}$ can be arranged in the following way:

$$\begin{cases} \text{(i)} \quad \text{If } i > j, \text{ we put } a_i \text{ on the left side of } a_j, \\ \text{(ii)} \quad \text{If } j < k, \text{ we put } h_{i,j} \text{ on the left side of } h_{w,k}, \\ \text{(iii)} \quad \text{If } i > w, \text{ we put } h_{i,j} \text{ on the left side of } h_{w,j}, \\ \text{(iv)} \quad \text{Apply the rules (ii) and (iii) to } b_{i,j}. \end{cases}$$

Then from (a)–(c) and (i)–(iv), the factors $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ in g must satisfy the following conditions:

$$\begin{cases} (1) \quad x_{1,j} \geq x_{2,j} \geq \cdots \geq x_{b,j}, \\ (2) \quad x_{i,0} \geq x_{i,1} \geq \cdots \geq x_{i,n}, \\ (3) \quad \text{If } y_{i,j-1} = 0 \text{ and } y_{i,j} = 1, \text{then for all k} < j \ y_{i,k} = 0, \\ (4) \quad \text{If } y_{i,j} = 1 \text{ and } y_{i,j+1} = 0, \text{ then for all } k > j \ y_{i,k} = 0, \\ (5) \quad y_{1,0} \geq y_{2,0} \geq \cdots \geq y_{v,0}, \\ (6) \quad \text{If } y_{i,0} = y_{i+1,0}, \ y_{i,1} = y_{i+1,1}, \cdots, y_{i,j} = y_{i+1,j}, \text{ then } y_{i,j+1} \geq y_{i+1,j+1}, \\ (7) \quad \text{Apply the similar rules (3)} \sim \text{(6) to } z_{i,j}. \end{cases} \tag{3.1}$$

From $\deg\, g = \sum_{i=1}^{b} \deg\, x_i + \sum_{i=1}^{v} \deg\, y_i + \sum_{i=1}^{l} \deg\, z_i$, by the properties of the p-adic number we get the following group of equations:

$$\begin{cases} x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{v,0} + 0 + \cdots + 0 = \bar{c}_0 + k_0 p, \\ x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{v,1} + z_{1,1} + \cdots + z_{l,1} = \bar{c}_1 + k_1 p - k_0, \\ \cdots \\ x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{v,n-1} + z_{1,n-1} + \cdots + z_{l,n-1} = \bar{c}_{n-1} + k_{n-1}p - k_{n-2}, \\ x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{v,n} + z_{1,n} + \cdots + z_{l,n} = \bar{c}_n - k_{n-1}, \end{cases} \tag{3.2}$$

where $k_i \geq 0$ for $0 \leq i \leq n-1$.

In the above group of equations, we get two integer sequences $K = (k_0, k_1, \cdots, k_{n-1})$ and $S = (\bar{c}_0 + k_0 p, \bar{c}_1 + k_1 p - k_0, \cdots, \bar{c}_n - k_{n-1})$ denoted by $(c_0, c_1, \cdots, c_n)$ which are determined by $(k_0, k_1, \cdots, k_{n-1})$ and $(\bar{c}_0, \bar{c}_1, \cdots, \bar{c}_n)$. We want to get the solutions of the group of equations (3.2) which satisfy the conditions (3.1).

Remark Since the values of $x_{i,j}$, $y_{i,j}$ and $z_{i,j}$ must be 0 or 1, to solve the group of equations (3.2) will be mechanical. Since we want to get the solutions of the group of equations (3.2) which satisfy the conditions (3.1), we can use the conditions (3.1) in solving the group of equations. For example, if $x_{1,0} = 0$ for x_1, using the conditions (1)–(2) of (3.1), we will get all $x_{i,j}=0$. By the method, to determine the solutions of the group of equations (3.2) which satisfy the conditions (3.1) will not be too difficult.

Notice that the elements x_i, y_i and z_i are uniquely determined by their degrees. A solution of (3.2) which satisfies (3.1) determines a generator g by setting deg x_i (respectively y_i and z_i) to be $(x_{i,0}+x_{i,1}p+\cdots+x_{i,n}p^n)q+1$ (respectively $(y_{i,0}+y_{i,1}p+\cdots+y_{i,n}p^n)q$ and $(0+z_{i,1}p+\cdots+z_{i,n})q)$. Thus for the $E_1^{s,t+b,*}$-term, where $t=(\bar{c}_0+\bar{c}_1p+\cdots+\bar{c}_np^n)q$ with $0\le\bar{c}_i<p$ $(0\le i<n)$, $0<\bar{c}_n<p$, $0\le b<q$, the determination of $E_1^{s,t+b,*}$ is reduced to the following three steps:

(1) List up all the possible (b,v,l) such that $b+v+2l=s$.

(2) For each given (b,v,l), list all the sequences $K=(k_0,k_1,\cdots,k_{n-1})$ and $S=(c_0,c_1,\cdots,c_n)$ such that $c_i\le b+v+l$ for all $0\le i\le n$.

(3) For each given (b,v,l), $K=(k_0,k_1,\cdots,k_{n-1})$ and $S=(c_0,c_1,\cdots,c_n)$, solve the group of equations (3.2) by virtue of (3.1), then determine all the generators of $E_1^{s,t+b,*}$ by setting the corresponding second degrees.

4. The proof of Theorem 1.2

In this section we first give two lemmas which are needed in the proof of Theorem 1.2. Then we give the proof of Theorem 1.2.

Lemma 4.1 [5, Lemma 3.1]. *For $p\ge 11$ and $0\le s<p-4$. Then the fourth Greek letter family element $\tilde{\delta}_{s+4}\in \mathrm{Ext}_A^{s+4,t_1(s)+s}(\mathbb{Z}_p,\mathbb{Z}_p)$ is represented by*

$$a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}\in E_1^{s+4,t_1(s)+s,*}$$

in the E_1-term of the MSS, where $\tilde{\delta}_{s+4}$ is actually $\tilde{\alpha}_{s+4}^{(4)}$ described in [9] and $t_1(s)=[(s+1)+(s+2)p+(s+3)p^2+(s+4)p^3]q$.

Proof. This lemma is essentially [5, Lemma 3.1]. The proof is omitted in [5]. Here, we give the proof for completeness. We only need to prove that in the MSS

$$E_1^{s+4,t_1(s)+s,*}=\mathbb{Z}_p\{a_4^sh_{4,0}h_{3,1}h_{2,2}h_{1,3}\}.$$

Consider $g\in E_1^{s+4,t_1(s)+s,*}$ with $(\bar{c}_0,\bar{c}_1,\bar{c}_2,\bar{c}_3)=(s+1,s+2,s+3,s+4)$. Note that dim $g=s+4$ and deg $g=t_1(s)+s$. Since dim $g<s+q$, by the assertion in Section 3, the number of x_i in g must be s. By the reason of dimension, all the possibilities of g can be listed as $x_1x_2\cdots x_sz_1z_2$, $x_1x_2\cdots x_sy_1y_2z_1$, $x_1x_2\cdots x_sy_1y_2y_3y_4$.

Case 1 $g=x_1x_2\cdots x_sz_1z_2$ or $x_1x_2\cdots x_sy_1y_2z_1$. Obviously, in the two cases, the sequence $S=(c_0,c_1,c_2,c_3)$ in the group of equations (3.2) is $(s+1,s+2,s+3,s+4)$. The corresponding group of equations (3.2) has no solution since the number of the factors in g is at most $s+3$ which is less than $c_3=s+4$. So, g is impossible to exist in these two cases.

Case 2 $g=x_1x_2\cdots x_sy_1y_2y_3y_4$. In this case, the sequence $S=(c_0,c_1,c_2,c_3)$ as in Case 1. We can use the method in Remark in Section 3 to solve the corresponding group of equations (3.2) by virtue of (3.1). In this case, the group of equations is

$$\begin{cases} x_{1,0} + \cdots + x_{s,0} + y_{1,0} + y_{2,0} + y_{3,0} + y_{4,0} = s+1, \\ x_{1,1} + \cdots + x_{s,1} + y_{1,1} + y_{2,1} + y_{3,1} + y_{4,1} = s+2, \\ x_{1,2} + \cdots + x_{s,2} + y_{1,2} + y_{2,2} + y_{3,2} + y_{4,2} = s+3, \\ x_{1,3} + \cdots + x_{s,3} + y_{1,3} + y_{2,3} + y_{3,3} + y_{4,3} = s+4. \end{cases}$$

From the fourth equation we get $x_{i,3} = y_{1,3} = y_{2,3} = y_{3,3} = y_{4,3} = 1$ for $1 \le i \le s$, by the conditions (1)–(2) of (3.1) we have $x_{i,j} = 1$ for $1 \le i \le s$ and $0 \le j \le 3$. From the first equation and the condition (5) of (3.1), we get $y_{1,0} = 1, y_{2,0} = y_{3,0} = y_{4,0} = 0$. By the conditions (3)–(4) of (3.1) we have $y_{1,1} = y_{1,2} = 1$. Then from the second equation we get $y_{2,1} = 1, y_{3,1} = y_{4,1} = 0$. By the conditions (3)–(4) of (3.1) we have $y_{2,2} = 1$. Then from the third equation we get $y_{3,2} = 1, y_{4,2} = 0$. It follows that $g = a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3}$ up to sign, showing that $E_1^{s+4,t_1(s)+s,*} = \mathbb{Z}_p\{a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3}\}$. □

Lemma 4.2 *Let $p \ge 11$, $n \ge 4$, $0 \le s < p-4$. Then the May E_1-term satisfies*

$$E_1^{s+6,t(s,n)+s,*} = \begin{cases} M & n = 4 \\ 0 & n \ge 5 \text{ and } 0 \le s < p-5 \\ K & n \ge 5 \text{ and } s = p-5 \end{cases}$$

for $0 < r \le s+7$. Here, $t(s,n) = [(s+2) + (s+4)p + (s+3)p^2 + (s+4)p^3 + p^n]q$, M is the $\mathbb{Z}_p$-module generated by the following twenty elements:

$$\begin{cases} \mathbf{g}1 = a_4^s h_{4,0} h_{2,0} h_{3,1} h_{1,3} b_{4,0}, & \mathbf{g}2 = a_4^s h_{5,0} h_{4,0} h_{3,1} h_{1,1} b_{1,2}, \\ \mathbf{g}3 = a_4^s h_{5,0} h_{4,0} h_{3,1} h_{1,3} b_{1,0}, & \mathbf{g}4 = a_4^s h_{4,0} h_{2,0} h_{4,1} h_{3,1} b_{1,2}, \\ \mathbf{g}5 = a_4^s h_{5,0} h_{4,0} h_{1,1} h_{1,3} b_{3,0}, & \mathbf{g}6 = a_5 a_4^{s-1} h_{4,0} h_{2,0} h_{3,1} h_{1,3} b_{3,0}, \\ \mathbf{g}7 = a_4^s h_{5,0} h_{2,0} h_{3,1} h_{1,3} b_{3,0}, & \mathbf{g}8 = a_4^s h_{4,0} h_{2,0} h_{3,1} h_{2,3} b_{3,0}, \\ \mathbf{g}9 = a_4^s h_{4,0} h_{2,0} h_{4,1} h_{1,3} b_{3,0}, & \mathbf{g}10 = a_4^{s-1} a_2 h_{5,0} h_{4,0} h_{3,1} h_{1,3} b_{3,0}, \\ \mathbf{g}11 = a_4^s h_{4,0} h_{2,0} h_{3,1} h_{2,1} h_{2,3} h_{1,3}, & \mathbf{g}12 = a_5 a_4^{s-1} h_{4,0} h_{2,0} h_{3,1} h_{1,1} h_{2,2} h_{1,3}, \\ \mathbf{g}13 = a_4^s h_{5,0} h_{2,0} h_{3,1} h_{1,1} h_{2,2} h_{1,3}, & \mathbf{g}14 = a_4^s h_{4,0} h_{2,0} h_{4,1} h_{1,1} h_{2,2} h_{1,3}, \\ \mathbf{g}15 = a_4^s h_{4,0} h_{2,0} h_{3,1} h_{1,1} h_{3,2} h_{1,3}, & \mathbf{g}16 = a_4^s h_{4,0} h_{2,0} h_{3,1} h_{1,1} h_{2,2} h_{2,3}, \\ \mathbf{g}17 = a_4^s h_{4,0} h_{1,0} h_{4,1} h_{3,1} h_{1,1} h_{1,3}, & \mathbf{g}18 = a_4^{s-1} a_2 h_{4,0} h_{2,0} h_{4,1} h_{3,1} h_{2,2} h_{1,3}, \\ \mathbf{g}19 = a_4^{s-1} a_2 h_{5,0} h_{4,0} h_{3,1} h_{1,1} h_{2,2} b_{1,3}, & \mathbf{g}20 = a_4^s h_{4,0} h_{3,0} h_{3,1} h_{1,1} h_{2,3} h_{1,3}, \end{cases}$$

where $M(\mathbf{g}1) = 9s + 7p + 16$, $M(\mathbf{g}i) = 9s + p + 22$ for $2 \le i \le 4$, $M(\mathbf{g}i) = 9s + 5p + 18$ for $5 \le i \le 10$, $M(\mathbf{g}i) = 9s + 22$ for $11 \le i \le 20$, and K is the $\mathbb{Z}_p$-module generated by two elements $\mathbf{g}21 = a_n^{p-5} h_{n,0} h_{5,0} h_{n-1,1} h_{1,1} h_{n-3,3} h_{n-4,4}, \mathbf{g}22 = a_n^{p-5} h_{n,0} h_{2,0} h_{n-1,1} h_{4,1} h_{n-3,3} h_{n-4,4}$, where $M(\mathbf{g}21) = M(\mathbf{g}22) = (2n+1)(p-5) + 8n - 10$.

Proof. Consider $g \in E_1^{s+6,t(s,n)+s,*}$, where $t(s,n) = [(s+2) + (s+4)p + (s+3)p^2 + (s+4)p^3 + p^n]q$ with $(\bar{c}_0, \bar{c}_1, \cdots, \bar{c}_n) = (s+2, s+4, s+3, s+4, 0, \cdots, 0, 1)$. Then $\dim g = s+6$ and $\deg g = t(s,n) + s$.

Since $s + 6 < s + q$, according to the assertion in Section 3, the number of x_i in g is s. By the reason of dimension, all the possibilities of g can be listed as $x_1 \cdots x_s z_1 z_2 z_3$, $x_1 \cdots x_s y_1 y_2 z_1 z_2$, $x_1 \cdots x_s y_1 y_2 y_3 y_4 z_1$, $x_1 \cdots x_s y_1 y_2 y_3 y_4 y_5 y_6$.

Case 1 $g = x_1x_2\cdots x_sz_1z_2z_3$. Note that $s < p-4$. It follows that $\sum\limits_{i=1}^{s} x_{i,0} \le s < s+2 = \bar{c}_0 \le c_0$, which is impossible. Thus the first equation of (3.2) has no solution, showing that such g is impossible to exist.

Case 2 $g = x_1x_2\cdots x_sy_1y_2z_1z_2$. By $s < p-4$, we have that $\sum\limits_{i=1}^{s} x_{i,j} + y_{1,j} + y_{2,j} + z_{1,j} + z_{2,j} \le s+4 < p$ for all $0 \le j \le n$. Thus the integer sequence $K = (k_0, k_1, \cdots, k_{n-1})$ in the corresponding group of equation (3.2) is $(0,0,\cdots,0)$ and $S = (c_0, c_1, \cdots, c_n) = (s+2, s+4, s+3, s+4, 0, \cdots, 0, 1)$. Since $\sum\limits_{i=1}^{s} x_{i,1} + y_{1,1} + y_{2,1} + z_{1,1} + z_{2,1} = s+4$ and $\sum\limits_{i=1}^{s} x_{i,3} + y_{1,3} + y_{2,3} + z_{1,3} + z_{2,3} = s+4$, we get $x_{i,1} = y_{1,1} = y_{2,1} = z_{1,1} = z_{2,1} = 1$ and $x_{i,3} = y_{1,3} = y_{2,3} = z_{1,3} = z_{2,3} = 1$ $(1 \le i \le s)$. Then by the conditions (2), (4) and (7) in (3.1), we get $x_{i,2} = y_{1,2} = y_{2,2} = z_{1,2} = z_{2,2} = 1$ $(1 \le i \le s)$, which is impossible because of $\sum\limits_{i=1}^{s} x_{i,2} + y_{1,2} + y_{2,2} + z_{1,2} + z_{2,2} = s+3$. So the corresponding group of equations (3.2) has no solution. It follows that g is impossible to exist.

Case 3 $g = x_1x_2\cdots x_sy_1y_2y_3y_4z_1$.

Subcase 3.1 $n = 4$. Similar to Case 2, we get $S = (c_0, c_1, c_2, c_3, c_4) = (s+2, s+4, s+3, s+4, 1)$. We solve the corresponding group of equations (3.2) by virtue of (3.1), and get the following ten nontrivial generators

$$\begin{cases} \mathbf{g}1 = a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,3}b_{4,0}, & \mathbf{g}2 = a_4^s h_{5,0}h_{4,0}h_{3,1}h_{1,1}b_{1,2}, \\ \mathbf{g}3 = a_4^s h_{5,0}h_{4,0}h_{3,1}h_{1,3}b_{1,0}, & \mathbf{g}4 = a_4^s h_{4,0}h_{2,0}h_{4,1}h_{3,1}b_{1,2}, \\ \mathbf{g}5 = a_4^s h_{5,0}h_{4,0}h_{1,1}h_{1,3}b_{3,0}, & \mathbf{g}6 = a_5a_4^{s-1}h_{4,0}h_{2,0}h_{3,1}h_{1,3}b_{3,0}, \\ \mathbf{g}7 = a_4^s h_{5,0}h_{2,0}h_{3,1}h_{1,3}b_{3,0}, & \mathbf{g}8 = a_4^s h_{4,0}h_{2,0}h_{3,1}h_{2,3}b_{3,0}, \\ \mathbf{g}9 = a_4^s h_{4,0}h_{2,0}h_{4,1}h_{1,3}b_{3,0}, & \mathbf{g}10 = a_4^{s-1}a_2h_{5,0}h_{4,0}h_{3,1}h_{1,3}b_{3,0}, \end{cases}$$

where $M(\mathbf{g}1) = 9s+7p+16$, $M(\mathbf{g}i) = 9s+p+22$ for $2 \le i \le 4$, $M(\mathbf{g}i) = 9s+5p+18$ for $5 \le i \le 10$.

Subcase 3.2 $n \ge 5$. Since $\sum\limits_{i=1}^{s} x_{i,j} + y_{1,j} + y_{2,j} + y_{3,j} + y_{4,j} + z_{1,j} \le s+5 \le p$ $(0 \le j \le n$ and $z_{1,0} = 0$), then all possibilities of the integer sequence $K = (k_0, k_1, \cdots, k_{n-1})$ in the corresponding group of equation (3.2) are

$K_1 = (0, 0, \cdots, 0)$,

$K_i = (0,0,0,0,0,\cdots,0,1^{(i)},1,\cdots,1)$ $(5 \le i \le n$ and $s = p-5)$,

where $1^{(i)}$ means that 1 is the i-th term of the sequence K_i. The corresponding sequence $S = (c_0, c_1, \cdots, c_n)$ are listed as

$S_1 = (s+2, s+4, s+3, s+4, 0, \cdots, 0, 1)$,

$S_i = (p-3, p-1, p-2, p-1, 0, \cdots, 0, p^{(i)}, p-1, \cdots, p-1, 0)$ $(5 \le i \le n$ and $s = p-5)$.

For S_1 and $S_i (5 \le i \le n)$, we can easily show that the corresponding group of equations (3.2) has no solution.

Case 4 $g = x_1x_2\cdots x_sy_1y_2y_3y_4y_5y_6$.

Subcase 4.1 $n = 4$. Similar to Case 2, we get $S = (c_0, c_1, c_2, c_3, c_4) = (s+2, s+4, s+3, s+4, 1)$. One can solve the corresponding group of equations (3.2) by virtue of (3.1), and get ten nontrivial generators as follows:

$$\begin{cases} \mathbf{g}11 = a_4^s h_{4,0}h_{2,0}h_{3,1}h_{2,1}h_{2,3}h_{1,3}, & \mathbf{g}12 = a_5 a_4^{s-1} h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}, \\ \mathbf{g}13 = a_4^s h_{5,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}, & \mathbf{g}14 = a_4^s h_{4,0}h_{2,0}h_{4,1}h_{1,1}h_{2,2}h_{1,3}, \\ \mathbf{g}15 = a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{1,3}, & \mathbf{g}16 = a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}, \\ \mathbf{g}17 = a_4^s h_{4,0}h_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{1,3}, & \mathbf{g}18 = a_4^{s-1} a_2 h_{4,0}h_{2,0}h_{4,1}h_{3,1}h_{2,2}h_{1,3}, \\ \mathbf{g}19 = a_4^{s-1} a_2 h_{5,0}h_{4,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}, & \mathbf{g}20 = a_4^s h_{4,0}h_{3,0}h_{3,1}h_{1,1}h_{2,3}h_{1,3}, \end{cases}$$

where $M(\mathbf{g}i) = 9s + 22$ for $11 \leq i \leq 20$.

Subcase 4.2 $n \geq 5$ and $0 \leq s < p - 6$. Similar to Case 2, we get the integer sequence $S = (c_0, c_1, \cdots, c_n) = (s+2, s+4, s+3, s+4, 0, \cdots, 0, 1)$. One can solve the corresponding group of equations (3.2) by virtue of (3.1), and get the following three generators $a_4^s h_{4,0}^2 h_{3,1}h_{1,1}h_{1,3}h_{1,n}$, $a_4^{s-1} a_2 h_{4,0}^2 h_{3,1}^2 h_{1,3}h_{1,n}$ and $a_4^s h_{4,0}h_{2,0}h_{3,1}^2 h_{1,3}h_{1,n}$ which are all trivial by $h_{4,0}^2 = h_{3,1}^2 = 0$.

Subcase 4.3 $n \geq 5$ and $s = p - 6$. Similar to Subcase 3.2, we get all the possibilities of $S = (c_0, c_1, \cdots, c_n)$:

$S_1 = (p-4, p-2, p-3, p-2, 0, \cdots, 0, 1)$,

$S_i = (p-4, p-2, p-3, p-2, 0, \cdots, 0, p^{(i)}, p-1, \cdots, p-1, 0) \quad (5 \leq i \leq n)$.

For S_1, we can solve the corresponding group of equations (3.2) by virtue of (3.1), and get three generators $a_4^{p-6} h_{4,0}^2 h_{3,1}h_{1,1}h_{1,3}h_{1,n}$, $a_4^{p-7} a_2 h_{4,0}^2 h_{3,1}^2 h_{1,3}h_{1,n}$, $a_4^{p-6} h_{4,0}h_{2,0}h_{3,1}^2 h_{1,3}h_{1,n}$ which are all trivial by $h_{4,0}^2 = h_{3,1}^2 = 0$. For $S_i (5 \leq i \leq n)$, by (3.1) we can show that the corresponding group of equations (3.2) has no solution.

Subcase 4.4 $n \geq 5$ and $s = p - 5$. Similar to Subcase 3.2, we get all the possibilities of $S = (c_0, c_1, \cdots, c_n)$:

$S_1 = (p-3, p-1, p-2, p-1, 0, \cdots, 0, 1)$,

$S_i = (p-3, p-1, p-2, p-1, 0, \cdots, 0, p^{(i)}, p-1, \cdots, p-1, 0) \quad (5 \leq i \leq n)$.

For S_1, one can solve the corresponding group of equations (3.2) by (3.1) and get three generators $a_4^{p-5} h_{4,0}^2 h_{3,1} h_{1,1}h_{1,3}h_{1,n}$, $a_4^{p-6} a_2 h_{4,0}^2 h_{3,1}^2 h_{1,3}h_{1,n}$ and $a_4^{p-5} h_{4,0}h_{2,0}h_{3,1}^2 h_{1,3}h_{1,n}$ which are all trivial by $h_{4,0}^2 = h_{3,1}^2 = 0$.

For $S = S_5$ and $n = 5$, we solve the corresponding group of equations (3.2) by virtue of (3.1), and get three generators $a_5^{p-5} h_{5,0}^2 h_{4,1}h_{1,1}h_{2,3}h_{1,4}$, $a_5^{p-6} a_2 h_{5,0}^2 h_{4,1}^2 h_{2,3}h_{1,4}$ and $a_5^{p-5} h_{5,0}h_{2,0}h_{4,1}^2 h_{2,3}h_{1,4}$ which are all trivial by $h_{5,0}^2 = h_{4,1}^2 = 0$.

For $S = S_5$ and $n > 5$, one can solve the corresponding group of equations (3.2) by virtue of (3.1), and get two nontrivial generators $\mathbf{g}21 = a_n^{p-5} h_{n,0}h_{5,0}h_{n-1,1}h_{1,1}h_{n-3,3}h_{n-4,4}$ and $\mathbf{g}22 = a_n^{p-5} h_{n,0}h_{2,0}h_{n-1,1}h_{4,1}h_{n-3,3} h_{n-4,4}$ with $M(\mathbf{g}21) = M(\mathbf{g}22) = (2n+1)(p-5) + 8n - 10$.

For $S_i (6 \leq i \leq n)$, by (3.1) one can show that the corresponding group of equations (3.2) has no solution.

This finishes the proof of Lemma 4.2. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2 (1) It is known that $h_{2,0}h_{1,1} \in E_1^{2,q(2p+1),*}$ and $h_{1,n} \in E_1^{1,qp^n,*}$ are permanent cocycles in the MSS and represent $k_0 \in \mathrm{Ext}_A^{2,2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$ and $h_n \in \mathrm{Ext}_A^{1,p^n q}(\mathbb{Z}_p, \mathbb{Z}_p)$ respectively. By Lemma

4.1, $\tilde{\delta}_{s+4}$ is represented by

$$a_4^s h_{4,0} h_{3,1} h_{2,2} h_{1,3} \in E_1^{s+4,*,*}$$

in the MSS. Then, we get that $h_{2,0}h_{1,1}h_{1,n}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+7,t(s,n)+s,9s+21}$ is a permanent cocycle and represents $k_0 h_n \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+7,t(s,n)+s}(\mathbb{Z}_p, \mathbb{Z}_p)$ in the MSS.

Case 1 $0 \le s < p-5$. From Lemma 4.2, the May E_1-term $E_1^{s+6,t(s,n)+s,*} = 0$, which implies that

$$E_r^{s+6,t(s,n)+s,*} = 0$$

for $r \ge 1$. Consequently, the permanent cocycle $h_{2,0}h_{1,1}h_{1,n}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential in the MSS. Thus in this case, $k_0 h_n \tilde{\delta}_{s+4} \ne 0$.

Case 2 $s = p-5$. By Lemma 4.2, in this case

$$E_1^{p+1,t(p-5,n)+p-5,*} = \mathbb{Z}_p\{\mathbf{g}21, \mathbf{g}22\}$$

and $M(\mathbf{g}21) = M(\mathbf{g}22) = (2n+1)(p-5) + 8n - 10$. By the reason of May filtration, we see that $h_{2,0}h_{1,1}h_{1,n}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_1(E_1^{p+1,t(p-5,n)+p-5,(2n+1)(p-5)+8n-10})$. By (2.4), we have the May differentials of the generators of $E_1^{p+1,t(p-5,n)+p-5,(2n+1)(p-5)+8n-10}$ as follows:

$$\begin{aligned}
d_1(\mathbf{g}21) =\ & d_1(a_n^{p-5} h_{n,0} h_{5,0} h_{n-1,1} h_{1,1} h_{n-3,3} h_{n-4,4}) \\
=\ & -a_n^{p-5} d_1(h_{n,0}) h_{5,0} h_{n-1,1} h_{1,1} h_{n-3,3} h_{n-4,4} + \cdots \\
=\ & -a_n^{p-5} h_{n-2,2} h_{2,0} h_{5,0} h_{n-1,1} h_{1,1} h_{n-3,3} h_{n-4,4} + \cdots \ne 0, \\
d_1(\mathbf{g}22) =\ & d_1(a_n^{p-5} h_{n,0} h_{2,0} h_{n-1,1} h_{4,1} h_{n-3,3} h_{n-4,4}) \\
=\ & a_n^{p-5} d_1(h_{n,0}) h_{2,0} h_{n-1,1} h_{4,1} h_{n-3,3} h_{n-4,4} + \cdots \\
=\ & a_n^{p-5} h_{n-5,5} h_{5,0} h_{2,0} h_{n-1,1} h_{4,1} h_{n-3,3} h_{n-4,4} + \cdots \ne 0.
\end{aligned}$$

We can see that the first May differential of each generator contains at least a term which is not in the first May differential of the other generator. It follows that the first May differential of the two generators is linearly independent. This means that

$$E_r^{p+1,t(p-5,n)+p-5,(2n+1)(p-5)+8n-10} = 0$$

for $r \ge 2$. Then $h_{2,0}h_{1,1}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ is not in $d_r(E_r^{p+1,t(p-5,n)+p-5,(2n+1)(p-5)+8n-10})$ for $r \ge 1$, which implies that the permanent cocycle $h_{2,0}h_{1,1}h_{1,n}a_4^{p-5}h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ cannot be hit by any May differential. Thus $k_0 h_n \tilde{\delta}_{p-1} \ne 0 \in \mathrm{Ext}_A^{p+2,t(p-5,n)+p-5}(\mathbb{Z}_p, \mathbb{Z}_p)$.

This completes the proof of Theorem 1.2 (1).

(2) Since $k_0 h_3 \tilde{\delta}_{s+4}$ is represented in the MSS by $h_{2,0}h_{1,1}h_{1,3}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3}$ which is trivial by $h_{1,3}^2 = 0$, it follows that $k_0 h_3 \tilde{\delta}_{s+4} = 0$.

Now we will show $k_0 h_4 \tilde{\delta}_{s+4} = 0$. It suffices to prove that $h_{2,0}h_{1,1}h_{1,4}a_4^s h_{4,0}h_{3,1}h_{2,2}h_{1,3} \in E_1^{s+6,t(s,4)+s,9s+21}$ which represents $k_0 h_n \tilde{\delta}_{s+4} \in \mathrm{Ext}_A^{s+7,t(s,4)+s}(\mathbb{Z}_p, \mathbb{Z}_p)$ is in $d_1(E_1^{s+6,t(s,4)+s,9s+22})$. From Lemma 4.2,

$$E_1^{s+6,t(s,4)+s,9s+22} = \mathbb{Z}_p\{\mathbf{g}11, \cdots, \mathbf{g}20\}.$$

By (2.4) we compute the first May differential of $\mathbf{g}i$ $(11 \leq i \leq 20)$ as follows (use the graded commutativity in the MSS, we arrange $d_1(\mathbf{g}i)$ for $11 \leq i \leq 20$ in the way of (i), (ii) and (iii) in Section 3)

$$
\begin{aligned}
d_1(\mathbf{g}11) = {} & (-1)^s(s\underline{a_4^{s-1}a_2h_{4,0}h_{2,0}h_{3,1}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_1 + \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,1}h_{1,1}h_{2,3}h_{1,3}}_2 \\
- {} & \underline{a_4^s h_{4,0}h_{2,0}h_{2,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_3 + \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{1,2}h_{2,3}h_{1,3}}_4), \\
d_1(\mathbf{g}12) = {} & (-1)^s(\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_5 + \underline{a_4^{s-1}a_1h_{4,0}h_{2,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_6 \\
+ {} & \underline{a_4^{s-1}a_2h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_7 - \underline{a_4^{s-1}a_3h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_8 \\
+ {} & \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_9), \\
d_1(\mathbf{g}13) = {} & (-1)^s(-s\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_5 - \underline{a_4^s h_{2,0}h_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{10} \\
- {} & \underline{a_4^s h_{3,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{11} + \underline{h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_9), \\
d_1(\mathbf{g}14) = {} & (-1)^s(-s\underline{a_4^{s-1}a_1h_{4,0}h_{2,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_6 + \underline{a_4^s h_{2,0}h_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{10} \\
- {} & \underline{a_4^s h_{4,0}h_{2,0}h_{2,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_3 + \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_9), \\
d_1(\mathbf{g}15) = {} & (-1)^s(-s\underline{a_4^{s-1}a_2h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_7 - \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{1,2}h_{2,3}h_{1,3}}_4 \\
+ {} & \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_9), \\
d_1(\mathbf{g}16) = {} & (-1)^s(s\underline{a_4^{s-1}a_3h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_8 + \underline{a_4^s h_{3,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{11} \\
+ {} & \underline{a_4^{s+2}h_{4,0}h_{2,0}h_{2,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_3 + \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{1,2}h_{2,3}h_{1,3}}_4 \\
+ {} & \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}h_{1,4}}_9), \\
d_1(\mathbf{g}17) = {} & (-1)^s(-s\underline{a_4^{s-1}a_2h_{4,0}h_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{12} - \underline{a_4^s h_{2,0}h_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{10} \\
+ {} & \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,1}h_{1,1}h_{2,3}h_{1,3}}_2), \\
d_1(\mathbf{g}18) = {} & (-1)^s a_4^{s-1}(\underline{a_1h_{4,0}h_{2,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_6 + \underline{a_4^{s-1}a_2h_{4,0}h_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{12} \\
- {} & \underline{a_4^{s-1}a_2h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_7 + \underline{a_4^{s-1}a_2h_{4,0}h_{2,0}h_{3,1}h_{2,1}h_{2,2}h_{2,3}h_{1,3}}_1), \\
d_1(\mathbf{g}19) = {} & (-1)^s(\underline{a_4^{s-1}a_0h_{5,0}h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_5 - \underline{a_4^{s-1}a_2h_{4,0}h_{1,0}h_{4,1}h_{3,1}h_{1,1}h_{2,2}h_{1,3}}_{12} \\
- {} & \underline{a_4^{s-1}a_2h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{3,2}h_{2,2}h_{1,3}}_7 + \underline{a_4^{s-1}a_2h_{4,0}h_{3,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{13}), \\
d_1(\mathbf{g}20) = {} & (-1)^s(s\underline{a_4^{s-1}a_2h_{4,0}h_{3,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{13} - \underline{a_4^s h_{3,0}h_{2,0}h_{3,1}h_{1,1}h_{2,2}h_{2,3}h_{1,3}}_{11} \\
- {} & \underline{a_4^s h_{4,0}h_{1,0}h_{3,1}h_{2,1}h_{1,1}h_{2,3}h_{1,3}}_2 + \underline{a_4^s h_{4,0}h_{2,0}h_{3,1}h_{1,1}h_{1,2}h_{2,3}h_{1,3}}_4).
\end{aligned}
$$

Without generality, we let s be even. Then we easily get

$$
\begin{pmatrix} d_1(\mathbf{g}11) \\ d_1(\mathbf{g}12) \\ d_1(\mathbf{g}13) \\ d_1(\mathbf{g}14) \\ d_1(\mathbf{g}15) \\ d_1(\mathbf{g}16) \\ d_1(\mathbf{g}17) \\ d_1(\mathbf{g}18) \\ d_1(\mathbf{g}19) \\ d_1(\mathbf{g}20) \end{pmatrix}
=
\begin{pmatrix}
s & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -s & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -s & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & -s & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & s & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -s & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & s
\end{pmatrix}
\begin{pmatrix} \underline{\quad}_1 \\ \underline{\quad}_2 \\ \underline{\quad}_3 \\ \underline{\quad}_4 \\ \underline{\quad}_5 \\ \underline{\quad}_6 \\ \underline{\quad}_7 \\ \underline{\quad}_8 \\ \underline{\quad}_9 \\ \underline{\quad}_{10} \\ \underline{\quad}_{11} \\ \underline{\quad}_{12} \\ \underline{\quad}_{13} \end{pmatrix}.
$$

By the knowledge of matrix, we can easily get that the rank of the middle matrix above is 8. To add a row $(0,0,0,0,0,0,0,0,1,0,0,0,0)$ to the matrix, we get that the rank of the new matrix is also 8, which implies that $(0,0,0,0,0,0,0,0,1,0,0,0,0)$ can be linearly represented by the other rows of the matrix and then

$\underline{a_4^s h_{4,0} h_{2,0} h_{3,1} h_{1,1} h_{2,2} h_{1,3} h_{1,4}}_9$ can be linearly represented by $d_1(\mathbf{g}11)$, $\cdots$, $d_1(\mathbf{g}20)$. So $h_{2,0}h_{1,1}h_{1,4}a_4^s h_{4,0}h_{3,1}$ $h_{2,2}h_{1,3}$ is in $d_1(E_1^{s+6,t(s,4)+s,9s+22})$, showing that $k_0 h_4 \tilde{\delta}_{s+4} = 0$.

This finishes the proof of Theorem 1.2. □

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Xiugui LIU
School of Mathematical Sciences and LPMC
Nankai University
Tianjin 300071, P. R. China
e-mail: xgliu@nankai.edu.cn

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He WANG
School of Mathematical Sciences
Nankai University
Tianjin 300071, P. R. China
e-mail: wanghe85@yahoo.com.cn