Turk J Math
35 (2011) , 323 – 331.
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doi:10.3906/mat-0905-26

Krull dimension of types in a class of first-order theories

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Abstract

We study a class of first-order theories whose complete quantifier-free types with one free variable either have a trivial positive part or are isolated by a positive quantifier-free formula—plus a few other technical requirements. The theory of vector spaces and the theory fields are examples. We prove the amalgamation property and the existence of a model-companion. We show that the model-companion is strongly minimal. We also prove that the length of any increasing sequence of prime types is bounded, so every formula has finite Krull dimension.

1. Introduction

Krull-minimal theories are defined in Definition 1 below. The main requirement is that every complete quantifier-free type with one free variable either has a trivial positive part or it is isolated by a positive quantifier-free formula. This means that the formula $x = x$ has Krull-dimension ≤ 1, as defined in Section 3 below.

We show that Krull-minimal theories have the amalgamation property in Theorem 11 and that they are model-companionable in Theorem 12. In Corollary 13 we show that the model-companion of a Krull-minimal theory is a strongly minimal theory. This reproduces in general the usual aguments used to prove elimination of quantifiers for vector spaces, torsion-free divisible groups (see e.g. Section 3.1 in [1]), and fields (see e.g. Section 3.1 in [1] and/or Section 1 in [2]). In Section 3 we consider two notions of dimension. We prove that in a Krull-minimal theory the length of any increasing sequence of prime types is bounded by the maximal degree of transcendence of its solutions. So every formula has finite Krull dimension.

In the past there has been some interest in first-order theories with formulas that satisfy descending chain conditions, see [5], [4], and references cited therein—as we found out when the final draft of this paper was ready. Our general setting is different, the most relevant difference is that we require strong properties to hold for one variable formulas and pose no requirement on formulas with many variables. Still, a question considered in this paper is not far in spirit from a question asked in [5], where [5] it is asked whether all 1-equational theories are n-equational (see Definition 2.1 in [4]). The aim of this paper is to try and understand which descending chain conditions can be deduced from good behaviour of formulas in one variable.

I wish to thank Anand Pillay for pointing out an inconsistency in a draft version of this paper and for drawing my attention to [5].

2000 *AMS Mathematics Subject Classification:* 03C60.

2. Krull-minimal theories

Throughout this paper T is a consistent theory: we shall say **model** for *model of* T, **consistent** for consistent *modulo* T, and **complete** for complete *modulo* T. The letters M, N, etc. denote models and A, B, etc. denote subsets of models. We assume that every model is contained in an infinite one and that T fixes the **characteristic of the models** i.e. all substructures generated by the empty sets are isomorphic. In other words, T is complete for quantifier-free sentences.

We say A-type, A-formula for type, respectively formula, over A. We write $\boldsymbol{\vdash_A q(z)}$, where $q(z)$ is an A-type, when $\forall z\, q(z)$ holds in every model containing the substructure generated by A. We omit the subscript when A is empty. The expression $\boldsymbol{p(z) \vdash_A q(z)}$ abbreviates $\vdash_A p(z) \to q(z)$. Let a be a possibly infinite tuple of elements of some model and let $p(w)$ the quantifier-free type of a; we shall use several times without further mention that $\vdash_a q(a,z)$ is equivalent to $p(w) \vdash q(w,z)$. The notation introduced in this paragraph is less common. From next section, when we know that a model-companion of T exists, one could substitute $\vdash_A q(z)$ with $U \vDash \forall z\, q(z)$, where U is some large saturated existentially-closed model of T.

An **positive quantifier-free formula** is a formula that contains only the connectives $\top$, $\bot$, $\wedge$ and $\vee$. An **positive quantifier-free type** is a set of positive quantifier-free formulas. Let $p(z)$ be an A-type (not necessarily positive quantifier-free). We say that $p(z)$ is **trivial over** $\boldsymbol{A}$ if $\vdash_A p(z)$. We say it is **consistent over** $\boldsymbol{A}$ if $\nvdash_A \neg p(z)$, that is, if it is realized in some model containing A or, in other words, if it is consistent with the quantifier-free type of A. We say that $p(z)$ is **maximal over** $\boldsymbol{A}$ if it is consistent over A and for every positive quantifier-free A-formula $\varphi(z)$ either $p(z) \vdash_A \varphi(z)$ or $p(z) \vdash_A \neg\varphi(z)$. It is **prime over** $\boldsymbol{A}$ if it is consistent over A and for every pair of positive quantifier-free A-formulas $\varphi(z)$ and $\psi(z)$,

$$p(z) \vdash_A \varphi(z) \vee \psi(z) \quad \Rightarrow \quad p(z) \vdash_A \varphi(z) \ \text{ or } \ p(z) \vdash_A \psi(z).$$

As expected, maximal implies prime. The specification 'over A' will be dropped when A is empty or clear from the context. We say that $p(z)$ is **principal** if there is an A-formula $\varphi(z)$ such that $\vdash_A \ p(z) \leftrightarrow \varphi(z)$. By compactness we can always assume that $\varphi(z)$ is a conjunction of formulas in $p(z)$. When $p(z)$ is maximal, we may say **isolated** for principal. The terminology could mislead the readers that uses ring of polynomials to guide their intuition. In fact, principal positive quantifier-free types correspond to finitely generated ideals in the ring of polynomials *not* to principal ideals (which do not have an analog in our setting).

Throughout this paper x denotes a single variable.

Definition 1 *We say that the theory T is* ***Krull-minimal*** *if for every tuple a,*

- D0 *the trivial type $x = x$ is prime over a;*
- D1 *for every consistent positive quantifier-free formula $\zeta(a,x)$ there is a quantifier-free formula $\vartheta(z)$ such that $\vdash_a \vartheta(a)$ and $\zeta(b,x)$ is consistent for every b such that $\vdash_b \vartheta(b)$;*
- D2 *every positive quantifier-free formula $\zeta(a,x)$ that is consistent over a, is consistent over any B containing a;*
- D3 *every non-trivial prime positive quantifier-free type $p(a,x)$ is maximal and principal.*

We say that T is ***locally Krull-minimal*** *if these conditions hold when a is a finite tuple.*

The heuristic is as follows: grosso modo axioms D2 and D3 ensure the amalgamation property of models and axiom D1 the existence of a model-companion. fail to be Axiom D0 is introduced for a smooth and non-trivial theory of dimension. It is only necessary for the definition of locally Krull-minimal theory in fact, when infinite tuples of parameters are allowed, it follows easily from D1. All results in this section are independent of D0.

Example 2 *The theory of integral domains is a Krull-minimal theory. In fact, observe that positive quantifier-free formulas in the language of rings are systems of equations. Then D0, D1 and D2 are obvious. To prove D3, let $t(a,x)$ be a polynomial of minimal deree among those such that $p(a,x) \vdash_a t(a,x)=0$. The maximality of the formula $t(a,x)=0$ is an easy consequence of Bézout identity—in the field of fractions generated by a (see e.g. section 12.3 of [3]).* □

Example 3 *The theory of modules over a fixed integral domain is a Krull-minimal theory.* □

The following examples show that the situation is different for locally Krull-minimal theories.

Example 4 *Consider a language that contains only a binary relation $r(x,y)$. Let T_{a}, T_{b}, and T_{c} be the theories that axiomatize the models that are*

a *disjoint union of tournaments and, respectively,*

b *disjoint union of dense linearly ordered sets without endpoints.*

c *disjoint union of tournaments and linearly ordered sets.*

It is easy to check that these are locally Krull-minimal theories. Observe that T_{a} is an unstable simple theory, T_{b} is an unstable theory without the independence property, and T_{c} is neither. □

Only one of the requirements in Definition 1 may fail in a locally Krull-minimal theory for some infinite tuple of parameters a: some non-trivial prime positive quantifier-free type $p(a,x)$ is non-isolated. In fact, infinite tuples of parameters are irrelevant for D0, D1 and D3 while the proposition below ensure that first claim in D3 holds also for infinite tuples of parameters.

Proposition 5 *Let T be a locally Krull-minimal theory and let $p(x)$ be a non-trivial prime positive quantifier-free A-type, for some possibly infinite set of parameters A. Then $p(x)$ is maximal.*

Proof. With $p_{\restriction B}(x)$ we denote the set of formulas in $p(x)$ with parameters in B. Note that the properties of being prime or maximal are local i.e., $p(x)$ is prime (maximal) if and only if $p_{\restriction B}(x)$ is prime (maximal) for every finite $B \subseteq A$. Then the proposition follows by compactness. □

We will often think of prime positive quantifier-free types as the positive part of a maximal quantifier-free type: this is precisely stated in point c of the following proposition. We need some notation: for every type $p(z)$ we define

$$\boldsymbol{p^{\bullet}(z)} \quad := \quad \Big\{ \neg\xi(z) \quad : \quad \xi(z) \text{ is a positive quantifier-free } A\text{-formula and } p(z) \nvdash_A \xi(z) \Big\},$$

$$\boldsymbol{p^\circ(z)} \quad := \quad \Big\{ \quad \xi(z) \quad : \quad \xi(z) \text{ is a positive quantifier-free } A\text{-formula and } p(z) \vdash_A \xi(z) \Big\}.$$

Note the dependency on A, however, as A will be always clear from the context, we omit it from the notation. When $p(z)$ is trivial then $p^\bullet(z)$ is called the **transcendental A-type** and is denoted by $\boldsymbol{o(z/A)}$. In general, it need not be a consistent type but, when it is consistent, it is also maximal. The following type is called the **positive quantifier-free type of b over A**:

$$\textbf{eqn-tp}(\boldsymbol{b/A}) \quad := \quad \Big\{ \quad \xi(z) \quad : \quad \xi(z) \text{ is a positive quantifier-free } A\text{-formula and } \vdash_{A,b} \xi(b) \Big\},$$

As usual, when A is empty we omit if from the notation.

Proposition 6 *Let $p(z)$ be any A-type. The following are equivalent:*

a *$p(z)$ is prime over A;*

b *$p^\bullet(z) \wedge p(z)$ is consistent, and consequently maximal, over A.*

c *$p^\circ(z) = \text{eqn-tp}(b/A)$ for some b.*

Proof. The implications a $\Rightarrow$ b $\Rightarrow$ c are clear by compactness. The implication c $\Rightarrow$ a amounts to claiming that $p^\circ(z) = \text{eqn-tp}(b/A)$ is prime over A. Suppose $p^\circ(z) \vdash_A \varphi(z) \vee \psi(z)$ for some positive quantifier-free A-formulas $\varphi(z)$ and $\psi(z)$. Let M be any model containing A, b, then $M \vDash \varphi(b) \vee \psi(b)$, so $M \vDash \varphi(b)$ or $M \vDash \psi(b)$. Observe that if $\varphi(z)$ is a positive quantifier-free A-formula and $M \vDash \varphi(b)$ for *some* model M containing A, b then $M \vDash \varphi(b)$ for *all* models M containing A, b and $\varphi(z)$ belongs to eqn-tp(b/A). For this it is essential to recall that by 'containing' A, b we understand 'containing the substructure generated by' A, b. Then $p^\circ(z) \vdash_A \varphi(z)$ or $p^\circ(z) \vdash_A \psi(z)$. □

Proposition 7 *We can rephrase condition* D3 *above as follows: for every complete quantifier-free type $p(a, x)$ either $p^\circ(a, x)$ is trivial or there is a positive quantifier-free formula $\xi(z, x)$ such that $\vdash_a \xi(a, x) \leftrightarrow p(a, x)$.* □

Prime types play the role of complete types when we restrict the attention to positive quantifier-free formulas. Precisely, we have the following proposition.

Proposition 8 *Let $q(z)$ be a positive quantifier-free A-type and let P be the set of prime positive quantifier-free A-types $p(z)$ such that $p(z) \vdash_A q(z)$. Then*

$$\vdash_A \; q(z) \;\leftrightarrow\; \bigvee_{p(z) \in P} p(z),$$

where if P is empty the disjunction is $\bot$.

Proof. It follows from c of Proposition 6. □

Proposition 9 *Let T be a Krull-minimal theory and let A be an arbitrary set of parameters (alternatively: T locally Krull-minimal and A finite). Let $p(x)$ be a non-trivial positive quantifier-free A-type. Then there are some positive quantifier-free A-formulas $\xi_1(x), \ldots, \xi_n(x)$ that are maximal and such that*

$$\vdash_A \; p(x) \;\leftrightarrow\; \bigvee_{i=1}^{n} \xi_i(x)\,.$$

Proof. By Proposition 8, axiom D3, and compactness. □

Theorem 10 *Let T be a Krull-minimal theory and let $\varphi(x)$ be a non-trivial positive quantifier-free A-formula. Then $\vdash_A \exists^{<n} x\, \varphi(x)$ for some n.*

Proof. Suppose for a contradiction that $\nvdash_A \varphi(x)$ and that there is an infinite set B such that $\vdash_{A,b} \varphi(b)$ for every $b \in B$. Apply Proposition 9 to obtain

$$\vdash_{A,B} \; \varphi(x) \;\leftrightarrow\; \bigvee_{i=1}^{n} \xi_i(x)\,,$$

for some positive quantifier-free A, B-formulas $\xi_1(x), \ldots, \xi_n(x)$ that are maximal over A, B. One of these formulas, say $\xi_i(x)$, is satisfied by two distinct elements in B, say b_1 and b_2. But $\xi_i(x)$ is maximal so it implies both $x = b_1$ and $x = b_1$, a contradiction. □

Theorem 11 *Let T be a locally Krull-minimal theory. The class of models of T has the amalgamation property (over sets).*

Proof. Let $A = M \cap N$ be a substructure of both M and N. We show that there is a model N' containing N and an embedding $f' : M \to N'$ that fixes A. Clearly it suffices to show that for any element $b \in M$ there a model N' containing N and some $b' \in N'$ such that $N, b \equiv_{A,\mathrm{qf}} N', b'$. Let a be a tuple that enumerates A. Let $p(z,x) = \text{eqn-tp}(a,b)$, which is prime by Proposition 6.b. Assume first that $p(a,x)$ is trivial. Then b realizes $o(x/A)$ in M. Take any N', b' realizing $o(x/N)$, which is consistent by D0. Now assume instead that $p(a,x)$ is non-trivial. By D2, the type $p(a,x)$ is consistent over N, so there is an N' containing N and $b' \in N'$ realizing $p(a,x)$. By Proposition 5, $p(a,x)$ is maximal over A, so $N, b \equiv_{A,\mathrm{qf}} N', b'$. □

Theorem 12 *Let T be a locally Krull-minimal theory. Then T has a model-companion that admits elimination of quantifiers.*

Proof. Let T_c be the theory of the existentially closed models of T and let M and N be two ω-saturated models of T_c, let a and d be finite tuples in M, respectively, N and such that $M, a \equiv_{\mathrm{qf}} N, d$. Let c be an element of M, we show that there is an e such that $M, a, c \equiv_{\mathrm{qf}} N, d, e$. Then elimination of quantifiers follows by back-and-forth (see e.g., Section 2.4 in [1]). Let $p(z,x) = \text{eqn-tp}(a,c)$, if $p(a,x)$ is trivial, then any e in N realizing $o(x/d)$ satisfies $M, a, c \equiv_{\mathrm{qf}} N, d, e$. In this case it suffices to observe that ω-saturation ensures that $o(x/d)$ is realized in N. Otherwise, since $p(a,x)$ is prime over a, by Proposition 7, there is a positive quantifier-free formula $\xi(z,x)$ such that $\vdash_a \xi(a,x) \leftrightarrow p^\bullet(a,x) \wedge p(a,x)$. As $M, a \equiv_{\mathrm{qf}} N, d$, the same holds with d substituted for a. Let $\vartheta(z)$ be as in D1. Then, for every b that s! atisfies $\vartheta(z)$, the formula $\xi(b,x)$ is consistent, so $\xi(b,x)$ satisfied in any existentially closed model containing b. Then the formula $\forall z[\vartheta(z) \to \exists x\, \xi(z,x)]$ belongs to T_c, so it holds in N. Finally, as $M, a \equiv_{\mathrm{qf}} N, d$ we obtain that $N \vDash \xi(d,e)$ for some e. As $\xi(d,x)$ is maximal over d, we obtain $M, a, c \equiv_{\mathrm{qf}} N, d, e$ as required. □

Corollary 13 *Let T be a Krull-minimal theory. Then the model-companion of T is a strongly minimal theory.*

Proof. By Theorem 10 and 12 every model of the model-companion of T is minimal. □

3. Krull dimension

Let $p(z)$ be a consistent positive quantifier-free A-type. Define **k-dim($p(z)/A$)**, the **Krull dimension** of $p(z)$ over A, as the maximal n such that there are some prime positive quantifier-free A-types $p_0(z),\dots,p_n(z)$ such that $p_i(z)\vdash_A p_{i+1}(z)\nvdash_A p_i(z)$ and $p_n(z)\vdash_A p(z)$. Directly form the definition we obtain that, if $p(z)$ is a prime positive quantifier-free type, then $\text{k-dim}(p(z)/A)=0$ if and only if $p(z)$ is maximal. Then, by Proposition 8, for any positive quantifier-free type: $\text{k-dim}(p(z)/A)=0$ if and only if $p(z)$ is a disjunction of maximal types.

We want to prove that $\text{k-dim}(p(z)/A)$ is bounded by the length of z. First we introduce another natural dimension $\text{o-dim}(p(z)/A)$. This is, roughly, the maximal degree of transcendence of a tuple satisfying $p(z)$. We will prove that $\text{k-dim}(p(z)/A)\leq\text{o-dim}(p(z)/A)$.

Recall that the consistency of $o(z/A)$ is equivalent to the primality over A of the trivial type $z=z$, that is, to requiring the validity of the following implication for every pair of positive quantifier-free A-formulas $\varphi(z)$ and $\psi(z)$:

d $$\vdash_A \varphi(z)\vee\psi(z) \quad\Rightarrow\quad \vdash_A \varphi(z) \text{ or } \vdash_A \psi(z)$$

If b is a tuple that realizes $o(z/A)$, we say that ***b* is transcendental over *A***. So, b is transcendental over A if for every positive quantifier-free A-formula $\vdash_{A,b}\varphi(b)\leftrightarrow\varphi(z)$.

Proposition 14 *Let T be a locally Krull-minimal theory and let A be an arbitrary set of parameters. The following facts hold:*

- a *$o(z/A)$ is consistent for every tuple of variables z;*
- b *$o(z/A)\subseteq o(z/B)$ whenever $A\subseteq B$;*
- c *$o(z/A)$ is non-trivial whenever A is non-empty or z is a tuple of length >1.*

Proof. To prove a we proceed by induction on the length of z. Assume that d above holds for tuple z and prove it holds for the tuple x,z. Suppose $\vdash_A \varphi(x,z)\vee\psi(x,z)$. Then, for a arbitrary, $\vdash_{A,a}\varphi(a,z)\vee\psi(a,z)$ so, from the induction hypothesis, either $\vdash_{A,a}\varphi(a,z)$ or $\vdash_{A,a}\psi(a,z)$. Now, let b be arbitrary and let a be transcendental over A,b. Suppose for definiteness that $\vdash_{A,a}\varphi(a,z)$ obtains. Then $o(x/A,b)\vdash_A\varphi(x,z)$ so, by compactness, $\vdash_{A,b}\xi(x)\vee\varphi(x,z)$ for some positive quantifier-free A,b-formula $\xi(x)$ such that $\nvdash_{A,b}\xi(x)$. Then, $\vdash_{A,b}\xi(x)\vee\varphi(x,b)$ so, by D0, either $\vdash_{A,b}\xi(x)$ or $\vdash_{A,b}\varphi(x,b)$. The first is contrary to the choice of $\xi(x)$, so $\vdash_{A,b}\varphi(x,b)$. Finally, by the arbitrarity of b, we conclude $\vdash_A\varphi(x,z)$.

Claim b is consequence of amalgamation, in fact, from Theorem 11 it follows that any quantifier-free formula consistent over A is consistent over any B containing A. Finally, to prove c observe that $\nvdash z_1=z_2$

and that $\nvdash x = a$ for any $a \in A$. □

In the sequel we work over a Krull-minimal theory T and by A we denote an arbitrary set of parameters. Alternatively, one can assume that T only locally Krull-minimal and that A is finite. The letter z denotes the tuple $z_0, \ldots, z_{n-1}$.

Let $I = \{i_1, \ldots, i_k\}$ for some $0 \leq i_1 < \cdots < i_k < n$. We write z_I for the tuple $z_{i_1}, \ldots, z_{i_k}$. Let $p(z)$ be a consistent A-type. Define **o-dim($p(z)/A$)**, which we call the **algebraic dimension** of $p(z)$ over A to be the largest cardinality of some I such that $o(z_I/A) \wedge p(z)$ is consistent. We agree that when I is empty $o(z_I/A)$ is trivial so o-dim$(p(z)/A)$ always defined and $0 \leq$ o-dim$(p(z)/A) \leq n$.

Proposition 15 *Let $q(z)$ be an A-type and let P be any set of A-types such that*

$$\vdash_A \; q(z) \; \leftrightarrow \bigvee_{p(z) \in P} p(z).$$

Then o-dim$(q(z)/A) = \max\{$o-dim$(p(z)/A) \; : \; p(z) \in P\}$. In particular, if $q(z)$ is positive quantifier-free, by Proposition 8, its dimension is the maximal dimension of a prime positive quantifier-free A-type $p(z)$ such that $p(z) \vdash_A q(z)$. □

Now, let $p(z, w)$ be an arbitrary A-type. Define

$$\boldsymbol{p(z,\text{-})} \; := \; \Big\{\varphi(z) \; : \; \varphi(z) \text{ is a positive quantifier-free } A\text{-formula and } p(z,x) \vdash_A \varphi(z)\Big\}.$$

Note the dependency on A, however, as A will be always clear from the context, we omit it from the notation. Note also that $p(z,\text{-})$ is by definition a positive quantifier-free type, independently of the complexity of $p(z, w)$.

Lemma 16 *Let $p(z)$ be a non-trivial positive quantifier-free A-type. Let $I \subseteq \{0, \ldots, n-1\}$ be a set of cardinality o-dim$(p(z)/A)$ such that $o(z_I/A) \wedge p(z)$ is consistent. Then*

$$o(z_I/A) \; \vdash_A \; p(z) \leftrightarrow \bigvee_{h=1}^{k} \xi_h(z)$$

for some positive quantifier-free A-formulas $\xi_h(z)$ such that $o(z_I/A) \wedge \xi_h(z)$ is maximal over A.

Proof. Observe that by Proposition 9, if $q(z)$ is maximal and $q(z) \nvdash_A p(z,x)$ then

$$q(z) \; \vdash_A \; p(z,x) \leftrightarrow \bigvee_{h=1}^{k} \xi_h(z,x)$$

for some positive quantifier-free formulas $\xi_h(z,x)$ such that $q(z) \wedge \xi_h(z,x)$ is maximal. This will be used below. Now, let I be as in the statement of the lemma and let J be maximal such that $I \subseteq J \subseteq \{0, \ldots, n-1\}$ and

$$o(z_I/A) \; \vdash_A \; p(z_J,\text{-}) \leftrightarrow \bigvee_{h=1}^{k} \xi_h(z_J)$$

for some for some positive quantifier-free formulas $\xi_h(z_J)$ such that $o(z_I/A) \wedge \xi_h(z_J)$ is maximal. Note that if we take $I = J$ then $p(z_J,\text{-})$ is a consequence of $o(z_I/A)$ and the requirement is satisfied with $\xi_h(z_J)$ trivial. So the required J exists. The lemma follows if we show that $J = \{0, \ldots, n-1\}$. Suppose not and let

$m \in \{0, \ldots, n-1\} \smallsetminus J$. Let $1 \leq h \leq k$ be arbitrary. We claim that $o(z_I/A) \wedge \xi_h(z_J) \nvdash_A p(z_J, z_m, \text{-})$: if not $p(z_J, z_m, \text{-})$ would be consistent with $o(z_I, z_m/A)$. Then also $p(z)$ would be consistent with $o(z_I, z_m/A)$ contradicting the maximality of I. So we can apply the proposition above

$$o(z_I/A) \wedge \xi_h(z_J) \vdash_A p(z_J, z_m, \text{-}) \leftrightarrow \bigvee_{l=1}^{k_h} \xi_{h,l}(z_J, z_m)$$

for some positive quantifier-free formulas $\xi_{h,l}(z)$ such that $o(z_I/A) \wedge \xi_h(z_J) \wedge \xi_{h,l}(z_J, z_m)$ is maximal. Then it follows that

$$o(z_I/A) \vdash_A p(z_J, z_m, \text{-}) \leftrightarrow \bigvee_{h=1}^{k} \bigvee_{l=1}^{k_h} \xi_h(z_J) \wedge \xi_{h,l}(z_J, z_m)$$

This contradicts the maximality of J proving the lemma. □

Proposition 17 *Suppose A is non-empty. Let $p(z)$ be a non-trivial positive quantifier-free A-type. Then the following are equivalent*

a *o-dim$(p(z)/A) = 0$;*

b *$p(z)$ is a disjunction of finitely many maximal positive quantifier-free formulas;*

c *$p(z)$ is a disjunction of possibly infinitely many maximal positive quantifier-free types.*

Proof. To prove a⇒b, suppose o-dim$(p(z)/A) = 0$. As $o(z_I/A)$ is trivial when I is empty, Lemma 16 implies that $p(z)$ is a disjunction a maximal formulas. The implication b⇒c is trivial. To prove c⇒a, assume c and suppose for a contradiction that $o(z_i/A) \wedge p(z)$ is consistent for some $0 \leq i < n$. Then $o(z_i/A)$ is consistent with some maximal positive quantifier-free type $q(z) \vdash p(z)$. Then $q(z_i, \text{-})$ is maximal positive quantifier-free type consistent with $o(z_i/A)$. It follows that $q(z_i, \text{-})$ is trivial. Then also $o(z_i/A)$, which is equivalent to it, is trivial. This cannot be by c of Proposition 14. □

Theorem 18 *Let $p(z)$ and $q(z)$ be non-trivial positive quantifier-free A-types such that $q(z) \vdash_A p(z) \nvdash_A q(z)$ and assume that $p(z)$ is prime. Then o-dim$(q(z)/A) <$ o-dim$(p(z)/A)$.*

Proof. Clearly o-dim$(q(z)/A) \leq$ o-dim$(p(z)/A)$. Suppose for a contradiction that equality holds. Fix I of cardinality o-dim$(q(z)/A)$ such that $o(z_I/A) \wedge q(z)$ is consistent. Then $o(z_I/A) \wedge p(z)$ is also consistent. It is easy to check that $o(z_I/A) \wedge p(z)$ is prime so, by Lemma 16, it is maximal. Let $\varphi(z)$ be a positive quantifier-free formula in $q(z)$ such that $p(z) \nvdash_A \varphi(z)$. As $o(z_I/A) \wedge p(z) \wedge \varphi(z)$ is consistent, by maximality, $o(z_I/A) \wedge p(z) \vdash_A \varphi(z)$, so $p(z) \vdash_A \varphi(z) \vee \psi(z_I)$ where $\psi(z_I)$ is a positive quantifier-free formula whose negation is in $o(z_I/A)$. By primality, either $p(z) \vdash_A \varphi(z)$ or $p(z) \vdash_A \psi(z_I)$. The first possibility contradicts the choice of $\varphi(z)$, the second contradicts the consistency of $o(z_I/A) \wedge p(z)$. □

Corollary 19 *From the theorem it follows that k-dim$(p(z)/A) \leq$ o-dim$(p(z)/A)$, so positive quantifier-free types have a finite Krull-dimension.* □

4. Final remarks and questions

The main question is whether (locally) Krull-minimal theories are (locally) Noetherian this meaning that every positive quantifier-free A-types $p(z)$ is principal (where A is finite in the local case). A second question is whether Krull and algebraic dimension agree.

Theorem 20 *Suppose that, for any set of parameters $B \supseteq A$, there is no positive quantifier-free type such that $q(z,x) \vdash_B o(x/B)$. Then k-dim$(p(z)/A)$ = o-dim$(p(z)/A)$ for every positive quantifier-free A-type $p(z)$.*

Proof. By Corollary 19 we have k-dim$(p(z)/A) \leq$ o-dim$(p(z)/A)$, so we only need to prove the converse inequality. As observed in 15, there is a prime type such that $p'(z) \vdash_A p(z)$ and o-dim$(p'(z)/A)$ = o-dim$(p(z)/A)$. As clearly k-dim$(p'(z)/A) \leq$ k-dim$(p(z)/A)$, it suffices to prove the inequality o-dim$(p'(z)/A) \leq$ k-dim$(p'(z)/A)$. We suppose o-dim$(p'(z)/A) = m+1$ and show that there is a positive quantifier-free A-type $q(z)$ such that $q(z) \vdash_A p'(z) \nvdash_A q(z)$ and o-dim$(q(z)/A) = m$. The theorem follows by induction. Let $I \subseteq \{0,\dots,n-1\}$ and $i \in \{0,\dots,n-1\} \smallsetminus I$ be such that $o(z_I, z_i/A)$ is consistent with $p'(z)$. Observe that $o(z_I/A) \wedge p'(z) \nvdash_A o(z_I, z_i/A)$. This follows from the hypothesis above after replacing z_I with some parameters b_I. Then there is a positive quantifier-free A-formula $\psi(z_I, z_i)$ such that $\nvdash_A \psi(z_I, z_i)$ and $o(z_I/A) \wedge p'(z) \wedge \psi(z_I, z_i)!$ is consistent. Then $m \leq$ o-dim$(p'(z) \wedge \psi(z_I, z_i)/A)$. As $p'(z)$ is consistent with $o(z_I, z_i/A)$ while $p'(z) \wedge \psi(z_I, z_i)$ is not, then $p'(z) \nvdash \psi(z_I, z_i)$. So $p'(z) \wedge \psi(z_I, z_i) \vdash_A p'(z) \nvdash_A p(z) \wedge \psi(z_I, z_i)$ and from Lemma 18 we obtain o-dim$(p'(z) \wedge \psi(z_I, z_i)/A) <$ o-dim$(p'(z)/A)$. So o-dim$(p'(z) \wedge \psi(z_I, z_i)/A) = m$ as required. □

When T is Noetherian and the model-companion of T is not ω-categorical the hypothesis of theorem above is satisfied. In fact, suppose $\zeta(z,x) \vdash_B o(x/B)$ for some positive quantifier-free B-formula and let U be an existentially closed saturated model containing B. By elimination of quantifier there is a pair of positive quantifier-free B-formulas $\varphi(x)$ and $\psi(x)$ such that $\varphi(x) \wedge \neg\psi(x) \rightarrow o(x/B)$ holds in U. Observe that $\varphi(x)$ has to be trivial, otherwise $o(x/B)$ would contain $\neg\varphi(x)$. Then $o(x/B)$ is principal and, together with D3, this implies that there are only finitely many quantifier-free B-types in x.

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Received: 26.05.2009