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Some properties of Associate and Presimplifiable rings

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Abstract

In this paper we study some properties of associate and presimplifiable rings. We give a characterization of the associate (resp., domainlike) pullback P of $R_1 \to R_3 \leftarrow R_2$, where R_1 and R_2 are two presimplifiable (resp., domainlike) rings. We prove that R is presimplifiable ring if and only if the factor ring $R/nil(R)$ is presimplifiable and the ideal $nil(R)$ is presimplifiable. Then we investigate the associate and presimplifiable property of the dual rings $R[x]/\langle x^2\rangle$ and its modules through the base ring R and its modules.

Key Words: Associate ring, presimplifiable ring, domainlike, pullback and dual ring.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with unity and all modules are unitary. If R is a ring, the Jacobson radical of R, the nilradical of R, the set of zero divisors of R and the set of units of R are denoted by $J(R), nil(R), Z(R)$ and $U(R)$, respectively. And the annihilator of a subset X of a module over R is denoted by $ann_R(X)$. Any unexplained terminology will be standard as in Hungerford [10].

Definition 1.1 *A ring R is called presimplifiable if, whenever for any $a, b \in R$ with a=ab, we have that $a = 0$ or $b \in U(R)$.*

One can easily verify that a ring R is presimplifiable if and only if $Z(R) \subseteq J(R)$. And every presimplifiable ring is indecomposable while the converse of this statement need not necessarily be true. Domainlike rings (i.e., $Z(R) \subseteq nil(R)$) and local rings are examples of presimplifiable rings.

Definition 1.2 *A ring R is called an associate ring if whenever any two elements a and b generates the same principal ideal of R there is a unit u such that $a = ub$.*

The class of associate rings contains a large class of rings such as presimplifiable rings, principal ideal rings, artinian rings, von Neumann regular rings and PP rings. This class of rings was originally studied by Kaplansky [12]. Then Bouvier studied presimplifiable rings in a series of papers [5]–[8]. Recently, the class of associate rings was studied more extensively by Anderson and Valdes Leon[1],[2], Spelman et al [13] and Anderson et al

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[3]. Our aim of this paper is to study some properties of presimplifiable and associate rings and to investigate the associate and presimplifiable properties of the dual ring $R[x]/\langle x^2\rangle$ and its modules.

2. Some properties of associate and presimplifiable rings

Definition 2.1 *A ring R is said to be superpresimplifiable (resp., superassociate) if every subring of R is presimplifiable(resp., associate).*

Remark 2.1 (Anderson et al [3]).

(1) Domainlike is superpresimplifiable.

(2) A subring of a presimplifiable (resp., associate) ring need not be presimplifiable (resp., associate).

(3) A superassociate ring need not be presimplifiable.

(4) A direct product of superassociate rings need not be superassociate.

Anderson et al in [3] gave $\mathbb{Z}\times\mathbb{Z}$ and $\mathbb{Z}_2\times\mathbb{Z}_2$ as an examples of superassociate rings. It is easy to generalize this result to a direct product of any two presimplifiable rings with the set of units $\{\pm 1\}$.

Theorem 2.1 *A direct product of any two presimplifiable rings with the set of units $\{\pm 1\}$ is superassociate.*

Proof.

Suppose that $R = R_1 \times R_2$ where R_1 and R_2 are two presimplifiable rings. Let S be a subring of R. Let (a, b) and (c, d) be two nonzero elements of S that generate the same principal ideal of S. Then $(a, b) = (x, y)(c, d)$ and $(c, d) = (m, n)(a, b)$ for some $(x, y), (m, n) \in S$. Then if $a \neq 0$ and $b \neq 0$ we have $(xm, yn) \in U(R)$ because the rings R_1 and R_2 are presimplifiable and $(a, b) = (xm, yn)(a, b)$. So $(x, y) \in U(R)$ and $(x, y)^2 = (1, 1)$. Hence $(x, y) \in U(S)$. And if $a = 0$ then $(y, y) \in U(S)$ and $(a, b) = (y, y)(c, d)$ because R_2 is presimplifiable ring with the set of units $\{\pm 1\}$ and $b = ynb$. If $b = 0$ then, likewise, $(x, x) \in U(S)$ with $(a, b) = (x, x)(c, d)$. □

However, the product of two presimplifiable rings with torsion units groups need not be superassociate. For example, $R = \{(n, f(x)) \in \mathbb{Z}\times\mathbb{Z}_5[x] : f(0) \equiv n \mod 5\}$ is a subring of $\mathbb{Z}\times\mathbb{Z}_5[x]$. And $\mathbb{Z}$, $\mathbb{Z}_5[x]$ are presimplifiable rings with the torsion units groups $U(\mathbb{Z}) = \{\pm 1\}$ and $U(\mathbb{Z}_5[x]) = \{1, 2, 3, 4\}$. But R is not associate because $(0, x)$ and $(0, 2x)$ generate the same principal ideal of R and there is no unit (a,b) satisfies $(0, x) = (a, b)(0, 2x)$ (see Anderson et al [3], page 1).

We consider now the pullback.

Definition 2.2 *Let R_1, R_2 and R_3 be any three rings with homomorphisms $p_i : R_i \to R_3$, $i = 1, 2$, which preserve the unity. The subring $P = \{(r_1, r_2) \in R_1 \times R_2 : p_1(r_1) = p_2(r_2)\}$ of the ring $R_1 \times R_2$ is called the pullback P of $R_1 \to R_3 \leftarrow R_2$ with the set of units $U(P) = \{(u_1, u_2) \in P : u_1 \in U(R_1)$ and $u_2 \in U(R_2)\}$.*

Anderson et al in [3] determined when the pullback P of $R_1 \to R_3 \leftarrow R_2$, where R_1 and R_2 are integral domains, is presimplifiable or associate.

Theorem 2.2 *Let R_1, R_2 and R_3 be any three rings with epimorphisms $p_i : R_i \to R_3$, $i = 1,2$, which are not isomorphisms. Suppose that R_1 and R_2 are integral domains. Then the pullback P of $R_1 \to R_3 \leftarrow R_2$ is presimplifiable (resp., associate) if and only if $p_i^{-1}(1) \subseteq U(R_i), i = 1,2$ (resp., $p_1(U(R_1)) = p_2(U(R_2))$.*

In the same manner we can prove that the pullback P of $R_1 \to R_3 \leftarrow R_2$ where R_1 and R_2 are presimplifiable rings.

Theorem 2.3 *Let R_1, R_2 and R_3 be any three rings with epimorphisms $p_i : R_i \to R_3$ $, i = 1,2$, which are not isomorphisms. Suppose that R_1 and R_2 are presimplifiable rings. Then the pullback P of $R_1 \to R_3 \leftarrow R_2$ is presimplifiable if and only if $p_i^{-1}(1) \subseteq U(R_i), i = 1,2$.*

Proof. Let $a \in p_1^{-1}(1)$ and $x \in Ker(p_2) - \{0\}$. Then $(a,1),(0,x) \in P$ with $(0,x)(a,1) = (0,x)$. Since P is presimplifiable we have $(a,1) \in U(P)$. Thus $a \in U(P)$.

Conversely, let $(a,b) = (x,y)(a,b)$ where $(a,b),(x,y) \in P$. Assume that $(a,b) \neq (0,0)$, say $a \neq 0$. Since $a = xa$ we have $x \in U(R_1)$. So there exists $t \in R_2$ such that $p_1(x^{-1}) = p_2(t)$. Hence $(x^{-1},t) \in P$. Therefore, $(x,y)(x^{-1},t) = (1,ty) \in P$ and hence $ty \in p_2^{-1}(1)$. But $p_2{}^{-1}(1) \subseteq U(R_2)$. So $(x,y) \in U(P)$ and hence P is presimplifiable. □

Theorem 2.4 *Let R_1, R_2 and R_3 be any three rings with epimorphisms $p_i : R_i \to R_3$ $, i = 1,2$, which are not isomorphisms. Suppose that R_1 and R_2 are two presimplifiable rings. Then the following statements are equivalent:*

(1) The pullback P of $R_1 \to R_3 \leftarrow R_2$ is associate; and

(2) $p_1(U(R_1)) = p_2(U(R_2))$ or $p_i(U(R_i)) \nsubseteq p_j(U(R_j))$ implies that $Ker(p_i) \subsetneq Z(R_i)$ and for any $u_i \in U(R_i)$ with $p_i(u_i) \in p_i(U(R_i)) - p_j(U(R_j))$ and $a \in Ker(p_i)$ there exists $v_i \in U(R_i)$ with $p_i(v_i) \in p_1(U(R_1)) \cap p_2(U(R_2))$ and $u_i - v_i \in ann_{R_i}(a)$.

Proof. (1)$\Longrightarrow$(2): Suppose that $p_1(U(R_1)) \neq p_2(U(R_2))$. WLOG assume that $p_1(U(R_1)) \nsubseteq p_2(U(R_2))$. Then by (Proposition 5, [3]), $Ker(p_1) \nsubseteq Z(R_1)$ implies that P is not associate. So we must assume that $Ker(p_1) \subseteq Z(R_1)$. Now let, $a \in Ker(p_1) - \{0\}$ and $u \in U(R_1)$ with $p_1(u) \in p_1(U(R_1)) - p_2(U(R_2))$. Let $t, r \in R_2$ with $p_1(u) = p_2(r)$ and $p_1(u^{-1}) = p_2(t)$. Therefore $(u,r),(u^{-1},t) \in P - U(P)$. Consequently, $(a,0) = (u^{-1},t)(au,0)$ and $(au,0) = (u,r)(a,0)$, so, $(a,0)$ and $(au,0)$ generate the same ideal of P. Since P is associate, there exists a unit $(m,n) \in P$ with $(au,0) = (m,n)(a,0)$, so, $a(u-m) = 0$ and hence, $u - m \in ann_{R_1}(a) \subseteq Z(R_1)$. Also, $u - m \in R_1 - Ker(p_1)$. Otherwise, $p_1(u) = p_1(m) = p_2(n)$, a contradiction. So $Ker(p_1) \subsetneq Z(R_1)$.

(2)$\Longrightarrow$(1): Let (a,b) and (c,d) be two nonzero elements of P with $(c,d) = (r,s)(a,b)$ and $(a,b) = (m,n)(c,d)$ for some $(m,n),(r,s) \in P$. If $a \neq 0$ and $b \neq 0$ then $(r,s),(m,n) \in U(P)$ because R_1 and R_2 are presimplifiable. So WLOG assume that $b = 0$ and $(r,s),(m,n) \in P - U(P)$. Then $r, m \in U(R_1)$ and $s, n \in R_2 - U(R_2)$. If there exists $x \in U(R_2)$ such that $p_1(m) = p_2(x)$, then $(m,x) \in U(P)$ with $(a,b) = (m,x)(c,d)$. Otherwise, $p_1(m) \in p_1(U(R_1)) - p_2(U(R_2))$. But $c \in Ker(p_1)$ because $c(1-rm) = 0$. So there exists $v \in U(R_1)$ such that $p_1(v) \in p_1(U(R_1)) \cap p_2(U(R_2))$ and $m - v \in ann_{R_1}(c)$. Then $p_1(v) = p_2(t)$ for some $t \in U(R_2)$. Hence $(a,b) = (v,t)(c,d)$, $(v,t) \in U(P)$. Thus P is associate. □

Example 2.1 *The pullback P of* $\mathbb{Z} \to \mathbb{Z}_8 \leftarrow \mathbb{Z}_{16}$ *with natural maps* $p_i, i = 1, 2$, *is associate eventhough* $p_1(U(\mathbb{Z})) = \{1, 7\} \neq \{1, 3, 5, 7\} = p_2(U(\mathbb{Z}_{16}))$ *since* $Ker(p_2) \subsetneq Z(\mathbb{Z}_{16})$ *and* $5 - 7,\ 3 - 1 \in ann_{R_2(8)}$.

Now, we determine when the pullback P of $R_1 \to R_3 \leftarrow R_2$ where R_1 and R_2 are domainlike is again domainlike.

Theorem 2.5 *Let* R_1, R_2 *and* R_3 *be rings with epimorphisms* $p_i : R_i \to R_3$ *,* $i = 1, 2$, *which are not isomorphisms. Suppose that* R_1 *and* R_2 *are domainlike. Then the pullback P of* $R_1 \to R_3 \leftarrow R_2$ *is domainlike if and only if* $Ker(p_i) \subseteq Z(R_i), i = 1, 2$.

Proof. Let $a \in Ker(p_1) - \{0\}$. Then $(a, 0) \in Z(P)$ because $(a, 0)(0, b) = (0, 0)$ for any $b \in Ker(p_2)$. But P is domainlike so $(a, 0)^n = (0, 0)$ and hence $a^n = 0$. Thus $a \in Z(R_1)$.

Conversely, let $(a, b) \in Z(P)$. If $a \neq 0, b \neq 0$, then $a \in Z(R_1)$ and $b \in Z(R_2)$. But R_1 and R_2 are domainlike, so $a^n = 0$ and $b^m = 0$ for some $m, n \in \mathbb{Z}$. Therefore, $(a, b)^{mn} = (0, 0)$ and hence $(a, b) \in nil(P)$. Also, if $b = 0$ then $a \in Ker(p_1)$. But $Ker(p_1) \subseteq Z(R_1)$ and R_1 is domainlike. So, we have $a \in nil(R_1)$. Thus $(a, b) \in nil(P)$. □

Example 2.2 *The pullback P of* $\mathbb{Z}_{p^\alpha} \to \mathbb{Z}_{p^\beta} \leftarrow \mathbb{Z}_{p^\gamma}$ *where* $\beta < \alpha$ *and* $\beta < \gamma$ *and p is prime number is domainlike because* $\mathbb{Z}_{p^\alpha}$ *and* $\mathbb{Z}_{p^\gamma}$ *are domainlike and* $Ker(p_i) \subseteq Z(R_i), i = 1, 2$.

The following is a consequence of Theorem 2.6 and Theorem 2.7.

Corollary 2.1 *If* R_1 *and* R_2 *are domainlike, then the pullback P of* $R_1 \to R_3 \leftarrow R_2$ *is associate but not domainlike if and only if* $Ker(p_i) \nsubseteq Z(R_i), i = 1$ *or* 2 *and* $p_1(U(R_1)) = p_2(U(R_2))$.

Example 2.3 *Let P be the pullback of* $\mathbb{Z}_9 \to \mathbb{Z}_3 \leftarrow \mathbb{Z}_9[x]$ *with the epimorphisms mapping* $p_1 : \mathbb{Z}_9 \to \mathbb{Z}_3$ *defined by* $p_1(x) = \bar{x}$ *and* $p_2 : \mathbb{Z}_9[x] \to \mathbb{Z}_3$ *defined by* $p_2(f(x)) = f(0)$. *Note that,* $p_1(U(\mathbb{Z}_9)) = p_2(U(\mathbb{Z}_9[x])) = \{1, 2\}$ *and* $Ker(p_2) \nsubseteq Z(\mathbb{Z}_9[x])$ *since* $3 + x \in Ker(p_2) - Z(\mathbb{Z}_9[x])$. *So P is associate but not domainlike. Moreover, P is not presimplifiable because* $(3, 0) = (1, 1 + x)(3, 0)$ *and* $1 + x \in \mathbb{Z}_9[x] - U(\mathbb{Z}_9[x])$.

Next we give a new characterization of presimplifiable rings. But first we introduce the following definition.

Definition 2.3 *Let R be a ring and I be an ideal of R. Then I is called a presimplifiable ideal if for every* $a \in I - \{0\}$ *and* $b \in R$ *with the property* $a = ba$ *implies that* $b \in U(R)$.

Clearly, every ideal of a presimplifiable ring is presimplifiable.

Lemma 2.1 *Let R be a ring and I be an ideal of R. Then I is a presimplifiable ideal if and only if* $ann_R(a) \subseteq J(R)$ *for every* $a \in I - \{0\}$.

Proof. Let $a \in I - \{0\}$ and $x \in ann_R(a)$. Then $x \cdot a = 0$. Hence $a(tx+1) = a$ for any $t \in R$. Therefore $tx + 1 \in U(R)$ and hence $x \in J(R)$.

Conversely, let $a = ba$ where $0 \neq a \in I$ and $b \in R$. Then $b - 1 \in ann_R(a)$, $ann_R(a) \subseteq J(R)$. Consequently, $b \in U(R)$. □

Theorem 2.6 *A ring R is presimplifiable if and only if the factor ring $R/nil(R)$ is presimplifiable and the ideal $nil(R)$ is presimplifiable.*

Proof. Let $\bar{a} \in Z(R/nil(R)) - \{\bar{0}\}$. Then $\bar{a}\bar{b} = \bar{0}$ for some $\bar{b} \in Z(R/nil(R)) - \{\bar{0}\}$. So there exits $m \in \mathbb{N}$ such that $a^m b^m = 0$ where $b^m \neq 0$. Therefore, $a \in Z(R)$. Since R is presimplifiable ring we have $a \in J(R)$. So $ax + 1 \in U(R)$ for every $x \in R$. Consequently, $\bar{a}\bar{x} + \bar{1} \in U(R/nil(R))$ for every $\bar{x} \in R/nil(R)$ i.e. $\bar{a} \in J(R/nil(R))$.

Conversely, let $ab = a$ where a and b are nonzeros belong to R. Then $a \in nil(R)$ or $\bar{0} \neq \bar{a} \in R/nil(R)$. Since $nil(R)$ is presimplifiable ideal and $R/nil(R)$ is presimplifiable ring, we have $b \in U(R)$. □

Anderson and Valdes Leon [3] extended the presimplifiable and associate property to modules. Recall that a module M is presimplifiable if for every $m \in M$ and $u \in R$, $m = um$ implies that $u \in U(R)$. While a module M is associate if for every $m, n \in M$ with $mR = nR$ implies that there exists $u \in U(R)$ such that $m = un$. It is easy to verify that every presimplifiable module is associate. And a module M is presimplifiable if and only if $ann_R(m) \subseteq J(R)$ for every $m \in M - \{0\}$. Recall that, a module M is torsion-free as an R-module if $ann_R(m) \subseteq Z(R)$ for every $m \in M - \{0\}$. Clearly, for any presimplifiable ring, a torsion-free R-module is presimplifiable. However the converse need not be true as the following example shows.

Theorem 2.7 *Let R be a ring and T be a maximal ideal of R with $T \nsubseteq Z(R)$. Let $M = R/T$ and $S = R - T$. Then M is presimplifiable module over a ring of fractions $S^{-1}R$ eventhough it is not torsion-free.*

Proof. Note that, $S^{-1}R$ is local ring so it is presimplifiable. And M is $S^{-1}R$ module under the multiplication $\frac{a}{b} \cdot \bar{m} = \overline{acm}$ where $\frac{a}{b} \in S^{-1}R$, $\bar{m} \in M$ and $\overline{bc} = \bar{1}$. Now, let $\bar{m} = \frac{a}{b} \cdot \bar{m}$ and $\frac{a}{b} \in S^{-1}R - U(S^{-1}R)$. Then $\bar{a} = \bar{0}$ since $a \in T$. Thus, $\bar{m} = \bar{0}$ and hence M is presimplifiable as an $S^{-1}R$ module. However, M is not torsion-free. To show this let $m \in T - Z(R)$. Therefore, $\frac{m}{1} \cdot \bar{1} = \bar{0}$. Consequently, $ann_{S^{-1}R}(\bar{1}) \nsubseteq Z(S^{-1}R)$. So the result holds. □

3. The ring of dual numbers

Let R be a commutative ring with unity. Then the ring $R_0 = R[x]/(x^2)$ is called the ring of dual numbers. This section is devoted to study the associate and presimplifiable properties of the dual ring R_0 and its modules via the basic ring R and its modules.

First, we need to prove the following important lemma.

Lemma 3.1 *Let $R_0 = R[x]/(x^2)$ be a ring of dual numbers. Then*

1. $U(R_0) = \{a + bx : a, b \in R \text{ and } a \in U(R)\}$

2. $J(R_0) = \{a + bx : a, b \in R \text{ and } a \in J(R)\}$

3. $Z(R_0) = \{a + bx : a, b \in R \text{ and } a \in Z(R)\}$

4. $nil(R_0) = \{a + bx : a, b \in R \text{ and } a \in nil(R)\}$.

Proof.

(1) Note that, $a + bx \in R_0$ and $a \in U(R)$ implies that $(a + bx)(a^{-1} + (-a^{-2}b)x) = 1$.

(2) $a+bx \in J(R_0) \Longleftrightarrow 1 + (a + bx)(c + dx) \in U(R_0)$ for every $c+dx \in R_0 \Longleftrightarrow 1 + ac \in U(R)$ for every $c \in R \Longleftrightarrow a \in J(R)$.

(3) Let $a \in Z(R)$. Then $a = 0$ gives $(a + bx)(a + bx) = 0$ for any $b \in R$. And $a \neq 0$ implies that there exists $0 \neq c \in Z(R)$ with $(a + bx)(cx) = 0$ for every $b \in R$. So, $a + bx \in Z(R_0)$.

(4) Let $a \in nil(R)$. Then $a^n = 0$ for some $n \in \mathbb{N}$. But $(a + bx)^{n+1} = a^{n+1} + (n+1)a^n bx$. Thus $a + bx \in nil(R_0)$.

□

As a simple consequence of Lemma 3.1 we obtain the following results.

Theorem 3.1 *If $R_0 = R[x]/(x^2)$ is the ring of dual numbers then*

1. *R_0 is presimplifiable if and only if R is presimplifiable.*

2. *R_0 is domainlike if and only if R is domainlike.*

Proof. Note that, $Z(R_0) \subseteq J(R_0)$ (resp., $Z(R_0) \subseteq nil(R_0)$) if and only if $Z(R) \subseteq J(R)$ (resp., $Z(R) \subseteq nil(R)$). □

Theorem 3.2 *Let $R_0 = R[x]/(x^2)$ be the ring of dual numbers. Then R_0 is associate implies that R is associate.*

Proof. Suppose that a and b are two nonzero elements belonging to R with $aR = bR$. Then $aR_0 = bR_0$ and hence $a = (c + dx)b$ for some $c + dx \in U(R_0)$. So, $a = cb, c \in U(R)$. Therefore, R is associate. □

This raises the question: if the ring R is associate, is it true that the ring of the dual numbers R_0 is also associate? However, next we will give some examples of associate rings R for which the rings of dual numbers R_0 are also associate.

Recall that a ring R is a stable-range1 if for any $a, x, b \in R$ satisfying $ax + b = 1$, there exists $y \in R$ such that $a + by \in U(R)$. This class of rings is studied extensively in the literature, see, for example, Bass [4], Mental [11] and Chen [9]. It is easy to prove that a ring of stable range1 is associate.

Theorem 3.3 *Let $R_0 = R[x]/(x^2)$ be a ring of satble-range1. If R is stable-range1, then so is R_0.*

Proof. Let $a + bx, c + dx$ and $m + nx \in R_0$ satisfying $(a + bx) + (m + nx)(c + dx) = 1$. Then $a + mc = 1$. Since R is a stable-range1 ring, there exist $v \in R$ and $u \in U(R)$ with $av + c = u$. So, $(a + bx)v + (c + dx) = u + (bv + d)x \in U(R_0)$. □

Theorem 3.4 *Let* $\{R_\lambda : \lambda \in \Gamma\}$ *be a nonempty family of rings then* $(\prod R_\lambda)_0 \cong \prod (R_\lambda)_0$

Proof. Define $\Psi : (\prod R_\lambda)_0 \to \prod (R_\lambda)_0$ by the relation

$$\Psi((a_\lambda) + (b_\lambda)x) = (a_\lambda + b_\lambda x).$$

Then it is easy to verify that Ψ is a ring homomorphism. □

Since the direct product of associate rings is associate ([13], Theorem 6) using Theorems 3.1 and 3.3, we obtain the following results.

Corollary 3.1 *Let* $\{R_\lambda : \lambda \in \Gamma\}$ *be a family of presimplifiable rings or stable-range1 rings then* $(\prod R_\lambda)_0$ *is associate ring.*

Corollary 3.2 *For any principal ideal ring R, the ring of the dual numbers* R_0 *is associate.*

Proof. Note that, a principal ideal ring R is a direct product of principal ideal domains and special principal ideal rings ([14], Theorem 33, page 245), and special principal ideal rings are presimplifiable since they are local. So, R_0 is a direct product of presimplifiable rings. Hence R_0 is associate ring. □

Next, we investigate the relationship between presimplifiable (resp., associate) R_0-module and presimplifiable (resp., associate) R-module. In the following theorem a characterization of R_0-module is given.

Theorem 3.5 *If* $R_0 = R[x]/(x^2)$ *is the ring of dual numbers then M is an* R_0*-module if and only if M is an R-module and there exists an R-homomorphism* $\alpha : M \to M$ *such that* $\alpha^2 = 0$.

Proof. Define $\alpha : M \to M$ by $\alpha(m) = x.m$ then α is an R homomorphism with $\alpha^2 = 0$.

Conversely, let M be an R-module with R homomorphism $\alpha : M \to M$ with $\alpha^2 = 0$. Then M is an R_0-module under multiplication $(a + bx) \cdot m = a \cdot m + b \cdot \alpha(m), m \in M$. □

Theorem 3.6 *Let M be an* R_0*-module. Then M is presimplifiable* R_0*-module if and only if M is presimplifiable R-module.*

Proof. Suppose that M is a presimplifiable R_0 module. Let $m \in M - \{0\}$ and $a \in R$ with $am = 0$ then, $a \in ann_{R_0}(m)$ and hence, $a \in J(R_0)$. Thus, $a \in J(R)$ and hence, $ann_R(m) \subseteq J(R)$.

Conversely, let $m \in M - \{0\}$ and $a + bx \in R_0$ with $(a + bx) \cdot m = 0$. Therefore, $ax \cdot m = 0$. Now, if $x \cdot m \neq 0$ then $a \in ann_R(x \cdot m) \subseteq J(R)$ and if $x \cdot m = 0$ then $a \cdot m = 0$ and hence $a \in ann_R(m) \subseteq J(R)$. So, $a + bx \in J(R_0)$. □

Theorem 3.7 *Let M be an* R_0*-module. Then M is a torsion-free R-module if and only if M is a torsion-free* R_0*-module.*

Proof. Suppose that M is torsion-free R-module. Let $a + bx \in R_0$ and $m \in M - \{0\}$ with $(a + bx)m = 0$. Then $a \cdot m = 0$ or $ax \cdot m = 0$. Hence $a \in ann_R(m)$ or $a \in ann_R(x \cdot m)$ and thus $a \in Z(R)$. Therefore, $a + bx \in Z(R_0)$. □

Anderson et al. [3] proved that for $R = \mathbb{Z}$, M is presimplifiable R-module if and only if M is torsion-free. So we can conclude the following.

Corollary 3.3 *Let $R = \mathbb{Z}$ and M be an R_0 module. Then M is presimplifiable if and only if M is torsion-free.*

However, if M is associate as R_0 module, it need not be the case that M is associate as R module.

Example 3.1 *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_9$ with $x \cdot \bar{1} = \bar{3}$. Then it is easy to see that M is associate R_0 module. However, M is nonassociate R-module otherwise $M = T \bigoplus F$ where $4T = 0$ or $6T = 0$ and F is torsion-free ([3], Theorem 12).*

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