

Nilpotent elements and reduced rings*

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Abstract

In this paper, we show the following results: (1) R is a min-leftsemicentral ring if and only if $eR(1 - e)Re = 0$ for all $e \in ME_l(R)$; (2) Quasi-normal rings, NI rings and weakly reversible rings are all min-leftsemicentral ring; (3) R is left $MC2$ ring if and only if $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in R$ if and only if every projective simple left R -module is MUP -injective; (4) R is reduced if and only if R is n -regular and quasi-normal if and only if R is n -regular and weakly reversible; (5) R is strongly regular if and only if R is regular and quasi-normal if and only if R is regular and weakly reversible.

Key Words: Min-leftsemicentral rings, quasi-normal rings. NCI rings, weakly reversible rings, left $MC2$ rings, directly finite rings, regular rings.

1. Introduction

Throughout this paper every ring is associative with identity. Let R be a ring, we use $P(R), N^*(R), N(R), J(R), E(R)$ and $U(R)$ to denote the prime radical (i.e., the intersection of all prime ideals), the nilradical (i.e., the sum of all nil ideals), the set of all nilpotent elements, the Jacobson radical, the set of all idempotent elements and the set of all invertible elements of R , respectively. Note $P(R) \subseteq N^*(R) \subseteq N(R)$. Due to Marks [9], a ring R is called NI if $N^*(R) = N(R)$, and R is *reduced* if $N(R) = 0$. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced.

Hwang et al. [4] call a ring R NCI if $N(R)$ contains a nonzero ideal of R whenever $N(R) \neq 0$. Clearly, NI rings are NCI , But the converse need not be true by [4, Example 1.2].

According to [14], an element k of a ring R is called *left minimal* if Rk is a minimal left ideal of R , and an idempotent e of R is said to be *left minimal idempotent* if e is a left minimal element of R . We use $M_l(R)$ and $ME_l(R)$ to denote the set of all left minimal elements and the set of all left minimal idempotents elements of R , respectively.

A ring R is called

- (1) *min-leftsemicentral* if every element of $ME_l(R)$ is left semicentral in R ,

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- (2) *strongly min – leftsemicentral* if every element of $ME_l(R)$ is central in R ,
 (3) *left MC2* if for any $k \in M_l(R)$, Rk is a summand in ${}_R R$, whenever Rk is projective as left R -module.

In [14], min-leftsemicentral rings are also called left min-abel and strongly min-leftsemicentral rings are also called strongly left min-abel. Clearly, these rings are proper generalization of Abelian rings (i.e., every idempotent element of R is central). [14, Theorem 1.2] shows that R is a left quasi-duo ring if and only if R is a min-leftsemicentral MELT ring, where a ring R is called *left quasi-duo* (*MELT*, respectively) if every maximal left ideal (essential maximal left ideal, respectively) of R is an ideal. In this paper, we show that (1) *NI* rings are min-leftsemicentral, but *NCI* rings need not be (Theorem 2.2); (2) R is left *MC2* ring if and only if $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in R$ if and only if every projective simple left R -module is *MUP*-injective (Theorem 2.13) and (Theorem 3.8). Where a left R -module M is called *MUP*-injective [13] if for any complement left ideal C of R , $a \in R$, any left R -monomorphism $g : Ca \rightarrow M$, there exists $y \in M$ such that $g(ca) = cay$ for all $c \in C$.

A ring R is called *quasi-normal* if $eR(1-e)Re = 0$ for all $e \in E(R)$. According to Theorem 2.6, (e.g., R is Abelian if and only if R is left idempotent reflexive and quasi-normal), these rings are proper generalization of Abelian rings and Theorem 3.1 gives some characterization of quasi-normal rings. In term of quasi-normal rings, we show that R is strongly regular if and only if R is regular and quasi-normal (Corollary 2.12).

A ring R is called *semicommutative* [1, 8, 10] if for all $a, b \in R$, $ab = 0$ implies $aRb = 0$. This is equivalent to the definition that any left (right) annihilator over R is an ideal of R [6, Lemma 1.1]. Clearly, semicommutative rings are Abelian and *NI*, so semicommutative rings are left *MC2* and *NCI*. Since Abelian *NI* rings need not be semicommutative, left *MC2* *NCI* rings need not be semicommutative. Therefore Theorem 2.15 generalizes [7, Theorem 4].

Call a left R -module M

- (1) *nil*-injective [15] if for each $k \in N(R)$ and any left R -morphism $Rk \rightarrow M$ extends to R .
 (2) *Wnil*-injective [16] if for each $k \in N(R)$, there exists $n \geq 1$ such that $k^n \neq 0$ and any left R -morphism $Rk^n \rightarrow M$ extends to R .
 (3) *GP*-injective [10] if for each $k \in R$, there exists a positive integer n such that $k^n \neq 0$ and any left R -morphism $Rk^n \rightarrow M$ extends to R .

Clearly, *GP*-injective modules and *nil*-injective modules are *Wnil*-injective. [7, Lemma 3] shows that if R is a semicommutative ring whose every simple singular left R -module is *GP*-injective, then R is a reduced ring. We show that a ring R is reduced if and only if R is left *MC2* weakly reversible ring whose every simple singular left module is *nil*-injective, where a ring R is called *weakly reversible* [17] if $ab = 0$ implies that $Rbra$ is a nil left ideal of R for all $a, b, r \in R$. Clearly semicommutative rings are weakly reversible.

2. *NCI* Rings and reduced rings

A ring R is called *NI* if $N(R) = N^*(R)$. *NI* rings are almost completely characterized by Marks [9]. But surprisingly, some very natural characterizations of such rings seemed to have so far escaped notice. We begin by offering some of these new characterizations of *NI* rings.

Proposition 2.1 *The following conditions are equivalent for a ring R :*

- (1) R is NI .
- (2) $N(R)$ is a left ideal of R .
- (3) $N(R)$ is a right ideal of R .

Proof. (1) \implies (i), $i = 1, 2$ is evident.

(2) \implies (1) Let $a \in N(R)$ and $b \in R$. By (2), $ba \in N(R)$, so there exists $n \geq 1$ such that $(ba)^n = 0$. Clearly, $(ab)^{n+1} = a(ba)^n b = 0$, which implies $ab \in N(R)$. Therefore $N(R)$ is a right ideal of R .

Similarly, we can show that (3) \implies (1). □

Theorem 2.2 (1) *Let R be a NI ring. Then R is min-leftsemicentral.*

(2) *There exists a NCI ring which is not min-leftsemicentral.*

Proof. (1) Let $e \in ME_l(R)$ and $a \in R$. Write $h = ae - eae$. If $h \neq 0$, then $eh = 0$, $he = h$ and so we have $h^2 = 0$ and $Rh = Re$. Since R is a NI ring, $Rh \subseteq N(R)$, which implies $e \in N(R)$. This is impossible, so $h = 0$. Hence e is left semicentral in R and so R is a min-leftsemicentral ring.

(2) Let D be a division ring and the 2-by-2 upper triangular matrix ring $S = UT_2(D) = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$.

By [4, Lemma 1.1(1)], S is NCI . By [2, Lemma 1.1(7)], the matrix ring $S^{2 \times 2} = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$ is NCI . But $S^{2 \times 2}$ is not min-leftsemicentral. □

According to [4], the subrings of NCI rings need not be NCI . But we have the following proposition.

Proposition 2.3 *Let R be a NCI ring and $e = e^2 \in R$. If $ReR = R$, then eRe is NCI .*

Proof. If $N(eRe) = 0$, we are done. Hence we assume that $N(eRe) \neq 0$. Thus $N(R) \neq 0$ and so there exists a nonzero ideal I of R contained in $N(R)$ by hypothesis. Since $I = RIR = ReRIReR = ReIeR$, $eIe \neq 0$. Since eIe is an ideal of eRe and $eIe \subseteq N(eRe)$, eRe is a NCI ring. □

An ideal I of a ring R is called *quasi-normal* if idempotents can be lifted modulo I and for any $e \in E(R)$, $eR(1 - e)Re \subseteq I$, and if zero ideal is quasi-normal, then R is called *quasi-normal ring*. Clearly, semiabelian rings [3] (e.g., every idempotent element of R is either left semicentral or right semicentral) and so Abelian rings are quasi-normal.

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for all $a, b \in R$. According to [4], NCI rings need not be directly finite, but [2, Proposition 2.10] points out that NI rings must be directly finite. We have the following theorem.

Theorem 2.4 (1) *$R/N^*(R)$ is directly finite if and only if R is directly finite.*

(2) *Let I be an ideal of R and idempotents can be lifted modulo I . Then R/I is quasi-normal if and only if I is a quasi-normal ideal.*

(3) *Let I be a quasi-normal ideal of R and $a, b \in R$. If $1 - ab \in I$, then $1 - ba \in I$.*

(4) R is min-leftsemicentral if and only if $eR(1-e)Re = 0$ for all $e \in ME_l(R)$.

Proof. (1) and (2) are trivial.

(3) Let $1 - ab \in I$. Then in $\bar{R} = R/I$, $\bar{a}\bar{b} = \bar{1}$. Write $\bar{b}\bar{a} = x$. Then $x^2 = x$. Since I is a quasi-normal ideal of R , there exists an idempotent $e \in R$ such that $\bar{e} = x$, that is $e - ba \in I$. Therefore $e ba e - e \in I$. Since $eR(1-e)Re \subseteq I$, $eb(1-e)ae \in I$. Hence $e ba e - e ba e \in I$ and so $e - e ba e \in I$. Since $1 - ab \in I$, $bea - babea \in I$. Thus $bea - e ba e \in I$ because $ba - e \in I$ and $a - ae \in I$. Therefore $bea - e \in I$ and then $bea - ba \in I$. This implies $ab(1-e)ab = abab - abeab = a(ba - bea)b \in I$, so $1 - ba = 1 - e \in I$ because $1 - ab \in I$.

(4) Let R be a min-leftsemicentral ring and $e \in ME_l(R)$. Then e is left semicentral in R , which implies $eR(1-e)Re = eR(1-e)eRe = 0$.

Conversely, for any $e \in ME_l(R)$, by hypothesis, $eR(1-e)Re = 0$. If $(1-e)Re \neq 0$, then $R(1-e)Re = Re$, which implies $eRe = eR(1-e)Re = 0$, this is impossible. Hence $(1-e)Re = 0$, which shows that e is left semicentral. Therefore R is a min-leftsemicentral ring. \square

Corollary 2.5 (1) If R is quasi-normal, then R is directly finite.

(2) If R is NI , then R is directly finite.

(3) If R is quasi-normal, then R is min-leftsemicentral.

Proof. (1) This is a direct result of Theorem 2.4(3).

(2) Since R is NI , $N(R) = N^*(R)$ and $R/N^*(R)$ is a reduced ring. Hence $R/N^*(R)$ is quasi-normal, by (1), $R/N^*(R)$ is directly finite. By Theorem 2.4(1), R is directly finite.

(3) It is an immediate consequence of Theorem 2.4(4) and (2). \square

Call an ideal I of a ring R left idempotent reflexive [5], if for any $e \in E(R)$ and $a \in R$, $aRe \subseteq I$ implies $eRa \subseteq I$. If zero ideal is left idempotent reflexive, then R is called left idempotent reflexive ring. Clearly, Abelian rings and reflexive rings (e.g., $aRb = 0$ implies $bRa = 0$ for all $a, b \in R$) are left idempotent reflexive. Evidently, we have the following theorem.

Theorem 2.6 (1) Let I be an ideal of R and idempotents can be lifted modulo I . Then R/I is Abelian if and only if I is a quasi-normal ideal and left idempotent reflexive ideal of R .

(2) R is Abelian if and only if R is quasi-normal and left idempotent reflexive.

(3) The following conditions are equivalent for a ring R :

(a) R is left idempotent reflexive.

(b) For any $a \in N(R)$, $e \in E(R)$, $aRe = 0$ implies $eRa = 0$.

(c) For any $a \in P(R)$, $e \in E(R)$, $aRe = 0$ implies $eRa = 0$.

(d) For any $a \in N^*(R)$, $e \in E(R)$, $aRe = 0$ implies $eRa = 0$.

(e) For any $a \in J(R)$, $e \in E(R)$, $aRe = 0$ implies $eRa = 0$.

Let R be a ring. Write $T_n(R) = \left\{ \left(\begin{array}{cccccc} a & a_{12} & a_{13} & \cdots & a_{1n-1} & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n-1} & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n-1} & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a & a_{n-1n} \\ 0 & 0 & 0 & \cdots & 0 & a \end{array} \right) \mid a, a_{ij} \in R \right\}$ and $ST_n(R) = \left\{ \left(\begin{array}{cccccc} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & a_0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ 0 & 0 & a_0 & \cdots & a_{n-4} & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_0 & a_1 \\ 0 & 0 & 0 & \cdots & 0 & a_0 \end{array} \right) \mid a_i \in R, i = 0, 1, 2, \dots, n-1 \right\}$. Clearly, $T_n(R)$ and $ST_n(R)$ have the nonzero nilpotent ideals $\left(\begin{array}{ccccc} 0 & 0 & \cdots & 0 & R \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 \end{array} \right)$. Hence, we have.

Proposition 2.7 *For any ring R (possibly without identity), $T_n(R)$ and $ST_n(R)$ are NCI for $n \geq 2$.*

Given a ring R , the polynomial ring over R is denoted by $R[x]$. Then $R[x]/(x^n) \cong ST_n(R)$, where (x^n) is an ideal of $R[x]$ generated by x^n . Therefore we obtain the following corollary.

Corollary 2.8 *Let R be a ring. Then $R[x]/(x^n)$ is NCI.*

Let R be a ring, σ, τ ring endomorphisms of R and M a bimodule over R . The (σ, τ) extension of R and M is $R \times M_\sigma^\tau = \{(a, x) \mid a \in R, x \in M\}$ with addition defined componentwise and multiplication defined by $(a, x)(b, y) = (ab, \sigma(a)y + x\tau(b))$. Clearly $R \times M_\sigma^\tau$ is a ring. If $\sigma = \tau = id_R$, we obtain the trivial extension $R \times M = R \times M_{id_R}^{id_R}$ of R and M . Clearly, $0 \times M_\sigma^\tau = \{(0, x) \mid x \in M\}$ is a nonzero nilpotent ideal $R \times M_\sigma^\tau$.

Let R be a ring, M a bimodule over R . Write $T(R, M) = \left\{ \left(\begin{array}{cc} c & x \\ 0 & c \end{array} \right) \mid c \in R, x \in M \right\}$, then $T(R, M)$ is a ring and $T(R, M) \cong R \times M$. Especially, $T(R, R) \cong R \times R \cong R[x]/(x^2)$.

Let R be a ring and $\sigma : R \rightarrow R$ a ring endomorphism, let $R[x; \sigma]$ denote the ring of skew polynomials over R ; that is all formal polynomials in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$. Note that if $R(\sigma)$ is the (R, R) -bimodule defined by ${}_R R(\sigma) =_R R$ and $m \circ r = m\sigma(r)$ for all $m \in R(\sigma)$ and $r \in R$, then $R[x; \sigma]/(x^2) \cong R \times R(\sigma)$.

Proposition 2.9 *Let R be a ring, σ, τ ring endomorphisms of R and M a bimodule over R . Then*

- (1) $R \times M_\sigma^\tau$ is NCI.
- (2) $T(R, M)$ is NCI
- (3) $R[x, \sigma]/(x^2)$ is NCI.
- (4) $R \times M$ is NCI.
- (5) $R \times R$ is NCI.

Let R, S be rings and ${}_R M_S$ a (R, S) -bimodule. Let $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ with the addition componentwise and the usual matrix multiplication. Then E is a ring which is called the trivial extension of R and S by M . Clearly, $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is a nonzero nilpotent ideal of E . Hence we have the following proposition.

Proposition 2.10 *Let R, S be rings and ${}_R M_S$ a (R, S) -bimodule. Then $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is NCI .*

A ring R is called

regular if $a \in aRa$ for all $a \in R$.

unit-regular if for any $a \in R$, $a = aua$ for some $u \in U(R)$, where $U(R)$ denotes the group of units of R .

strongly regular if $a \in a^2R$ for all $a \in R$.

n -regular [15] if $a \in aRa$ for all $a \in N(R)$.

weakly regular if $a \in RaRa \cap aRaR$ for all $a \in R$.

left $Nduo$ if Ra is an ideal of R for all $a \in N(R)$.

2-primal if $N(R) = P(R)$.

Clearly, (1) reduced \implies left $Nduo$; (2) reduced \implies 2-primal $\implies NI \implies NCI$; (3) reduced $\implies n$ -regular; (4) reduced \implies weakly reversible.

The following theorem gives some new characterization of reduced rings, which also generalizes [4, Proposition 1.4].

Theorem 2.11 *The following conditions are equivalent for a ring R .*

- (1) R is reduced.
- (2) R is n -regular and left $Nduo$.
- (3) R is n -regular and Abelian.
- (4) R is n -regular and semiabelian.
- (5) R is n -regular and quasi-normal.
- (6) R is n -regular and 2-primal.
- (7) R is n -regular and NI .
- (8) R is n -regular and NCI .
- (9) R is n -regular and weakly reversible.

Proof. (1) \iff (3) \iff (6) \iff (7) are proved in [16, Theorem 2.7] and (1) \implies (i), $i = 2, 6, 9$; (3) \implies (4) \implies (5); (7) \implies (8) are trivial.

(2) \implies (3) Let $e \in E(R)$. For $a \in R$, write $h = ae - eae$. Clearly $he = h, eh = 0$ and $h^2 = 0$. By (2), $h = hch$ for some $c \in R$ and $hR \subseteq Rh$. This implies $hc \in Rh$ and so $h = hch \in Rh^2 = 0$. Hence $ae = eae$ for any $a \in R$, thus R is Abelian.

(5) \implies (1) Since n -regular rings are semiprime, n -regular rings are left idempotent reflexive. Since R is quasi-normal, by Theorem 2.6(3), R is Abelian. By [16, Theorem 2.7], R is reduced.

(8) \implies (1) If $N(R) = 0$, we are done. Otherwise there exists a nonzero ideal I of R contained in $N(R)$ by the NCI ness of R . Let $0 \neq a \in I$, then $a = aca$ for some $c \in R$ because R is n -regular. Since $ca \in E(R)$ and $ca \in I$, this is impossible. Hence $N(R) = 0$.

(9) \implies (1) In “(2) \implies (3)”, since R is weakly reversible and $eh = 0$, $Rhre$ is nil left ideal of R for all $r \in R$. This implies Rh is a nil left ideal of R . Since R is n -regular, this is impossible. Which shows that R is Abelian, so R is reduced. \square

Since R is strongly regular if and only if R is reduced and regular, we have the following corollary.

Corollary 2.12 *The following conditions are equivalent for a ring R .*

- (1) R is strongly regular.
- (2) R is regular and left $Nduo$.
- (3) R is regular and semiabelian.
- (4) R is regular and quasi-normal.
- (5) R is regular and 2-primal.
- (6) R is regular and NI .
- (7) R is regular and NCI .
- (8) R is regular and weakly reversible.

Proof. Since strongly regular rings are always regular and every principally left ideal is an ideal, (1) \implies (2) is clear.

(2) \implies (1) By (2), R is n -regular and left $Nduo$. By Theorem 2.11, R is Abelian. Since Abelian regular rings are always strongly regular, (2) \implies (1) is evident. \square

[14, Theorem 1.6] and [14, Theorem 1.8] show that (1) R is left $MC2$ if and only if $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in M_l(R)$; (2) R is strongly min-leftsemicentral if and only if R is min-leftsemicentral and left $MC2$.

Let D be a division ring. Then the 2-by-2 upper triangular matrix ring $UT_2(D) = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ is a weakly reversible ring by [17, Proposition 2.3]. Clearly $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in ME_l(UT_2(D))$. Since $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & D \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D & D \\ 0 & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \neq 0$, $UT_2(D)$ is not a left $MC2$ ring. Hence weakly reversible rings are not necessarily strongly min-leftsemicentral, so weakly reversible rings are not necessarily Abelian. However, We have the following theorem.

Theorem 2.13 (1) *Weakly reversible rings are directly finite.*

- (2) *Weakly reversible rings are min-leftsemicentral.*
- (3) *The following conditions are equivalent for a ring R :*
 - (a) R is left $MC2$.
 - (b) $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in R$.

- (c) $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in N(R)$.
 (d) $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in J(R)$.
 (e) $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in N^*(R)$.
 (f) $aRe = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in P(R)$.

Proof. (1) Let R be a weakly reversible ring and $ab = 1$. Write $e = ba$. Then $(1 - e)b = 0$, so we have $Rb(1 - e) \subseteq N^*(R)$ because R is a weakly reversible ring. Therefore $ab(1 - e) \in N^*(R) \subseteq N(R)$, which implies $1 - e \in N(R)$. Hence $ba = e = 1$.

(2) Let $e \in ME_l(R)$. If $eR(1 - e)Re \neq 0$, then $R(1 - e)Re = Re$ because Re is a minimal left ideal of R . Since $e(1 - e) = 0$, $R(1 - e)re$ is nil left ideal of R for all $r \in R$ by the weakly reversibility of R , Re is nil left ideal of R , which is a contradiction. Hence $eR(1 - e)Re = 0$, by Theorem 2.4(4), R is min-leftsemicentral.

(3) (a) \implies (b) Assume that $a \in R$ and $e \in ME_l(R)$ with $aRe = 0$. If $eRa \neq 0$, then there exists a $b \in R$ such that $eba \neq 0$. Since $eba \in M_l(R)$ and $ebaRe = 0$, by [14, Theorem 1.6], $eReba = 0$, which implies $eba = 0$, a contradiction. Hence $eRa = 0$.

(b) \implies (c) \implies (e) \implies (f) and (b) \implies (d) \implies (f) are trivial.

(f) \implies (a) Let $e \in ME_l(R)$ and $k \in M_l(R)$ with $kRe = 0$. If $eRk \neq 0$, then $ReRk = Rk$ by the minimality of Rk . Since $(Rk)^2 = RkRk = RkReRk = 0$, $k \in P(R)$. By (f), $eRk = 0$, which is a contradiction. Hence $eRk = 0$, by [14, Theorem 1.6], R is left $MC2$. □

A left $MC2$ ring R is called strongly left $MC2$ if R is also a weakly reversible ring. Clearly semicommutative rings are strongly left $MC2$ and strongly left $MC2$ rings are strongly min-leftsemicentral by Theorem 2.13(2) and [14, Theorem 1.8]. [7, Lemma 3] shows that if R is a semicommutative ring whose simple singular left modules are GP -injective, then R is a reduced ring. We can generalize this result as follows.

Theorem 2.14 *R is a reduced ring if and only if R is a strongly left $MC2$ ring whose simple singular left modules are $Wnil$ -injective.*

Proof. The necessity is evident.

Conversely, let $a^2 = 0$. Suppose that $a \neq 0$. Then there exists a maximal left ideal M of R containing $l(a)$. First observe that M is an essential left ideal of R . If not, then $M = l(e)$ for some $e \in ME_l(R)$. Since R is a strongly left $MC2$ ring, R is a strongly min-leftsemicentral ring, so we obtain e is central in R . Using $a \in l(a)$, we get $ea = ae = 0$. Hence $e \in l(a) \subseteq M = l(e)$, which is a contradiction. Therefore M must be an essential left ideal of R . Thus R/M is $Wnil$ -injective and so any R -homomorphism of Ra into R/M extends to one of R into R/M . Let $f : Ra \rightarrow R/M$ be defined by $f(ra) = r + M$. Note that f is a well-defined R -homomorphism. Since R/M is $Wnil$ -injective, there exists $c \in R$ such that $1 + M = f(a) = ac + M$. Since $a^2 = 0$, $Raca \subseteq N^*(R) \subseteq N(R)$. Hence $ac \in N(R)$ and so $1 - ac \in U(R)$, which implies $M = R$. This contradiction shows that $a = 0$. □

The following theorem generalizes [7, Theorem 4].

Theorem 2.15 *Let R be a strongly left $MC2$ ring. If every simple singular left R -module is GP -injective,*

then R is a reduced weakly regular ring.

Proof. By hypothesis and Theorem 2.14, R is a reduced ring, so R is a semicommutative ring. Therefore, by [7, Theorem 4], we obtain that R is a weakly regular ring. \square

Theorem 2.16 *Let R be a NCI ring. If R satisfies one of following conditions, then R is a reduced ring:*

- (1) R is left weakly regular.
- (2) Every simple left R -module is $Wnil$ -injective.
- (3) R is left MC2 whose every simple singular left module is $Wnil$ -injective.

Proof. If $N(R) \neq 0$, then there exists a nonzero ideal I of R contained in $N(R)$. Clearly, there exists a $0 \neq b \in I$ such that $b^2 = 0$ and so there exists a maximal left ideal M of R containing $l(b)$.

(1) If R is left weakly regular, then $b = cb$ for some $c \in RbR$. Since $RbR \subseteq I \subseteq N(R)$, there exists $n \geq 2$ such that $c^n = 0$. Hence $b = cb = c^2b = \cdots c^n b = 0$, which is a contradiction;

(2) By hypothesis, R/M is $Wnil$ -injective, hence every left R -homomorphism of Rb into R/M extends to one of R into R/M . Let $f : Rb \rightarrow R/M$ be defined by $f(rb) = r + M$. Then f is left R -homomorphism, so there exists $c \in R$ such that $1 - bc \in M$. Since $bc \in I \subseteq N(R)$, $1 - bc \in U(R)$, which is a contradiction.

(3) If M is not an essential left ideal of R , then $M = l(e)$ for some $e \in ME_l(R)$. Clearly $be = 0$. If $bRe \neq 0$, then $RbRe = Re$. But $RbRe \subseteq I \subseteq N(R)$, which is a contradiction because $e \notin N(R)$. Therefore $bRe = 0$. Since R is left MC2, $eRb = 0$, which implies $e \in l(b) \subseteq M = l(e)$, so $e = 0$. It is also a contradiction. Therefore M is essential, then R/M is $Wnil$ -injective. By the proof of (2), we know that a similar contradiction can be made.

All these imply $N(R) = 0$. \square

Corollary 2.17 *Let R be a NCI left MC2 ring whose every simple singular left module is GP-injective. Then:*

- (1) R is a weakly regular ring.
- (2) If R is also a MELT ring, then R is strongly regular.

Proof. (1) By Theorem 2.16, R is a reduced ring. By [6, Theorem 4], R is weakly regular.

(2) By [14, Theorem 1.2], R is a quasi-duo ring. By [7, Proposition 8], R is strongly regular. \square

3. Rings characterized by idempotents

Theorem 3.1 *The following conditions are equivalent for a ring R :*

- (1) R is quasi-normal.
- (2) $ea = 0$ implies $eRae = 0$ for any $a \in R$ and $e \in E(R)$.
- (3) $ea = 0$ implies $eRae = 0$ for any $a \in N(R)$ and $e \in E(R)$.

(4) $ae = 0$ implies $eaRe = 0$ for any $a \in R$ and $e \in E(R)$.

(5) $ae = 0$ implies $eaRe = 0$ for any $a \in N(R)$ and $e \in E(R)$.

Proof. (1) \implies (2) Let $ea = 0$. Then $a = (1 - e)a$, so $eRae = eR(1 - e)ae \subseteq eR(1 - e)Re$. By the definition of quasi-normal ring, we have $eRae = 0$.

(2) \implies (3) is trivial.

(3) \implies (1) Let $e \in E(R)$. For any $a \in R$, set $h = ae - eae$, then $h \in N(R)$ and $eh = 0$. By hypothesis, $eRhe = 0$. Since $h = he = (1 - e)ae$, $eR(1 - e)ae = eRhe = 0$ for any $a \in R$. Therefore $eR(1 - e)Re = 0$.

Similarly, we can show that (1) \iff (4) \iff (5). □

Since quasi-normal left idempotent reflexive rings are Abelian by Theorem 2.6(2) and there exists a semiprime ring which is not Abelian. Hence, “ $N(R)$ ”, as appears in Theorem 3.1, can not be replaced by any the conditions $P(R)$, $N^*(R)$ and $J(R)$, because semiprime rings are left idempotent reflexive. For min-leftsemicentral rings, we have the following result.

Theorem 3.2 *The following conditions are equivalent for a ring R :*

(1) R is min-leftsemicentral.

(2) $ea = 0$ implies $eRae = 0$ for any $a \in R$ and $e \in ME_l(R)$.

(3) $ea = 0$ implies $eRae = 0$ for any $a \in N(R)$ and $e \in ME_l(R)$.

(4) $ae = 0$ implies $eaRe = 0$ for any $a \in R$ and $e \in ME_l(R)$.

(5) $ae = 0$ implies $eaRe = 0$ for any $a \in N(R)$ and $e \in ME_l(R)$.

(6) $ae = 0$ implies $aRe = 0$ for any $a \in R$ and $e \in ME_l(R)$.

(7) $ae = 0$ implies $aRe = 0$ for any $a \in N(R)$ and $e \in ME_l(R)$.

Proof. By Theorem 2.4(4) and similar to the proof of Theorem 3.2, we have (1) \iff (2) \iff (3) \iff (4) \iff (5). By the definition of min-leftsemicentral rings, we have (1) \implies (6) \implies (7).

(7) \implies (1) Let $e \in ME_l(R)$ and $a \in R$. Write $h = ae - eae$. If $h \neq 0$, then $Rh = Re$, $h = he$, $eh = h$ and $h^2 = 0$. Set $e = ch$ for some $c \in R$. Then $h = he = hch = gh$, where $g = hc \in ME_l(R)$ and $hg = h^2c = 0$. By (7), $hRg = 0$. Since $hR = gR$, $gRg = hRg = 0$. Therefore $g = 0$, which is a contradiction. Hence $h = 0$, which implies $ae = eae$ for all $a \in R$. Hence R is min-leftsemicentral. □

A ring R is called left sub-abelian if $ea = 0$ implies $eRa = 0$ for all $e \in ME_l(R)$ and $a \in R$. Clearly, strongly min-leftsemicentral rings are left sub-abelian, and by Theorem 3.2, left sub-abelian rings are min-leftsemicentral. But there exists a min-leftsemicentral ring which is not left sub-abelian.

Let D be a division ring. Then the 2-by-2 upper triangular matrix ring $UT_2(D) = \begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ is a left quasi-duo ring, by [9, Theorem 1.2], $UT_2(D)$ is a min-leftsemicentral ring. Clearly, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in ME_l(UT_2(D))$

and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0$, but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} UT_2(D) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \neq 0$. Therefore, $UT_2(D)$ is not left sub-abelian. On the other hand, this example implies there exists a left quasi-duo ring which is not left sub-abelian. By [14, Theorem 1.8], we have the following theorem.

Theorem 3.3 *The following conditions are equivalent for a ring R :*

- (1) R is strongly min-leftsemicentral.
- (2) R is left sub-abelian left MC2.
- (3) $eRae = 0$ implies $ea = 0$ for any $a \in R$ and $e \in ME_l(R)$.
- (4) $eRae = 0$ implies $ea = 0$ for any $a \in N(R)$ and $e \in ME_l(R)$.

Proof. (1) \iff (2) is an immediate corollary of [14, Theorem 1.8]. (1) \implies (3) \implies (4) is trivial.

(4) \implies (1) Let $e \in ME_l(R)$ and $a \in R$. Write $h = ea - eae$. Then $eh = h, he = 0$ and $h^2 = 0$. Clearly, $eRhe = 0$, by (4), $h = eh = 0$. Therefore $ea = eae$, which implies e is right semicentral in R . By the remark between [14, Theorem 1.8] and [14, Theorem 1.9], we obtain that R is strongly min-leftsemicentral. \square

Contrast to Theorem 3.3, we have the following corollary.

Corollary 3.4 *The following conditions are equivalent for a ring R :*

- (1) R is min-leftsemicentral.
- (2) $eaRe = 0$ implies $ae = 0$ for any $a \in R$ and $e \in ME_l(R)$.
- (4) $eaRe = 0$ implies $ae = 0$ for any $a \in N(R)$ and $e \in ME_l(R)$.

Theorem 3.5 *The following conditions are equivalent for a ring R :*

- (1) R is left MC2.
- (2) $aRe = 0$ implies $ea = 0$ for all $e \in ME_l(R)$ and $a \in R$.
- (3) $aRe = 0$ implies $ea = 0$ for all $e \in ME_l(R)$ and $a \in N(R)$.
- (4) $aRe = 0$ implies $ea = 0$ for all $e \in ME_l(R)$ and $a \in J(R)$.
- (5) $aRe = 0$ implies $ea = 0$ for all $e \in ME_l(R)$ and $a \in N^*(R)$.
- (6) $aRe = 0$ implies $ea = 0$ for all $e \in ME_l(R)$ and $a \in P(R)$.

Proof. By Theorem 2.13, (1) \implies (2) \implies (3) \implies (5) \implies (6) and (2) \implies (4) \implies (6) are trivial.

(6) \implies (1) Now let $e \in ME_l(R)$ and $a \in P(R)$ with $aRe = 0$. If $eRa \neq 0$, then there exists a $b \in R$ such that $eba \neq 0$. Since $eba \in P(R)$ and $ebaRe = 0$, $e(eba) = 0$ by (6). Hence $eba = 0$, which is a contradiction. Therefore $eRa = 0$, by Theorem 2.13, R is left MC2. \square

Similarly, by Theorem 2.6, we have the following theorem.

Theorem 3.6 *The following conditions are equivalent for a ring R :*

- (a) R is left idempotent reflexive.
- (b) For any $a \in N(R)$, $e \in E(R)$, $aRe = 0$ implies $ea = 0$.
- (c) For any $a \in P(R)$, $e \in E(R)$, $aRe = 0$ implies $ea = 0$.
- (d) For any $a \in N^*(R)$, $e \in E(R)$, $aRe = 0$ implies $ea = 0$.
- (e) For any $a \in J(R)$, $e \in E(R)$, $aRe = 0$ implies $ea = 0$.

As for Abelian rings, we have the following theorem.

Theorem 3.7 *The following conditions are equivalent for a ring R :*

- (a) R is Abelian.
- (b) For any $a \in N(R)$, $e \in E(R)$, $eaRe = 0$ implies $ae = 0$.
- (c) For any $a \in R$, $e \in E(R)$, $eaRe = 0$ implies $ae = 0$.
- (d) For any $a \in R$, $e \in E(R)$, $eRae = 0$ implies $ea = 0$.
- (e) For any $a \in N(R)$, $e \in E(R)$, $eRae = 0$ implies $ea = 0$.

A ring R is called *left PS* [11] if for any $k \in M_l(R)$, ${}_R Rk$ is projective, and R is left universally mininjective [12] if for any $k \in M_l(R)$, $Rk = Re$ for some $e \in ME_l(R)$. Clearly, semiprime rings are left universally mininjective and left universally mininjective rings are left *PS*.

A left R -module M is called *MUP-injective* [13] if for any complement left ideal C of R , $a \in R$, any left R -monomorphism $g : Ca \rightarrow M$, there exists $y \in M$ such that $g(ca) = cay$ for all $c \in C$. In term of *MUP-injective* modules, we characterize left *MC2* rings and left universally mininjective rings as follows.

Theorem 3.8 (1) *R is left *MC2* if and only if every projective simple left R -module is *MUP-injective*.*

(2) *R is left universally mininjective if and only if R is left *PS* whose every simple left module is *MUP-injective*.*

Proof. (1) First, we assume that R is left *MC2* and M is a simple left R -module. Let C be a complement left ideal of R and $a \in R$, $f : Ca \rightarrow M$ be any nonzero left R -monomorphism. Then f is an isomorphism which implies Ca is a projective minimal left ideal of R . Since R is left *MC2*, $Ca = Re$ for some $e \in ME_l(R)$. Write $f(e) = y \in M$. Then $f(ca) = f(cae) = caf(e) = cay$ for all $c \in C$ which proves that ${}_R M$ is *MUP-injective*.

Conversely, let $k \in M_l(R)$ and ${}_R Rk$ be projective. By hypothesis, ${}_R Rk$ is *MUP-injective*. Since $l(k) = Re$ for some $e \in ME_l(R)$, $k = (1-e)k$. Since $R(1-e)$ is a complement left ideal of R , then the nonzero left R -monomorphism $f : R(1-e)k \rightarrow Rk$ defined by $f(r(1-e)k) = rk$ for all $r \in R$ can be write $f = \cdot y$ for for $y \in Rk$, that is $f(r(1-e)k) = r(1-e)ky$ for all $r \in R$. Especially, $k = f((1-e)k) = (1-e)ky = ky$. This implies $k \in kRk$ because $y \in Rk$. Hence $Rk = Rg$ for some $g \in ME_l(R)$. and so R is a left *MC2* ring.

(2) Since R is a left universally mininjective ring if and only if R is a left *PS* left *MC2* ring, by (1) and the necessity proof of (1), we can easily obtain (2) □

Finally, we generalize corner idempotents as follows: Call an idempotent e of R a left weakly corner element if $ReN = N$ for any left R -submodule N of Re . Clearly any central idempotent of a ring R is left weakly corner element. Let $e \in E(R)$ such that $ReR = R$, then e is also a left weakly corner element of R .

Theorem 3.9 *Let R be a strongly left *MC2* ring with $e \in E(R)$. If e satisfies one of the following conditions, then $S = eRe$ is strongly left *MC2*.*

- (1) e is a left weakly corner element of R .
- (2) e is contained in the central of R .
- (3) $ReR = R$.

Proof. (1) Note that any subring of a weakly reversible ring is weakly reversible so if R is weakly reversible then so is S . Now let $g \in ME_l(S)$ and $a \in S$ such that $aSg = 0$. Then $aRg = aeRg = aSg = 0$. We claim

that $g \in ME_l(R)$. In fact for any $x \in R$, if $xg \neq 0$, then $Rxg \subseteq Re$, so we have $ReRxg = Rxg$ because e is a left weakly corner element of R . Hence $eRxeg = eRxg \neq 0$. Since $eRxeg$ is a left ideal of $S = eRe$ and $g \in ME_l(S)$, $eRxg = eReg$. Therefore $Rg = Reg = ReReg = ReRxg \subseteq Rxg \subseteq Rg$, this means that $g \in ME_l(R)$. Since R is a left $MC2$ ring and $aRg = 0$, $gRa = 0$. Hence $gSa = 0$, this implies S is left $MC2$.

(2) and (3) are immediate results of (1) □

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References

- [1] Anderson, D.D., Camillo, V.: Semigroups and rings whose zero products commute. *Comm. Algebra* 27(6), 2847-2852 (1999).
- [2] Birkenmeier, G.F., Heatherly, H.E., Lee, E.K.: Completely prime ideals and associated radicals. *Proc. Biennial Ohio State-Denison Conference 1992.* (edited by Jain, S.K., Rizvi, S.T.: World Scientific, Singapore-New Jersey-London-Hong Kong 102-129 (1993).)
- [3] Chen, W.X.: On semiabelian π -regular rings. *Intern. J. Math. Sci.* 23, 1-10 (2007).
- [4] Hwang, S.U., Jeon, Y.C., Park, K.S.: On NCI rings. *Bull. Korean Math. Soc.* 44(2), 215-223 (2007).
- [5] Kim, J.Y.: Certain rings whose simple singular modules are GP -injective. *Proc. Japan. Acad.* 81, 125-128 (2005).
- [6] Kim, N.K., Lee, Y.: Extensions of reversible rings. *J. Pure Appl. Algebra* 185, 207-223 (2003).
- [7] Kim, N.K., Nam, S.B., Kim, J.Y.: On simple singular GP -injective modules. *Comm. Algebra* 27(5), 2087-2096 (1999).
- [8] Krempa, J., Niewieczeral, D.: Rings in which annihilators are ideals and their application to semigroup rings. *Bull. Acad. Polon. Sci. Ser. Sci., Math. Astronom. Phys.* 25, 851-856 (1977).
- [9] Marks, G.: On 2-primal Ore extensions. *Comm. Algebra* 29(5), 2113-2123 (2001).
- [10] Nam, S.B., Kim, N.K., Kim, J.Y.: On simple GP -injective modules. *Comm. Algebra* 23(14), 5437-5444 (1995).
- [11] Nicholson, W.K., Watters, J.F.: Rings with projective socle. *Proc. Amer. Math. Soc.* 102, 443-450 (1988).
- [12] Nicholson, W.K., Yousif, M.F.: Mininjective rings. *J. Algebra* 187, 548-578 (1997).
- [13] Roger, Y.C.M.: On von Neumann regular rings, VII. *Commentations Math. Univ. Carolinae* 23(2), 427-442 (1982).
- [14] Wei, J.C.: Certain rings whose simple singular modules are nil-injective. *Turk. J. Math.* 32, 393-408 (2008).
- [15] Wei, J.C., Chen, J.H.: Nil -injective rings. *Int. Electron. J. Algebra* 3, 1-21 (2007).
- [16] Wei, J.C., Chen, J.H.: NPP rings, reduced rings and SNF rings. *Int. Electron. J. Algebra* 4, 9-26 (2008).
- [17] Zhao, L., Yang, G.: On weakly reversible rings. *Acta. Math. Univ. Comenianae* 76(2), 189-192 (2007).

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