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Homology with respect to a kernel transformation

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Abstract

In this article we first give the relations between commonly used images of a morphism in a category. We then investigate d-homology in a category with certain properties, for a kernel transformation d. In particular, we show that, in an abelian category, d-homology, where d is induced by the subtraction operation, is the standard homology and that in more general categories the d-homology for a trivial d is zero. We also compute through examples the d-homology for certain kernel transformations d in such categories as R-modules, abelian groups and short exact sequences of R-modules. Finally, we characterize kernel transformations in the categories of R-modules, finitely generated R-modules, partial sets and pointed sets.

Key Words: Kernel, image, abelian category, standard homology, homology with respect to a kernel transformation, category of (finitely generated) R-modules, (finitely generated) abelian groups, partial sets, pointed sets.

1. Introduction

Since we have different definitions of an image of a morphism, which is a crucial entity in the definition of homology (see [2, 5, 6, 7, 9, 10, 12, 14]), we introduce all the usual images in a category in Section 2, and we investigate the relations between them. Also in this section, we give a few illustrative examples. In Section 3, for some general categories, we consider image and kernel as functors and for a pair $A \xrightarrow{f} B \xrightarrow{g} C$ with $gf = 0$, and give a functorial map from image of f to kernel of g. The homology with respect to a particular natural transformation $d : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$, called kernel transformation, where $\bar{\mathcal{C}}$ is the arrow category of $\mathcal{C}$, (see [13]), K is the kernel functor and S is the squaring functor, is investigated in Section 4, proving it is the standard homology, when the category is abelian and d is given by the subtraction operation and that it is zero when d is a trivial transformation, i.e., the projections or the zero transformation. Several examples are given in this section, computing the d-homology in the category, $Rmod$, of R-modules for $d = +(r \times s)$, with $r, s \in R$ and in the category, Sh_R, of short exact sequences of R-modules, for certain kernel transformations d. Finally in Section 4 we show for R a commutative ring with unity, the only kernel transformations in the category $Rmod$ are the ones of the form $+(r \times s)$ for some $r, s \in R$ and if, in addition, R is neotherian, these are the only transformations in the category, $FGRmod$, of finitely generated R-modules. We also prove the

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only kernel transformations in the categories $\overrightarrow{Set}$ of partial sets, (see [1, 4, 8]), and Set_* of pointed sets (see [10]), are the trivial ones.

2. Image and kernel of morphisms

Using the notation $K_f \xrightarrow{k_f} A$ for kernel, $P_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A$ for the kernel pair, and $B \underset{\nu_2}{\overset{\nu_1}{\rightrightarrows}} Q_f$ for the cokernel pair of a map $f : A \longrightarrow B$; and $Equ(f,g) \xrightarrow{equ(f,g)} A$ for the equalizer and $B \xrightarrow{coe(f,g)} Coe(f,g)$ for the coequalizer of a pair $A \underset{g}{\overset{f}{\rightrightarrows}} B$ we have the following definition.

Definition 2.1 *See [3, 11, 13]. Let $f : A \longrightarrow B$ be a morphism in a category $\mathcal{C}$. Each of the following defines an image of f (as an object).*

(a) $I_f^k = K_{c_f}$.

(b) $I_f^c = C_{k_f}$.

(c) $I_f^b = Coe(\pi_1, \pi_2)$, *where (π_1, π_2) is the kernel pair of f.*

(d) $I_f^o = Equ(\nu_1, \nu_2)$, *where (ν_1, ν_2) is the cokernel pair of f.*

Lemma 2.2 *If in the following diagram the left squares commute and the top and bottom rows are coequalizers, then there is a unique map i making the right square commute. Furthermore, i is a regular epi.*

$$\begin{array}{ccccc} A & \underset{g}{\overset{f}{\rightrightarrows}} & B & \xrightarrow{e} & C \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle 1_B} & & \downarrow{\scriptstyle i} \\ A' & \underset{g'}{\overset{f'}{\rightrightarrows}} & B & \xrightarrow[e']{} & C' \end{array}$$

Proof. Existence follows easily. Some computations show the diagram $A \underset{eg'}{\overset{ef'}{\rightrightarrows}} C \xrightarrow{i} C'$ is a coequalizer. □

Theorem 2.3 *Let $\mathcal{C}$ be a category with a zero object, pullbacks and pushouts and $f : A \longrightarrow B$ be a map in $\mathcal{C}$. Then we have the following diagram, in which all the three and four sided subdiagrams commute:*

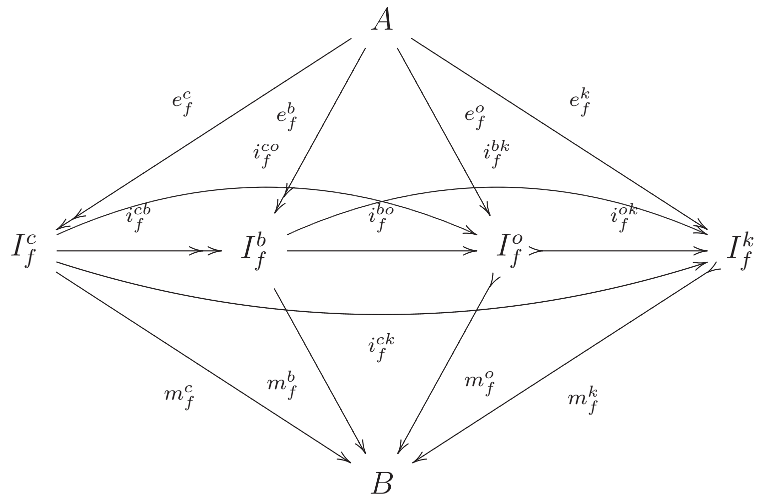
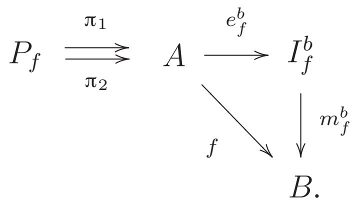


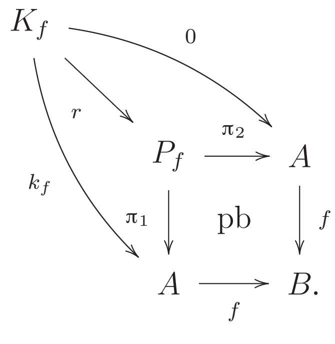
Diagram I

Furthermore, $f = m_f^b e_f^b = m_f^c e_f^c = m_f^k e_f^k = m_f^o e_f^o$ *. In addition* e_f^b, e_f^c *and* i_f^{cb} *are regular epis and* m_f^o, m_f^k *and* i_f^{ok} *are regular monos.*

Proof. To prove the existence of $i_f^{cb} : I_f^c \longrightarrow I_f^b$, we know the diagram $P_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} A \xrightarrow{e_f^b} I_f^b$ is a coequalizer and $\pi_1 f = \pi_2 f$, so there is a unique m_f^b making the following triangle commute.



Since $fk_f = 0$, there is a unique r making the following triangles commute:



Now, by the above lemma, there exists a unique regular epi i_f^{cb} making the following triangle commute:

$$\begin{array}{ccccc} K_f & \overset{k_f}{\underset{0}{\rightrightarrows}} & A & \xrightarrow{e_f^c} & I_f^c \\ & & & \searrow^{e_f^b} & \downarrow{\scriptstyle i_f^{cb}} \\ & & & & I_f^b\,. \end{array}$$

We have $m_f^c e_f^c = m_f^b e_f^b = m_f^b i_f^{cb} e_f^c$. Since e_f^c is epic, $m_f^c = m_f^b i_f^{cb}$.

We dually get maps $m_f^o : I_f^o \longrightarrow B$ and $e_f^o : A \longrightarrow I_f^o$ such that $f = m_f^o e_f^o$ and then the regular mono $i_f^{ok} : I_f^o \longrightarrow I_f^k$ with the commutative triangles $m_f^k i_f^{ok} = m_f^o$ and $i_f^{ok} e_f^o = e_f^k$.

To get $i_f^{bo} : I_f^b \longrightarrow I_f^o$, we have $m_f^o e_f^o \pi_1 = f\pi_1 = f\pi_2 = m_f^o e_f^o \pi_2$, with m_f^o monic. So $e_f^o \pi_1 = e_f^o \pi_2$ and thus e_f^o factors through e_f^b by a unique map i_f^{bo} satisfying $i_f^{bo} e_f^b = e_f^o$. We also have $m_f^o i_f^{bo} e_f^b = m_f^o e_f^o = f = m_f^b e_f^b$ with e_f^b epic. So $m_f^o i_f^{bo} = m_f^b$.

The maps i_f^{co}, i_f^{bk} and i_f^{ck} can be obtained by similar arguments as above or we can define them as $i_f^{co} = i_f^{bo} i_f^{cb}$, $i_f^{bk} = i_f^{ok} i_f^{bo}$ and $i_f^{ck} = i_f^{ok} i_f^{bo} i_f^{cb}$. Commutativity of the corresponding diagrams follows easily. □

Corollary 2.4 *Let $\mathcal{C}$ be a category with a zero object, pullbacks and pushouts and $f : A \longrightarrow B$ be a map in $\mathcal{C}$.*

(a) If m_f^c is monic, then $i_f^{cb} : I_f^c \cong I_f^b$ is an isomorphism and the maps m_f^b, i_f^{bo}, i_f^{co}, i_f^{bk} and i_f^{ck} are monic.

(b) If e_f^k is epic, then $i_f^{ok} : I_f^o \cong I_f^k$ is an isomorphism and the maps e_f^o, i_f^{bo}, i_f^{bk}, i_f^{co} and i_f^{ck} are epic.

Proof. Using Diagram I and Theorem 2.3, the result follows easily. □

Example 2.5 *In an abelian category $\mathcal{C}$, for a map $f : A \longrightarrow B$ we have*

$$I_f^c \cong I_f^b \cong I_f^o \cong I_f^k.$$

Since in an abelian category, every epi is a cokernel and every mono is a kernel (see [13]), m_f^c is monic and e_f^k is epic. By Corollary 2.4, $i_f^{cb} : I_f^c \cong I_f^b$ and $i_f^{ok} : I_f^o \cong I_f^k$ are isomorphisms and $i_f^{bo} : I_f^b \cong I_f^o$ is a bimorphism and hence also an isomorphism, since abelian categories are balanced.

Example 2.6 *In the category Grp of groups, since every epi is a cokernel, (see [14]), m_f^c is monic for every f, and so $I_f^c \cong I_f^b$.*

Now consider $f : \mathbb{Z}_2 \longrightarrow S_3$ such that $f(\bar{1}) = (1,2)$. Then $I_f^c = I_f^b = \{I, (1,2)\}$ and I_f^k is easily seen to be the normal closure of I_f^c, which is S_3. By Theorem 2.3, we have monos $I_f^b \overset{i_f^{bo}}{\rightarrowtail} I_f^o \overset{i_f^{ok}}{\rightarrowtail} I_f^k$. Since there is no group between $\{I, (1,2)\}$ and S_3, $I_f^o = I_f^b = \{I, (1,2)\}$ or $I_f^o = I_f^k = S_3$. Since f is not epi, $\nu_1 \neq \nu_2$, and so $equ(\nu_1, \nu_2) \neq 1$. It follows that $Equ(\nu_1, \nu_2) \neq S_3$ and so $I_f^o \neq I_f^k$. Therefore $I_f^o = I_f^c = \{I, (1,2)\}$.

Example 2.7 *In the category Set_* of pointed sets, since every mono is a kernel, (see [14]), e_f^k is epic for every f, and so $I_f^o \cong I_f^k$.*

Now for any $f:(X,x_0)\longrightarrow(Y,y_0)$, $I_f^c=(X/R,[x_0])$, where R is the equivalence relation on X defined by x_1Rx_2 if and only if $x_1=x_2$ or $x_1,x_2\in K_f$. On the other hand, $I_f^b=(f(X),y_0)$ and so, obviously, $I_f^c\neq I_f^b$ in some cases.

Example 2.8 *In the category, Sh_R of short exact sequences of R-modules, neither every epi is a cokernel, nor every mono is a kernel, see [13], page 177. We show for every F, m_F^c is monic and e_F^k is epic. Hence By Theorem 2.3, $i_F^{cb}: I_F^c\cong I_F^b$ and $i_F^{ok}: I_F^o\cong I_F^k$ are isomorphisms and $i_F^{bo}: I_F^b\cong I_F^o$ is a bimorphism.*

With $F=(\alpha,\beta,\gamma):M\longrightarrow N$ as shown below, we have

$$M \xrightarrow{F} N \qquad K_F \quad \text{and} \quad I_F^c = C_{k_F} \quad \text{so} \quad M \xrightarrow{e_F^c} I_F^c \xrightarrow{m_F^c} N$$

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
A & \xrightarrow{\alpha} & A' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\
B & \xrightarrow{\beta} & B' \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\
C & \xrightarrow[\gamma]{} & C' \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\qquad
\begin{array}{c}
0 \\ \downarrow \\ K_\alpha \\ {\scriptstyle f}\downarrow \\ K_\beta \\ {\scriptstyle g}\downarrow \\ I_{g\beta} \\ \downarrow \\ 0
\end{array}
\qquad
\begin{array}{c}
0 \\ \downarrow \\ \frac{I_f+K_\beta}{K_\beta} \\ \downarrow \\ \frac{B}{K_\beta} \\ \downarrow \\ \frac{C}{I_{g\beta}} \\ \downarrow \\ 0
\end{array}
\qquad
\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{\bar{f}} & \frac{I_f+K_\beta}{K_\beta} & \xrightarrow{\bar{\alpha}} & A' \\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle f'} \\
B & \longrightarrow & \frac{B}{K_\beta} & \xrightarrow{\bar{\beta}} & B' \\
{\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle g'} \\
C & \longrightarrow & \frac{C}{I_{g\beta}} & \xrightarrow[\bar{\gamma}]{} & C' \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0,
\end{array}$$

where $I_{g\beta}$ is the image of the restriction of g on K_β, $\bar{f}(a)=f(a)+K_\beta$ and $\bar{\alpha}(f(a)+K_\beta)=\alpha(a)$. To show m_F^c is monic, suppose $m_F^c h = m_F^c k$, where $h=(h_1,h_2,h_3)$ and $k=(k_1,k_2,k_3)$. Since $\bar{\alpha}$ and $\bar{\beta}$ are monic, $h_1=k_1$ and $h_2=k_2$. Since $h_3g''=gh_2=gk_2=k_3g''$ and g'' is epi, $h_3=k_3$ and hence $h=k$. So m_F^c is monic.

Next we have

$$C_F \quad \text{and} \quad I_F^k = K_{c_F} \quad \text{and then} \quad M \xrightarrow{e_F^k} I_F^k \xrightarrow{m_F^k} N$$

$$\begin{array}{c}
0\\ \downarrow \\ \frac{I_{f'}+I_\beta}{I_\beta} \\ \downarrow \\ \frac{B'}{I_\beta} \\ \downarrow \\ \frac{C'}{I_\gamma} \\ \downarrow \\ 0
\end{array}
\qquad
\begin{array}{c}
0\\ \downarrow \\ D \\ \downarrow \\ I_\beta \\ \downarrow \\ I_\gamma \\ \downarrow \\ 0
\end{array}
\qquad
\begin{array}{ccccc}
0 & & 0 & & 0\\
\downarrow & & \downarrow & & \downarrow\\
A & \xrightarrow{\alpha} & D & \longrightarrow & A'\\
{\scriptstyle f}\downarrow & & \downarrow & & \downarrow{\scriptstyle f'}\\
B & \xrightarrow{\beta} & I_\beta & \longrightarrow & B'\\
{\scriptstyle g}\downarrow & & \downarrow & & \downarrow{\scriptstyle g'}\\
C & \xrightarrow{\gamma} & I_\gamma & \longrightarrow & C'\\
\downarrow & & \downarrow & & \downarrow\\
0 & & 0 & & 0,
\end{array}$$

where $D = \{a \in A' | f'(a) \in I_\beta\}$. To show e_F^k is epic, suppose $he_F^k = ke_f^k$, where $h = (h_1, h_2, h_3)$ and $k = (k_1, k_2, k_3)$. We know that $\hat{\beta}$ and $\hat{\gamma}$ are epic, so $h_2 = k_2$ and $h_3 = k_3$. Since $f''h_1 = h_2 f' = k_2 f' = f''k_1$ and f'' is monic, $h_1 = k_1$ and so $h = k$. Therefore e_F^k is epic.

3. Image and kernel as functors

Let $\mathcal{C}$ be a category with a zero object, kernels, kernel pairs and coequalizers of kernel pairs.

Let $\bar{\mathcal{C}}$ be the arrow category of $\mathcal{C}$, see [13], with objects the morphisms of $\mathcal{C}$ and with morphisms from $f : A \longrightarrow B$ to $f' : A' \longrightarrow B'$ the pairs of morphisms (α, β) making the following square commutative:

$$\begin{array}{ccc}
A & \xrightarrow{f} & B\\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta}\\
A' & \xrightarrow[f']{} & B',
\end{array}$$

and let $\hat{\mathcal{C}}$ be the pair-chain category of $\mathcal{C}$, with objects the pair-chains, i.e., the composable pairs, (f, g), of morphisms of $\mathcal{C}$, such that $gf = 0$ and with morphisms from (f, g) to (f', g') the triples (α, β, γ) making the following squares commutative:

$$\begin{array}{ccccc} A & \stackrel{f}{\longrightarrow} & B & \stackrel{g}{\longrightarrow} & C \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ A' & \underset{f'}{\longrightarrow} & B' & \underset{g'}{\longrightarrow} & C'. \end{array}$$

The next two theorems follow easily.

Theorem 3.1 *Let $f : A \longrightarrow B$ and $f' : A' \longrightarrow B'$ be objects in $\bar{\mathcal{C}}$. The mapping $K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ that takes the object f to K_f and the morphism $(\alpha, \beta) : f \longrightarrow f'$ to $K(\alpha, \beta)$, where $K(\alpha, \beta)$ is the unique map making the square*

$$\begin{array}{ccc} K_f & \stackrel{k_f}{\longrightarrow} & A \\ {\scriptstyle K(\alpha,\beta)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ K_{f'} & \underset{k_{f'}}{\longrightarrow} & A' \end{array}$$

commutative, is a functor.

Theorem 3.2 *Let $f : A \longrightarrow B$ and $f' : A' \longrightarrow B'$ be objects in $\bar{\mathcal{C}}$. The mapping $I : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ that takes $f : A \longrightarrow B$ to $I_f = I_f^b$ and $(\alpha, \beta) : f \longrightarrow f'$ to $I(\alpha, \beta)$, where $I(\alpha, \beta)$ is the unique map making the square*

$$\begin{array}{ccc} A & \stackrel{e_f}{\longrightarrow} & I_f \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle I(\alpha,\beta)} \\ A' & \underset{e_{f'}}{\longrightarrow} & I_{f'} \end{array}$$

commutative, is a functor.

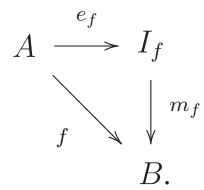
Lemma 3.3 *We have:*

(a) For each object (f, g) in $\hat{\mathcal{C}}$, there is a map $j_{fg} : I_f \longrightarrow K_g$ in $\mathcal{C}$.

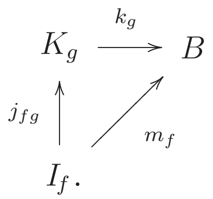
(b) For each morphism $(\alpha, \beta, \gamma) : (f, g) \longrightarrow (f', g')$ in $\hat{\mathcal{C}}$, the following square commutes.

$$\begin{array}{ccc} I_f & \stackrel{j_{fg}}{\longrightarrow} & K_g \\ {\scriptstyle I(\alpha,\beta)}\downarrow & & \downarrow{\scriptstyle K(\beta,\gamma)} \\ I_{f'} & \underset{j_{f'g'}}{\longrightarrow} & K_{g'}. \end{array}$$

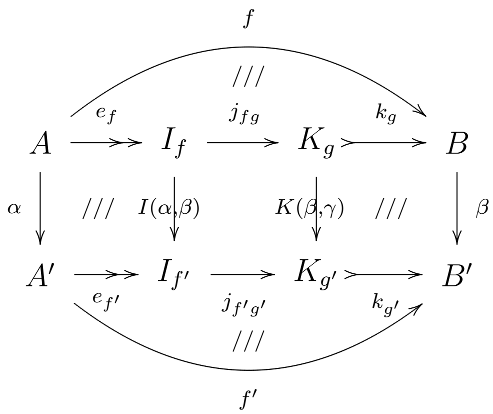
Proof. (a) Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an object of $\hat{\mathcal{C}}$. With e_f the coequalizer of the kernel pair of f, there is an m_f making the following triangle commutative:



Since $gm_fe_f = gf = 0$ and e_f is epic, $gm_f = 0$. So there is a unique map j_{fg} making the following triangle commutative:



(b) We have the following diagram:



in which, the left, the right and the outer squares commute. Since $k_{g'}$ is monic and e_f is epic, the middle square commutes. □

We now easily get the following theorem.

Theorem 3.4 *The mapping* $j : \hat{\mathcal{C}} \longrightarrow \bar{\mathcal{C}}$ *that takes the object* $(f, g) \in \hat{\mathcal{C}}$ *to* j_{fg} *and the morphism* (α, β, γ) *to* $(I(\alpha, \beta), K(\beta, \gamma))$ *is a functor.*

Remark 3.5 Let $(f, g) \in \hat{\mathcal{C}}$. By the proof of Lemma 3.3, $k_g j_{fg} = m_f$ and k_f is monic. So j_{fg} is monic if and only if m_f is monic. Therefore by Corollary 2.4, if m_f^c is monic, then so is j_{fg}.

4. Homology with respect to a kernel transformation

Definition 4.1 *Let $S : \mathcal{C} \longrightarrow \mathcal{C}$ be the squaring functor, i.e., the functor that takes $a \xrightarrow{f} b$ to $a^2 \xrightarrow{f^2} b^2$.*

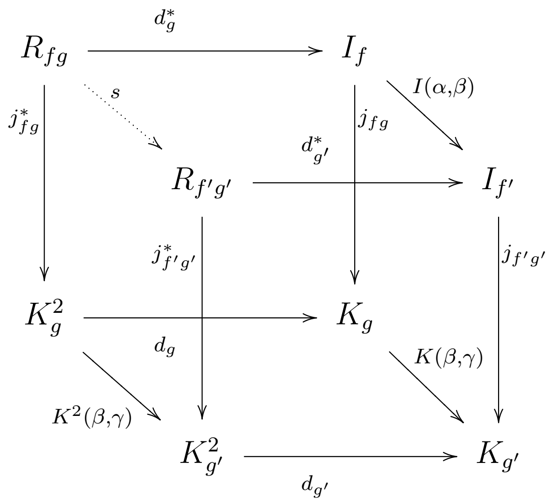
Definition 4.2 *A kernel transformation in a category $\mathcal{C}$ is a natural transformation $d : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ such that for all (f,g) in $\hat{\mathcal{C}}$, the pullback $j^*_{fg} : R_{fg} \longrightarrow K^2_g$, of j_{fg} along d_g and the coequalizer of the pair $j_1 = pr_1 j^*_{fg}$ and $j_2 = pr_2 j^*_{fg}$ exist, where pr_1 and pr_2 are the projection maps.*

With $H^d_{fg} = Coe(j_1, j_2)$ and $q = coe(j_1, j_2)$, we have the following lemma.

Lemma 4.3 *Let $d : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ be a kernel transformation. For each morphism $(\alpha, \beta, \gamma) : (f,g) \longrightarrow (f', g')$ in $\hat{\mathcal{C}}$, there exists a unique morphism $H^d(\alpha, \beta, \gamma) : H^d_{fg} \longrightarrow H^d_{f'g'}$ making the following diagram commutative:*

$$\begin{array}{ccc} K_g & \xrightarrow{q} & H^d_{fg} \\ {\scriptstyle K(\beta,\gamma)} \downarrow & & \downarrow {\scriptstyle H^d(\alpha,\beta,\gamma)} \\ K_{g'} & \xrightarrow[q']{} & H^d_{f'g'} . \end{array}$$

Proof. Let (α, β, γ) be a morphism in $\hat{\mathcal{C}}$ from (f,g) to (f', g'). Since in the diagram



the bottom square commutes by naturality of d, the right square commutes by Lemma 3.3, the front and the back squares are pullbacks, and we get a unique s making the top and the left squares commutative.

The naturality of pr_i yields $K(\beta,\gamma) pr_i = pr_i K^2(\beta,\gamma)$. So the left and the middle squares in the following diagram commute:

$$\begin{array}{ccccccc}
R_{fg} & \xrightarrow{j_{fg}} & K_g^2 & \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} & K_g & \xrightarrow{q} & H_{fg}^d \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle K^2(\beta,\gamma)} & & \downarrow{\scriptstyle K(\beta,\gamma)} & & \downarrow{\scriptstyle H^d(\alpha,\beta,\gamma)} \\
R_{f'g'} & \xrightarrow[j_{f'g'}]{} & K_{g'}^2 & \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} & K_{g'} & \xrightarrow[q']{} & H_{f'g'}^d .
\end{array}$$

Since $q = coe(j_1, j_2)$, we get the desired map $H^d(\alpha,\beta,\gamma)$ making the right square in the above diagram commutative. □

Now we easily get the following theorem.

Theorem 4.4 *The mapping* $H^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ *that takes the object* $(f,g) \in \hat{\mathcal{C}}$ *to* H_{fg}^d *and the morphism* (α,β,γ) *to* $H^d(\alpha,\beta,\gamma)$ *is a functor.*

The functor $H^d : \hat{\mathcal{C}} \longrightarrow \mathcal{C}$ *is called the* d*-homology or the homology with respect to the kernel transformation* d*.*

Let $\mathcal{C}$ be an abelian category. For each $A \in \mathcal{C}$, the projections $A^2 \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} A$ yield $pr_1 - pr_2$ which we denote by $-_A : A^2 \longrightarrow A$. It can be easily verified that these maps define a natural transformation $- : S \longrightarrow I : \mathcal{C} \longrightarrow \mathcal{C}$. So we get the kernel transformation $- \circ K : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$. Denoting $- \circ K$ also by $-$, we have the following theorem.

Lemma 4.5 *For any abelian category* $\mathcal{C}$*, the kernel transformation* $- : S \circ K \longrightarrow K : \bar{\mathcal{C}} \longrightarrow \mathcal{C}$ *is pointwise split epic.*

Proof. For each $f : A \longrightarrow B$, the right inverse of $-_f$ is the morphism $< 1, 0 >: K_f \longrightarrow K_f^2$. □

The homology of a pair $(f,g) \in \hat{\mathcal{C}}$, as defined in [13] is $Coker(j_{fg})$. We call this homology the standard homology of f *and* g and we denote it by H_{fg}^s. The corresponding functor is denoted by H^s.

Theorem 4.6 *In an abelian category* $\mathcal{C}$*,* $H^- = H^s$*.*

Proof. Since H_{fg}^- is the coequalizer $coe(j_1, j_2) : K_g \longrightarrow H_{fg}^-$ and $\mathcal{C}$ is an abelian category, we have $coe(j_1,j_2) = coker(j_1 - j_2) = coker((pr_1 - pr_2)j^*) = coker(-_g j^*) = coker(j_{fg} -_g^*)$. Now $-_g^*$, being the pullback of the split epi $-_g$, is a split epi, so $coker(j_{fg} -_g^*) = coker(j_{fg}) = H_{fg}^s$. □

Lemma 4.7 *Let* $\mathcal{C}$ *be a category with a zero object, finite products and coequalizers. If* $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$ *is an epi sink, then the coequalizer of* $A \times B \underset{\beta pr_2}{\overset{\alpha pr_1}{\rightrightarrows}} C$ *is* $C \longrightarrow 0$*. In particular, for any object* A*, the coequalizer of* $A^2 \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} A$ *is* $A \longrightarrow 0$*.*

Proof. Follows from the fact that for any morphism f with $f\alpha pr_1 = f\beta pr_2$, we have $f\alpha = f\alpha pr_1 < 1,0 >= f\beta pr_2 < 1,0 >= 0$ and $f\beta = f\beta pr_2 < 0,1 >= f\alpha pr_1 < 0,1 >= 0$ and so $f = 0$. □

Corollary 4.8 *Let (f,g) be a pair-chain. If $j^*_{fg} = \alpha \times \beta$ with (α,β) an epi sink or if j^*_{fg} is an epi, then $H^d_{fg} = 0$.*

Proof. In the former case we have, $H^d_{fg} = Coe(pr_1 j^*_{fg}, pr_2 j^*_{fg}) = Coe(\alpha pr_1, \beta pr_2) = 0$ and in the latter case, $H^d_{fg} = Coe(pr_1 j^*_{fg}, pr_2 j^*_{fg}) = Coe(pr_1, pr_2) = 0$. □

Calling the projection transformations and the zero transformation the trivial transformations, we have the following theorem.

Theorem 4.9 *Let $\mathcal{C}$ be a category with a zero object, pullbacks and coequalizers. If d is a trivial kernel transformation, then $H^d = 0$.*

Proof. Let (f,g) be any pair-chain. For $d = pr_1$, we get $j^*_{fg} = j_{fg} \times 1$, for $d = pr_2$, we get $j^*_{fg} = 1 \times j_{fg}$ and for $d = 0$, we get $j^*_{fg} = pr_1 : K^2_g \times K_{j_{fg}} \longrightarrow K^2_g$. Since $(j_{fg}, 1)$ and $(1, j_{fg})$ are epi sinks, and pr_1 is epic, the result follows from Corollary 4.8. □

Example 4.10 *Let $\mathcal{C} = Rmod$ and $d = rpr_1 + spr_2 = +(r \times s)$ with $r,s \in R$. Let (f,g) be a pair-chain. Then $R_{fg} = \{(a,b) \in K^2_g | ra + sb \in I_f\}$, j^* is the inclusion and $H_{fg} = \frac{K_g}{(j_1 - j_2)(R_{fg})} = \{[a] : a \in K_g\}$, where $[a] = \{b | r(a-b) \in (r+s)K_g + I_f\} = \{b | s(a-b) \in (r+s)K_g + I_f\}$ is the equivalence class under the equivalence relation $a \sim b$ if and only if $\exists m,n \in K_g$ such that $a - b = m - n$ and $rm + sn \in I_f$.*

Example 4.11 *As a special case of Example 4.10, for $d = +(r \times 1)$ or $d = +(1 \times r)$ with $r \in R$, we have $H^d_{fg} = \frac{K_g}{(1+r)K_g + I_f}$.*

Example 4.12 *As another special case of Example 4.10, let $\mathcal{C} = Abgrp$. For $d = +(r \times s)$, with $r,s \in \mathbb{Z}$, and any pair-chain (f,g) such that $K_g = \mathbb{Z}$ and $I_f = n\mathbb{Z}$, $H^d_{fg} = \mathbb{Z}_l$, where $l = \frac{(r+s,n)}{(r,(r+s,n))}$, with $(r+s,n)$ denoting the greatest common divisor of $r+s$ and n, etc.*

Example 4.13 *Let $\mathcal{C} = Sh_{\mathbb{Z}}$, $d = -$, and the pair-chain (f,g) be as in the following diagram with n an even integer. Then we have:*

$$
\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \quad & I_f & \xrightarrow{j_{fg}} & K_g & \quad \text{and} \quad & R_{fg} \\
\\
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & & \mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} & & \{(a,b,b-na)|a,b\in\mathbb{Z}\} \\
{\scriptstyle 2}\downarrow & & {\scriptstyle 2-}\downarrow & & \downarrow{\scriptstyle 2-} & & {\scriptstyle 2-}\downarrow & & \downarrow{\scriptstyle 2-} & & \downarrow{\scriptstyle (2-)^3} \\
\mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & & \mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} & & \{(a,b,b-na)|a,b\in\mathbb{Z}\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_2 & & \mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_2 & & \mathbb{Z}_2\times\{(0,0),(1,1)\} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 & & 0.
\end{array}
$$

Therefore $H^d_{fg} = Coker(j_1 - j_2)$ *is:*

$$
\begin{array}{ccccc}
R_{fg} & \xrightarrow{j_1 \quad j_2} & K_g & \xrightarrow{q} & H^d_{fg} \\
\\
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\{(a,b,b-na)|a,b\in\mathbb{Z}\} & \xrightarrow{na} & \mathbb{Z} & \longrightarrow & 2\mathbb{Z}_n \\
{\scriptstyle (2-)^3}\downarrow & & \downarrow{\scriptstyle 2} & & \downarrow \\
\{(a,b,b-na)|a,b\in\mathbb{Z}\} & \xrightarrow{na} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_2\times\{(0,0),(1,1)\} & \xrightarrow{0} & \mathbb{Z}_2 & \longrightarrow & \mathbb{Z}_2 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}
$$

Example 4.14 *Consider* $\mathcal{C} = Sh_{\mathbb{Z}}$*,* $d = +(r \times 1)$ *with* r *odd, and the pair-chain* (f, g) *as in the following diagram with* n *odd. Then we have:*

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad I_f \xrightarrow{j_{fg}} K_g \quad \text{and} \quad R_{fg}$$

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
{\scriptstyle 2-}\downarrow & & {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle 2-} \\
\mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 & \xrightarrow{0} & \mathbb{Z}_2 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}
\qquad
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} \\
{\scriptstyle 2-}\downarrow & & \downarrow{\scriptstyle 2-} \\
\mathbb{Z} & \xrightarrow{n-} & \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}_2 & \xrightarrow{1} & \mathbb{Z}_2 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\qquad
\begin{array}{c}
0 \\
\downarrow \\
\{(a, b, na - rb) | a, b \in \mathbb{Z}\} \\
\downarrow{\scriptstyle (2\,-)^3} \\
\{(a, b, na - rb) | a, b \in \mathbb{Z}\} \\
\downarrow \\
\{(a, b, c) | a - b - c = 0a, b, c \in \mathbb{Z}_2\} \\
\downarrow \\
0.
\end{array}$$

Therefore $H^d_{fg} = Coker(j_1 - j_2)$ *is:*

$$R_{fg} \xrightarrow{j_1 \quad j_2} K_g \xrightarrow{q} H^d_{fg}$$

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\{(a, b, na - rb) | a, b \in \mathbb{Z}\} & \xrightarrow{na-(1+r)b} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_{(n,1+r)} \\
{\scriptstyle (2\quad)^3}\downarrow & & \downarrow{\scriptstyle 2-} & & \downarrow \\
\{(a, b, na - rb) | a, b \in \mathbb{Z}\} & \xrightarrow{na-(1+r)b} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_{(n,1+r)} \\
\downarrow & & \downarrow & & \downarrow \\
\{(a, b, c) | a - b - c = 0a, b, c \in \mathbb{Z}_2\} & \xrightarrow{b-c} & \mathbb{Z}_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0.
\end{array}$$

5. Kernel transformations in some categories

Theorem 5.1 *Let R be a commutative ring with unity. Any natural transformation $- : S \longrightarrow I : Rmod \longrightarrow Rmod$ is of the form $d = +(r \times s)$, for some $r, s \in R$. In particular, any such natural transformation d in Abgrp is of the form $d = +(r \times s)$, for some $r, s \in \mathbb{Z}$.*

Proof. We first prove $d_A = +(r \times s)$ for a free R-module A. We know d_R being an R-module homomorphism from R^2 to R is of the form $d_R = +(r \times s)$. Let $A = \bigoplus_I R$ and $\pi_j : A = \bigoplus_I R \longrightarrow R$ be the jth projection. Since d is a natural transformation and π_j is an R-module homomorphism, $\pi_j d_A = \pi_j^2 d_R = +(r \times s)\pi_j^2 = \pi_j + (r \times s)$. Since $\bigoplus_I R \subseteq \prod_I R$, the projections $(\pi_j)_I$ form a mono source. It follows that $d_A = +(r \times s)$. The result then follows from the facts that every R-module is a homomorphic image of a free R-module and the square of an epi is an epi. □

Theorem 5.1 yields the following corollary.

Corollary 5.2 *Let R be a commutative ring with unity. Any kernel transformation in Rmod is of the form $d = +(r \times s)$, for some $r, s \in R$. In particular, any kernel transformation d in Abgrp is of the form $d = +(r \times s)$, for some $r, s \in \mathbb{Z}$.*

It can be easily verified that for a noetherian ring R, the category, *FGRmod*, of finitely generated R-modules is an abelian category. We have the following theorem.

Theorem 5.3 *Let R be a noetherian commutative ring with unity. Any kernel transformation d in FGRmod is of the form $d = +(r \times s)$ for some $r, s \in R$. In particular any kernel transformation d in FGAbgrp is of the form $d = +(r \times s)$ for some $r, s \in \mathbb{Z}$.*

Proof. Since R is noetherian, every finitely generated R-module M is noetherian and hence any submodule of M is finitely generated. The result follows by letting I be a finite set in the proof of Theorem 5.1. □

The categories, $\overrightarrow{Set}$, of partial sets, see [1, 4, 8], and, Set_*, of pointed sets see [10] have a zero object, finite limits and finite colimits and we have this theorem:

Theorem 5.4 *The only natural transformations, $d : S \longrightarrow I : \mathcal{C} \longrightarrow \mathcal{C}$, are the trivial ones for the categories:*

(a) $\mathcal{C} = \overrightarrow{Set}$, and

(b) $\mathcal{C} = Set_*$.

Proof. (a) Let $\vec{d} : S \longrightarrow I : \overrightarrow{Set} \longrightarrow \overrightarrow{Set}$ be a natural transformation. Denoting by $\vec{\times}$ the product in $\overrightarrow{Set}$, we have the following commutative diagram for every partial map $\vec{f} : X \longrightarrow Y$:

$$\begin{array}{ccc} X \vec{\times} X & \xrightarrow{\vec{d_X}} & X \\ {\scriptstyle \vec{f}\vec{\times}\vec{f}} \downarrow & & \downarrow {\scriptstyle \vec{f}} \\ Y \vec{\times} Y & \xrightarrow[\vec{d_Y}]{} & Y \end{array}$$

Diagram II

This gives the equality of the following two partial maps, in which $D = (D_f \coprod (D_f \times (X - D_f))) \coprod (D_f \coprod ((X - D_f) \times D_f)) \coprod (D_f \times D_f)$, $g = (f \bigoplus fpr_1) \coprod (f \bigoplus fpr_2) \coprod (f \times f)$ and all the vertical arrows are the inclusions,.

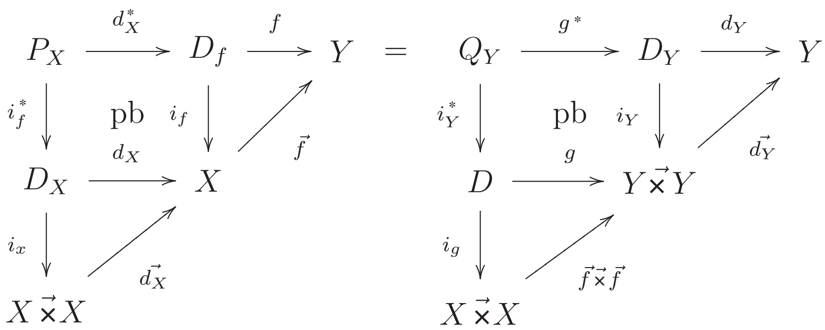
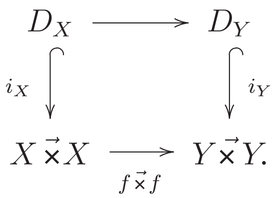


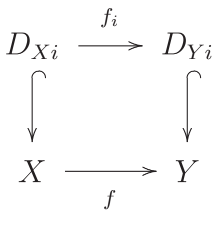
Diagram III

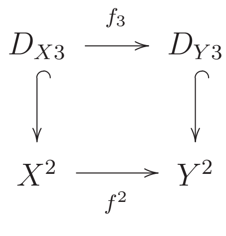
Therefore $P_X = Q_Y$ and $fd_X^* = d_Y g^*$.

This, for a whole map $f : X \longrightarrow Y$, yields the following pullback diagram:



Since $X \vec{\times} X = X \coprod X \coprod X^2$ and for a whole map f, $f \vec{\times} f = f \coprod f \coprod f^2$, we can decompose D_X as $D_{X1} \coprod D_{X2} \coprod D_{X3}$ and the above pullback diagram yields the following pullback diagrams.





where $i = 1, 2$ in the left diagram. Now we prove:

(i) For $i = 1, 2$, either for all X, $D_{Xi} = \emptyset$ or for all X, $D_{Xi} = X$, and either for all X, $D_{X3} = \emptyset$ or for all X, $D_{X3} = X^2$.

To prove the first assertion, given X, let Y be a singleton and $f : X \longrightarrow Y$ be the unique map. Then D_{Yi} is either $\emptyset$ or Y. Therefore by the above left pullback diagram D_{Xi} is either $\emptyset$ or X. This proves for all X, D_{Xi} is either $\emptyset$ or X. Since the above left diagram is a pullback for all f, the result then easily follows. The proof of the second assertion is similar.

The commutativity of Diagram II, for a whole map $f : X \longrightarrow Y$ yields, for $i = 1, 2, 3$, the commutativity of the following diagram:

$$\begin{array}{ccc} D_{Xi} & \xrightarrow{d_{Xi}} & X \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ D_{Yi} & \xrightarrow[d_{Yi}]{} & Y \end{array}$$

Diagram IV

Writing $d_X = d_{X1} \oplus d_{X2} \oplus d_{X3}$, we have:

(ii) For $i = 1, 2$, in the case for all X, $D_{Xi} = X$, then for all X, $d_{Xi} = 1_X$, and in the case for all X, $D_{X3} = X^2$, then for all X, $d_{X3}\Delta_X = 1_X$, with Δ_X the diagonal map.

To prove the first assertion, given Y, pick X to be the singleton, the commutativity of the above diagram for every whole map yields d_{Yi} is the identity function. The proof of the second assertion is similar.

By (i) and (ii) we have the following cases:

Case 1) For all X, $d_X : D_X = \emptyset \longrightarrow X$.

In this case $\vec{d} = 0$.

Case 2) For all X, $d_X = d_{X1} = 1_X : D_X = X \longrightarrow X$.

Pick $\vec{f} : X \longrightarrow Y$ such that cardinality of D_f is 2, i.e., $|D_f| = 2$ and $|X| = 3$. Using Diagram III, we get $P_X \subseteq X$ and $Q_Y = D_f \bigsqcup (D_f \times (X - D_f))$, so that $|P_X| \leq 3$ and $|Q_Y| = 4$. Therefore $P_X \neq Q_Y$, a contradiction.

Case 3) For all X, $d_X = d_{X2} = 1_X : D_X = X \longrightarrow X$.

Similar to case 2 we get a contradiction.

Case 4) For all X, $d_X = d_{X1} \oplus d_{X2} = 1_X \oplus 1_X : D_X = X \coprod X \longrightarrow X$.

Pick $\vec{f} : X \longrightarrow Y$ such that $|D_f| = 1$, $|X| = 2$. Using Diagram III, we see $P_X = D_f \coprod D_f$, while $Q_Y = D = (D_f \bigsqcup (D_f \times (X - D_f))) \bigsqcup (D_f \bigsqcup ((X - D_f) \times D_f))$. So that $|P_X| = 2$ while $|Q_Y| = 4$, a contradiction.

Case 5) For all X, $d_X = d_{X3} : D_X = X^2 \longrightarrow X$.

Pick $D_f = \{a\}$, $X = \{a, b\}$. Using Diagram III, we see $P_X = Q_Y = D_f \times D_f = (a, a)$. It follows that $d_X(a, b) = b$. Next pick $D_f = \{b\}$ to get $d_X(a, b) = a$. So $a = b$, a contradiction.

Case 6) For all X,

$d_X = d_{X1} \oplus d_{X3} = 1_X \oplus d_{X3} : D_X = X \coprod X^2 \longrightarrow X$.

In this case, $\vec{d_X} = \vec{pr}_1$. To prove this, for any $\vec{f} : X \longrightarrow Y$, by Diagram III, $P_X = D_f \coprod P_{X3}$, where P_{X3} is obtained by the pullback

$$\begin{array}{ccc} P_{X3} & \longrightarrow & D_f \\ \big\downarrow & & \big\downarrow{\scriptstyle i_f} \\ X^2 & \xrightarrow[d_{X3}]{} & X \end{array}$$

and $Q_Y = D_f \coprod D_f \times (X - D_f) \coprod D_f \times D_f = D_f \coprod (D_f \times X)$. Let $X = \{a, b\}$ and $D_f = \{a\}$. We have, $(a, b) \in D_f \times (X - D_f) \subseteq Q_Y = P_X$. Therefore $(a, b) \in P_{X3}$, and so by the above pullback diagram, $d_{X3}(a,b) \in D_f = \{a\}$. It follows that $d_{X3}(a,b) = a$. On the other hand $d_{X3}(b,a) = b$, since otherwise, $d_{X3}(b,a) = a$ and by the above pullback diagram and the second assertion of (ii), we get $P_{X3} = \{(a,a),(a,b),(b,a)\} \neq \{(a,a),(a,b)\} = D_f \times X$, a contradiction to $P_X = Q_Y$. This proves for $X = \{a,b\}$, $d_{X3} = pr_1$.

Now Let Y be any set, pick a whole $f : X \longrightarrow Y$. Diagram IV for $i = 3$ yields the following commutative diagram:

$$\begin{array}{ccc} X^2 & \xrightarrow{pr_1} & X \\ {\scriptstyle f^2}\downarrow & & \downarrow{\scriptstyle f} \\ Y^2 & \xrightarrow[d_{Y3}]{} & Y. \end{array}$$

Given $(y_1,y_2) \in Y^2$, pick f so that $f(a) = y_1, f(b) = y_2$. We have $d_{Y3}(y_1,y_2) = d_{Y3}(f(a),f(b)) = d_{Y3}f^2(a,b) = fpr_1(a,b) = f(a) = y_1$. Therefore $d_{Y3} = pr_1$. This proves the assertion.

Case 7): For all X,

$$d_X = d_{X2} \oplus d_{X3} = 1_X \oplus d_{X3} : D_X = X \coprod X^2 \longrightarrow X.$$

Similar argument as in the case 6, shows $\vec{d_X} = \vec{pr}_2$.

Case 8): For all X,

$$d_X = d_{X1} \oplus d_{X2} \oplus d_{X3} = 1_X \oplus 1_X \oplus d_{X3} : D_X = X \vec{\times} X \longrightarrow X.$$

In this case, $P_X = Q_Y$ yields, $P_{X3} = D_f \times (X - D_f) \coprod (X - D_f) \times D_f \coprod D_f \times D_f$. Let $X = \{a,b\}$. Picking $D_f = \{a\}$, we get $(a,b) \in P_{X3}$ and so $d_{X3}(a,b) \in D_f = \{a\}$, and so $d_{X3}(a,b) = a$. On the other hand, by picking $D_f = \{b\}$, we get $d_{X3}(a,b) = b$, a contradiction.

(b) Let $(X,x_0) = (\{x_0,x_1,x_2\},x_0)$. Then $d_{(X,x_0)}$ takes (x_1,x_2) to x_0, x_1 or x_2. Suppose $d_{(X,x_0)}(x_1,x_2) = x_0$. Let $(Y,y_0) \in Set_*$ and pick $y_1,y_2 \in Y$. Let the mapping $f : (X,x_0) \longrightarrow (Y,y_0)$ in Set_* take x_i to y_i for each $i = 1,2,3$. Naturality of d implies $d_{(Y,y_0)}(y_1,y_2) = y_0$. Since y_1 and y_2 were arbitrary, $d_{(Y,y_0)}$ is the constant map with value y_0. So $d = 0$.

Similar argument shows that in the two cases that $d_{(X,x_0)}(x_1,x_2) = x_1$ or $d_{(X,x_0)}(x_1,x_2) = x_2$, $d_{(Y,y_0)}$ is the projection to the first, respectively second, factor. So that $d = pr_1$ or $d = pr_2$. □

Finally by Theorem 5.4 we get:

Corollary 5.5 *The only kernel transformations in the categories $\overrightarrow{Set}$ and Set_* are the trivial ones.*

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