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Properties of RD-projective and RD-injective modules

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Abstract

In this paper, we first study RD-projective and RD-injective modules using, among other things, covers and envelopes. Some new characterizations for them are obtained. Then we introduce the RD-projective and RD-injective dimensions for modules and rings. The relations between the RD-homological dimensions and other homological dimensions are also investigated.

Key word and phrases: RD-projective module, RD-injective module, RD-flat module, RD-projective dimension, RD-injective dimension, (pre)envelope, (pre)cover.

1. Introduction

Following [20], an exact sequence $0 \to A \to B \to C \to 0$ of left R-modules is called *RD-exact* if for every $a \in R$, the sequence $\mathrm{Hom}(R/Ra, B) \to \mathrm{Hom}(R/Ra, C) \to 0$ is exact, or equivalently, the sequence $0 \to (R/aR) \otimes A \to (R/aR) \otimes B$ is exact. A left R-module M is said to be *RD-projective* if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ is exact. A left R-module N is called *RD-injective* if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) \to 0$ is exact. According to [3], a right R-module F is called *RD-flat* if for every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules, the sequence $0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0$ is exact. For more details about RD-projective, RD-injective and RD-flat modules, we refer the reader to [2, 3, 6, 15, 16, 19, 20].

Though the RD-property is most important and well known in the commutative case, so far not much is known about the RD-property in the theory of modules over non-commutative rings. In this paper, we will establish several basic results for RD-projective, RD-injective and RD-flat modules over a general ring.

In Section 2 of this paper, we obtain some properties of RD-projective and RD-injective modules in terms of, among other things, covers and envelopes. New characterizations for them are presented. For example, we prove that, if M is a submodule of an RD-injective left R-module E, then E is an RD-injective hull M in the sense of Warfield if and only if the inclusion $M \to E$ is an RD-injective envelope in the sense of Enochs. Also, we show that M is an RD-projective left R-module if and only if M is projective relative to every RD-exact sequence $0 \to K \to E \to F \to 0$ of left R-modules with E RD-injective. Dually, M is an RD-injective

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left R-module if and only if M is injective relative to every RD-exact sequence $0 \to K \to P \to L \to 0$ of left R-modules with P RD-projective. In addition, we get that the class of RD-injective left R-modules is closed under extensions if and only if every Warfield cotorsion left R-module is RD-injective. Finally, we prove that the following are equivalent for a ring R and an integer $n \geq 0$: (1) Every RD-flat left R-module has flat dimension $\leq n$. (2) Every RD-projective left R-module has flat dimension $\leq n$. (3) Every RD-injective right R-module has injective dimension $\leq n$. As a consequence, we obtain several new characterizations of left PP rings and von Neumann regular rings.

In Section 3, we introduce and study the RD-derived functor $\mathrm{Ext}^n_{RD}(-,-)$ of $\mathrm{Hom}(-,-)$, and RD-projective and RD-injective dimensions of modules and rings. We first prove that $\mathrm{Ext}^1_{RD}(M,N) \to \mathrm{Ext}^1(M,N)$ is a monomorphism for any ring R; R is a von Neumann regular ring if and only if $\mathrm{Ext}^1_{RD}(M,N) \cong \mathrm{Ext}^1(M,N)$ for all left R-modules M and N. Then we get that the left global RD-projective dimension $lRD-PD(R)$ is equal to the left global RD-injective dimension $lRD-ID(R)$. For a left strongly P-coherent ring R, we prove that $\sup\{id(M) : M$ is any divisible left R-module$\} \leq lRD-ID(R)$, and $\sup\{pd(M) : M$ is any torsionfree left R-module$\} \leq lRD-PD(R)$. Finally, it is shown that $lD(R) \leq lRD-ID(R) + \sup\{id(M) : M$ is any RD-injective left R-module$\} \leq lRD-ID(R) + wD(R)$.

Throughout this paper, R is an associative ring with identity and all modules are unitary. We write ${}_RM$ to indicate a left R-module. The character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M is denoted by M^+. $lD(R)$ (resp. $wD(R)$) stands for the left (resp. the weak) global dimension of R. $pd(M)$ (resp. $id(M)$, $fd(M)$) denotes the projective (resp. injective, flat) dimension of M. Let M and N be R-modules. $\mathrm{Hom}(M,N)$ (resp. $\mathrm{Ext}^n(M,N)$) means $\mathrm{Hom}_R(M,N)$ (resp. $\mathrm{Ext}^n_R(M,N)$), and similarly $M \otimes N$ (resp. $\mathrm{Tor}_n(M,N)$) denotes $M \otimes_R N$ (resp. $\mathrm{Tor}^R_n(M,N)$) for an integer $n \geq 1$. For unexplained concepts and notations, we refer the reader to [1, 5, 6, 7, 11, 17, 21, 22].

2. RD-projective and RD-injective modules

We begin with the following lemmas.

Lemma 2.1 *Let R be a ring.*

(1) [6, Lemma VI 12.1] For any left R-module M, there exists an RD-exact sequence $0 \to N \to C \to M \to 0$, where C is a direct sum of cyclically presented left R-modules.

(2) [20, Corollary 1] and [3, Proposition 1.3] A left R-module M is RD-projective if and only if M is a direct summand of a direct sum of cyclically presented left R-modules if and only if M is RD-flat and pure-projective.

(3) [3, Proposition 1.4] A right R-module F is RD-flat if and only if F^+ is RD-injective.

Lemma 2.2 *The following are equivalent:*

(1) $0 \to A \to B \to C \to 0$ is an RD-exact sequence of left R-modules.

(2) *The sequence* $0 \to \mathrm{Hom}(M, A) \to \mathrm{Hom}(M, B) \to \mathrm{Hom}(M, C) \to 0$ *is exact for any* RD*-projective left* R*-module* M*.*

(3) *The sequence* $0 \to \mathrm{Hom}(C, N) \to \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) \to 0$ *is exact for any* RD*-injective left* R*-module* N*.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1) is clear since R/Ra is RD-projective for any $a \in R$.

(3) $\Rightarrow$ (1) Let $a \in R$. By Lemma 2.1 (3), $(R/aR)^+$ is RD-injective. So by (3), we get the exact sequence

$$\mathrm{Hom}(B, (R/aR)^+) \to \mathrm{Hom}(A, (R/aR)^+) \to 0,$$

which gives the exactness of the sequence

$$((R/aR) \otimes B)^+ \to ((R/aR) \otimes A)^+ \to 0.$$

Therefore we obtain the exact sequence

$$0 \to (R/aR) \otimes A \to (R/aR) \otimes B.$$

So the sequence $0 \to A \to B \to C \to 0$ is RD-exact. □

According to [8, 11], a left R-module M is said to be *divisible* if $\mathrm{Ext}^1(R/Ra, M) = 0$ for all $a \in R$. A right R-module N is called *torsionfree* if $\mathrm{Tor}_1(N, R/Ra) = 0$ for all $a \in R$. It is clear that a right R-module N is torsionfree if and only if N^+ is divisible by the standard isomorphism $\mathrm{Ext}^1(R/Ra, N^+) \cong \mathrm{Tor}_1(N, R/Ra)^+$ for all $a \in R$.

Next we characterize divisible and torsion-free modules in terms of RD-projective and RD-injective modules.

Proposition 2.3 *The following are equivalent for a left* R*-module* M*:*

(1) M *is divisible.*

(2) *Every left* R*-module exact sequence* $0 \to M \to E \to F \to 0$ *is* RD*-exact.*

(3) *There exists an* RD*-exact sequence* $0 \to M \to B \to C \to 0$ *with* B *divisible.*

(4) $\mathrm{Ext}^1(N, M) = 0$ *for any* RD*-projective left* R*-module* N*.*

(5) *For every* RD*-injective left* R*-module* G*, any homomorphism* $M \to G$ *factors through an injective left* R*-module.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) are routine.

(1) $\Rightarrow$ (4) follows from Lemma 2.1 (2). (4) $\Rightarrow$ (1) is clear.

(2) $\Rightarrow$ (5) is easy since M embeds in an injective R-module.

(5) $\Rightarrow$ (3) There exists an exact sequence $0 \to M \xrightarrow{i} E \to L \to 0$ with E injective. Let $a \in R$. Then $(R/aR)^+$ is RD-injective. For any $f : M \to (R/aR)^+$, there exist an injective left R-module Q and

$g : M \to Q$ and $h : Q \to (R/aR)^+$ such that $f = hg$ by (5). Thus there exists $\alpha : E \to Q$ such that $g = \alpha i$, and so $f = (h\alpha)i$. Therefore we get the exact sequence

$$\mathrm{Hom}(E, (R/aR)^+) \to \mathrm{Hom}(M, (R/aR)^+) \to 0,$$

which leads to the exactness of the sequence

$$((R/aR) \otimes E)^+ \to ((R/aR) \otimes M)^+ \to 0.$$

It follows that $0 \to (R/aR) \otimes M \to (R/aR) \otimes E$ is exact, as required. □

Proposition 2.4 *The following are equivalent for a right R-module N:*

(1) N is torsionfree.

(2) Every right R-module exact sequence $0 \to K \to P \to N \to 0$ is RD-exact.

(3) There exists a right R-module RD-exact sequence $0 \to K \to T \to N \to 0$ with T torsionfree.

(4) $\mathrm{Ext}^1(N, M) = 0$ for any RD-injective right R-module M.

(5) For every RD-projective right R-module F, every homomorphism $f : F \to N$ factors through a projective right R-module.

(6) $\mathrm{Tor}_1(N, M) = 0$ for any RD-flat left R-module M.

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) are straightforward.

(2) ⇒ (5) is clear since there is an exact sequence $P \to N \to 0$ with P projective.

(5) ⇒ (1) follows from [13, Lemma 3.9].

(1) ⇔ (6) holds by the fact that every RD-flat module is a direct limit of finite direct sums of cyclically presented modules (see [3, Proposition I.1]). □

Corollary 2.5 *The following are true for any ring R:*

(1) A divisible RD-injective left R-module is injective.

(2) A torsionfree RD-projective right R-module is projective.

(3) A torsionfree RD-flat right R-module is flat.

Proof. (1) follows from Proposition 2.3. (2) holds by Proposition 2.4.

(3) Let N be a torsionfree RD-flat right R-module. Then N^+ is divisible RD-injective by Lemma 2.1 (3), and so is injective by (1). Thus N is flat. □

Following [6], an *RD-injective hull of an R-module M* is defined as an RD-injective R-module E such that M is an RD-essential submodule of E, where M is called an *RD-essential submodule of E* if M is an RD-submodule of E, and there is no nonzero submodule K of E with $K \cap M = 0$ and $(K+M)/K$ an RD-submodule of E/K.

By [6, Theorem 1.6], any R-module admits an RD-injective hull.

Let $\mathcal{C}$ be a class of R-modules and M an R-module. According to Enochs [4], a homomorphism $\phi : C \to M$ is a *$\mathcal{C}$-precover of M* if $C \in \mathcal{C}$ and the abelian group homomorphism $\mathrm{Hom}(C', \phi) : \mathrm{Hom}(C', C) \to \mathrm{Hom}(C', M)$ is surjective for every $C' \in \mathcal{C}$. A $\mathcal{C}$-precover $\phi : C \to M$ is said to be a *$\mathcal{C}$-cover of M* if every endomorphism $g : C \to C$ such that $\phi g = \phi$ is an isomorphism. Dually we have the definitions of a $\mathcal{C}$-preenvelope and a $\mathcal{C}$-envelope. $\mathcal{C}$-covers ($\mathcal{C}$-envelopes) may not exist in general, but if they exist, they are unique up to isomorphism.

Theorem 2.6 *Let R be a ring.*

(1) Every R-module has an RD-projective precover.

(2) Every R-module has an RD-flat cover.

(3) Every R-module has an RD-injective envelope.

Proof. (1) follows from Lemma 2.1 (1).

(2) We first prove that the class of RD-flat R-modules is closed under pure quotient modules. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence with B RD-flat. Then we get the split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$. Since B^+ is RD-injective by Lemma 2.1 (3), C^+ is RD-injective. So C is RD-flat. In addition, the class of RD-flat R-modules is clearly closed under direct limits. Thus every R-module has an RD-flat cover by [9, Theorem 2.5].

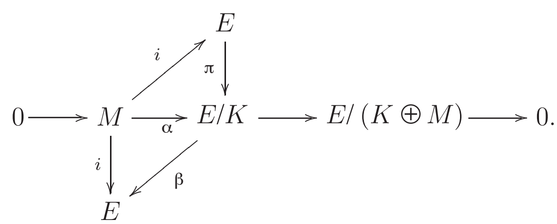
(3) Since every R-module admits an RD-injective hull, every R-module admits an RD-injective preenvelope. On the other hand, any direct limit of RD-exact sequences is RD-exact (see [6, Exercise I 7.15]). By a proof similar to that of [22, Theorem 2.3.8 or 2.2.6], every R-module has an RD-injective envelope. □

Theorem 2.7 *Suppose that M is a submodule of an RD-injective left R-module E. Then the following are equivalent:*

(1) $i : M \to E$ is an RD-injective envelope (here i is the inclusion).

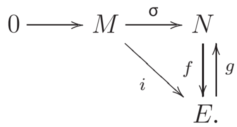
(2) E is an RD-injective hull of M.

Proof. (1) $\Rightarrow$ (2) Suppose that there is a nonzero submodule K of E such that $K \cap M = 0$ and $(K+M)/K$ is an RD-submodule of E/K. Since $(K+M)/K \cong M$ and E is RD-injective, there is $\beta : E/K \to E$ such that the following diagram is commutative, where $\pi : E \to E/K$ is the natural map:



Hence $\beta\pi i = i$. Since i is an envelope, $\beta\pi$ is an isomorphism, whence π is an isomorphism. But this is impossible because $\pi(K) = 0$. So E is an RD-injective hull of M.

(2) $\Rightarrow$ (1) Let E be an RD-injective hull of M. Clearly the inclusion $i : M \to E$ is an RD-injective preenvelope. By Theorem 2.6 (3), M has an RD-injective envelope $\sigma : M \to N$. Thus there exist $f : N \to E$ and $g : E \to N$ such that the following diagram is commutative.



So $gf\sigma = gi = \sigma$. Hence gf is an isomorphism. Without loss of generality, we may assume $gf = 1$. Thus $E = \mathrm{im}(f) \oplus \ker(g)$. Note that $M \cap \ker(g) = 0$ and M is an RD-submodule of $\mathrm{im}(f)$. So $(M \oplus \ker(g))/\ker(g)$ is an RD-submodule of $E/\ker(g)$ by [6, p.39]. Hence $\ker(g) = 0$ by (2). Thus g is an isomorphism. Therefore $i : M \to E$ is an RD-injective envelope. □

Now we give new characterizations of RD-projective and RD-injective modules.

Theorem 2.8 *The following are equivalent for a left R-module M:*

(1) M is RD-projective.

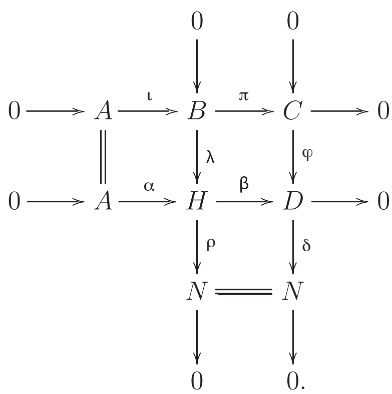
(2) Every RD-exact sequence $0 \to K \to N \to M \to 0$ of left R-modules is split.

(3) M is projective relative to every RD-exact sequence $0 \to K \to E \to F \to 0$ of left R-modules with E RD-injective.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1) By Lemma 2.1 (1), there exists an RD-exact sequence $0 \to N \to C \to M \to 0$ with C RD-projective. So M is RD-projective by (2).

(3) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an RD-exact sequence of left R-modules. By Theorem 2.6 (3), B has an RD-injective envelope $\lambda : B \to H$. Then we have the following pushout diagram:



Thus $\alpha = \lambda\iota$, and so $0 \to A \to H \to D \to 0$ is an RD-exact sequence. Let $\psi : M \to C$ be any homomorphism. By (3), there exists $\gamma : M \to H$ such that $\beta\gamma = \varphi\psi$. Since $\rho\gamma = \delta\beta\gamma = \delta\varphi\psi = 0$, we have $\mathrm{im}(\gamma) \subseteq \ker(\rho) = \mathrm{im}(\lambda)$. So we can define $\theta : M \to B$ by

$$\theta(x) = \lambda^{-1}\gamma(x) \text{ for any } x \in M.$$

Thus

$$\varphi\psi = \beta\gamma = \beta\lambda\theta = \varphi\pi\theta.$$

So $\psi = \pi\theta$ since φ is monic. Hence M is RD-projective. □

Theorem 2.9 *The following are equivalent for a left R-module M:*

(1) M is RD-injective.

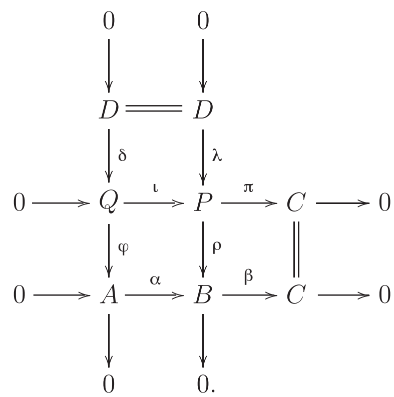
(1) Every RD-exact sequence $0 \to M \to E \to F \to 0$ of left R-modules is split.

(2) M is injective relative to every RD-exact sequence $0 \to K \to P \to L \to 0$ of left R-modules with P RD-projective.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are clear.

(2) $\Rightarrow$ (1) By [6, Theorem 1.6], there exists an RD-exact sequence $0 \to M \to B \to N \to 0$ with B RD-injective. So M is RD-injective by (2).

(3) $\Rightarrow$ (1) Let $0 \to A \to B \to C \to 0$ be an RD-exact sequence of left R-modules. By Lemma 2.1 (1), there is an RD-exact sequence $0 \to D \to P \to B \to 0$ with P RD-projective. Then we have the following pullback diagram:



Thus $\pi = \beta\rho$, and so $0 \to Q \to P \to C \to 0$ is an RD-exact sequence. Let $\psi : A \to M$ be any homomorphism. By (3), there exists $\gamma : P \to M$ such that $\psi\varphi = \gamma\iota$. Since $\gamma\iota\delta = \psi\varphi\delta = 0$, we have

$$\ker(\rho) = \mathrm{im}(\lambda) = \mathrm{im}(\iota\delta) \subseteq \ker(\gamma).$$

So there exists $\theta : B \to M$ such that $\theta\rho = \gamma$. Thus

$$\psi\varphi = \theta\rho\iota = \theta\alpha\varphi.$$

Therefore $\psi = \theta\alpha$ since φ is epic. Hence M is RD-injective. □

RD-injective and RD-flat modules over a commutative ring can be characterized as follows.

Proposition 2.10 *Let R be a commutative ring. The following are equivalent for an R-module M:*

(3) M is an RD-injective R-module.

(4) $\mathrm{Hom}(F, M)$ is an RD-injective R-module for any flat R-module F.

Proof. (1) $\Rightarrow$ (2) Let $0 \to A \to B \to C \to 0$ be an RD-exact sequence of R-modules. For any flat R-module F, we get the exact sequence

$$0 \to F \otimes A \to F \otimes B \to F \otimes C \to 0.$$

It is easy to verify that the sequence is RD-exact. Since M is RD-injective, we obtain the exact sequence

$$\mathrm{Hom}(F \otimes B, M) \to \mathrm{Hom}(F \otimes A, M) \to 0,$$

which yields the exact sequence

$$\mathrm{Hom}(B, \mathrm{Hom}(F, M)) \to \mathrm{Hom}(A, \mathrm{Hom}(F, M)) \to 0.$$

Thus $\mathrm{Hom}(F, M)$ is an RD-injective R-module.

(2) $\Rightarrow$ (1) is clear by letting $F = R$. □

Proposition 2.11 *Let R be a commutative ring. The following are equivalent for an R-module N:*

(1) N is an RD-flat R-module.

(2) $\mathrm{Hom}(N, E)$ is an RD-injective R-module for any injective R-module E.

Proof. (1) $\Rightarrow$ (2) Let E be any injective R-module. Then there is a split exact sequence

$$0 \to E \to \Pi R^+.$$

So we get the split exact sequence

$$0 \to \mathrm{Hom}(N, E) \to \mathrm{Hom}(N, \Pi R^+) \cong \Pi \mathrm{Hom}(N, R^+) \cong \Pi N^+.$$

By (1), N^+ is RD-injective, and so ΠN^+ is RD-injective. Thus $\mathrm{Hom}(N, E)$ is RD-injective.

(2) $\Rightarrow$ (1) is obvious by letting $E = R^+$. □

Recall that a right R-module M is *Warfield cotorsion* [6, 7] if $\mathrm{Ext}^1(F, M) = 0$ for every torsionfree right R-module F. Clearly, any RD-injective module is Warfield cotorsion by Proposition 2.4.

The following theorem exhibits the homological property of RD-projective, RD-injective and RD-flat modules.

Theorem 2.12 *The following are equivalent for a ring R and an integer $n \geq 0$:*

(1) Every RD-flat left R-module has flat dimension $\leq n$.

(2) Every RD-projective left R-module has flat dimension $\leq n$.

(3) Every Warfield cotorsion right R-module has injective dimension $\leq n$.

(4) Every RD-injective right R-module has injective dimension $\leq n$.

Proof. (1) $\Rightarrow$ (2) is clear by Lemma 2.1 (2).

(2) $\Rightarrow$ (3) Let M be a Warfield cotorsion right R-module and N any right R-module. Then there is an exact sequence

$$0 \to K_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to N \to 0$$

with each P_i projective. By (2), for any $a \in R$, we have

$$\mathrm{Tor}_1(K_n, R/Ra) \cong \mathrm{Tor}_{n+1}(N, R/Ra) = 0.$$

Thus K_n is torsionfree, and so

$$\mathrm{Ext}^{n+1}(N, M) \cong \mathrm{Ext}^1(K_n, M) = 0.$$

It follows that M has injective dimension $\leq n$.

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (1) For every RD-flat left R-module A, A^+ is RD-injective. By (4), for every right R-module B, we have

$$\mathrm{Tor}_{n+1}(B, A)^+ \cong \mathrm{Ext}^{n+1}(B, A^+) = 0.$$

So $\mathrm{Tor}_{n+1}(B, A) = 0$, and hence A has flat dimension $\leq n$. □

Recall that a ring R is *left PP* if every principal left ideal of R is projective. R is called *left P-coherent* [15] in case each principal left ideal of R is finitely presented.

Corollary 2.13 *The following are equivalent for a ring R:*

(1) R is a left PP ring.

(2) R is a left P-coherent ring and every submodule of a torsionfree right R-module is torsionfree.

(3) Every quotient module of a divisible left R-module is divisible.

(4) Every RD-projective left R-module has projective dimension ≤ 1.

(5) R is a left P-coherent ring and every RD-injective right R-module has injective dimension ≤ 1.

(6) R is a left P-coherent ring and every RD-flat left R-module has flat dimension ≤ 1.

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) hold by [14, Theorem 5.1].

(3) $\Rightarrow$ (4) Let M be an RD-projective left R-module and N any left R-module. Then there is an exact sequence $0 \to N \to E \to L \to 0$ with E injective. By (3), L is divisible, and so $\mathrm{Ext}^2(M, N) \cong \mathrm{Ext}^1(M, L) = 0$ by Proposition 2.3. It follows that M has projective dimension ≤ 1.

(4) $\Rightarrow$ (1) Let $a \in R$. Since R/Ra has projective dimension ≤ 1, Ra is projective.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) follow from Theorem 2.12 and the equivalence of (4) and (1).

(6) $\Rightarrow$ (1) Let $a \in R$. Since R/Ra has flat dimension ≤ 1, Ra is flat. So Ra is projective since Ra is finitely presented. □

In general, RD-projective (RD-injective) modules need not be projective (injective). For example, $\mathbb{Z}_2$ is an RD-projective (RD-injective) $\mathbb{Z}$-module, but it is not a projective (injective) $\mathbb{Z}$-module. In fact, we have the following result.

Corollary 2.14 *The following are equivalent for a ring R:*

(1) R is a von Neumann regular ring.

(2) Every RD-projective left R-module is projective.

(3) Every RD-flat left R-module is flat.

(4) Every RD-injective right R-module is injective.

(5) Every left R-module exact sequence $0 \to A \to B \to C \to 0$ is RD-exact.

Proof. (1) $\Rightarrow$ (2) By Lemma 2.1 (2), an RD-projective left R-module is a direct summand of a direct sum of cyclically presented left R-modules. Since every cyclically presented left R-module is projective by (1), every RD-projective left R-module is projective.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) follow from Theorem 2.12 by letting $n = 0$.

(4) $\Rightarrow$ (5) holds by Lemma 2.2.

(5) $\Rightarrow$ (1) By (5) and Proposition 2.3, every left R-module is divisible. So R is a von Neumann regular ring. □

Recall that a left R-module M is *absolutely pure* [12] if M is a pure submodule of every module which contains M as a submodule.

Proposition 2.15 *Consider the following conditions for a ring R:*

(1) Every RD-exact sequence $0 \to A \to B \to C \to 0$ of left R-modules is pure.

(2) Every pure injective left R-module is RD-injective.

(3) Every pure projective left R-module is RD-projective.

(4) Every finitely presented left R-module is a summand of a direct sum of cyclically presented left R-modules.

(5) every divisible left R-module is absolutely pure.

Then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ *(5).*

Proof. The equivalence of (1) through (4) follow from [3, Theorem I.4].

(1) $\Rightarrow$ (5) holds by Proposition 2.3. □

In [2], some examples of pure-injective modules that fail to be RD-injective were given for commutative rings. The following example gives an RD-exact sequence which is not pure over a non-commutative ring, and so there exists a pure-injective left module, which is not RD-injective.

Example 2.16 Let K be a field and ρ an isomorphism of K onto a subfield L such that $K \neq L$ and K has finite vector space dimension over L. $K[X;\rho]$ will denote the ring of twisted right polynomials over K, i.e., $K[X;\rho]$ is the set of all formal polynomials in commuting indeterminate X with coefficients from K write on the right. Equality and addition are defined in the usual fashion and multiplication by assuming the associate and distributive laws and the rule

$$aX = X\rho(a)$$

for all $a \in K$.

Let $R = K[X;\rho]/(X^2)$. Then by [18, Example 1], ${}_RR$ is divisible, and R is a two-sided Artinian ring, but is not a quasi-Frobenius ring. Thus ${}_RR$ is not absolutely pure (and so is not RD-injective by Corollary 2.5 (1)). Let $E({}_RR)$ denote the injective envelope of ${}_RR$. Then by Proposition 2.3, the left R-module exact sequence

$$0 \to {}_RR \to E({}_RR) \to E({}_RR)/{}_RR \to 0$$

is an RD-exact sequence, but it is not pure. Thus by Proposition 2.15, there exists a pure injective left R-module which is not RD-injective, and there exists a pure projective left R-module which is not RD-projective.

By the way, the class of RD-flat left R-modules coincides with the class of RD-projective left R-modules by [3, Theorem III.1] since R is left Artinian.

Remark 2.17 We note that some properties of RD-projective and RD-injective modules over commutative rings can be generalized to non-commutative cases. For example, by [6, Theorem XIII 1.1 and Example VI 12.5], for a commutative domain R, every RD-injective R-module has injective dimension ≤ 1, and every RD-projective R-module has projective dimension ≤ 1. By replacing "commutative domain" with "left PP ring", Corollary 2.13 extends the above result to a more general setting.

However, there seems to be some difference between the commutative and the non-commutative cases when we consider the projectivity and injectivity for RD. For instance, if R is a commutative domain, then by [6, Proposition IX 3.4 and Theorem XIII 2.8], all conditions in Proposition 2.15 are equivalent (which exactly characterizes Prüfer domain). But for a non-commutative ring, we do not know whether the conditions (4) and (5) in Proposition 2.15 are equivalent. However, by [7, Corollary 3.2.4], the condition (5) in Proposition 2.15 is equivalent to the condition that every finitely presented left R-module is a direct summand in a left R-module N such that N is a union of a continuous chain, $(N_\alpha : \alpha < \lambda)$, for a cardinal λ, $N_0 = 0$ and $N_{\alpha+1}/N_\alpha$ is cyclically presented for all $\alpha < \lambda$.

Although the class of RD-injective left R-modules is closed under direct products and direct summands, the class of RD-injective left R-modules is not closed under direct sums in general. In fact, if R is not a left Artinian ring, then the class of RD-injective left R-modules is not closed under direct sums by [3, Theorem II. 1].

Next we will consider when the class of RD-injective left R-modules is closed under extensions.

Theorem 2.18 *The following are equivalent for a ring R:*

(1) The class of RD-injective left R-modules is closed under extensions.

(2) Every Warfield cotorsion left R-module is RD-injective.

Proof. (1) ⇒ (2) Let M be a Warfield cotorsion left R-module. Then by Theorem 2.6 (3), we have an exact sequence $0 \to M \to N \to L \to 0$, where $M \to N$ is an RD-injective envelope of M. By (1) and Wakamatsu's Lemma (see [22, Lemma 2.1.2]), $\mathrm{Ext}^1(L, C) = 0$ for every RD-injective left R-module C, and so L is torsionfree by Proposition 2.4. Therefore $\mathrm{Ext}^1(L, M) = 0$, and hence the exact sequence $0 \to M \to N \to L \to 0$ is split. Thus M is RD-injective.

(2) ⇒ (1) is obvious because the class of Warfield cotorsion left R-modules is closed under extensions. □

Remark 2.19 (1) In general, the class of RD-injective R-modules is not closed under extensions. For example, [22, p. 75, Example] constructs a cotorsion $\mathbb{Z}$-module which is not pure injective. Since torsionfree $\mathbb{Z}$-modules coincide with flat $\mathbb{Z}$-modules, Warfield cotorsion $\mathbb{Z}$-modules need not be RD-injective. So the class of RD-injective $\mathbb{Z}$-modules is not closed under extensions by Theorem 2.18.

(2) If R is a left pure-semisimple ring, then the equivalent conditions of Theorem 2.18 are clearly satisfied.

(3) If R is a von Neumann regular ring, then every RD-injective left R-module is injective by Corollary 2.14. So the equivalent conditions of Theorem 2.18 are also satisfied.

(4) If R is a Prüfer domain, then the equivalent conditions of Theorem 2.18 hold if and only if the class of RD-injective R-modules is closed under cokernels of monomorphisms by [16, Proposition 4.5] and [22, Theorem 3.5.1].

3. RD-derived functors of $\mathrm{Hom}(-,-)$ and RD-homological dimensions

By Theorem 2.6 (1), every left R-module has an RD-projective precover. So every left R-module M has a *left RD-projective resolution*, that is, there is an exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each P_i RD-projective and such that $\mathrm{Hom}(N, -)$ leaves the sequence exact whenever N is an RD-projective left R-module, equivalently, there exists an RD-exact sequence $\cdots \to P_1 \to P_0 \to M \to 0$ with each P_i RD-projective by Lemma 2.2. Write $K_0 = M, K_1 = \ker(P_0 \to M), K_i = \ker(P_{i-1} \to P_{i-2})$ for $i \geq 2$. The nth kernel K_n $(n \geq 0)$ is called the *nth RD-projective syzygy of M*.

Dually, by Theorem 2.6 (3), every left R-module N has an RD-injective envelope. So N has a *right RD-injective resolution*, that is, there is an exact sequence $0 \to N \to E^0 \to E^1 \to \cdots$ with each E^i RD-injective and such that $\mathrm{Hom}(-, M)$ leaves the sequence exact whenever M is an RD-injective left R-module, equivalently, there is an RD-exact sequence $0 \to N \to E^0 \to E^1 \to \cdots$ with each E^i RD-injective by Lemma 2.2. Write $L^0 = N, L^1 = \mathrm{coker}(N \to E^0), L^i = \mathrm{coker}(E^{i-2} \to E^{i-1})$ for $i \geq 2$. The nth cokernel L^n $(n \geq 0)$ is called the *nth RD-injective cosyzygy of N*.

Note that $\mathrm{Hom}(-,-)$ is right balanced by {the class of all RD-projective left R-modules} × {the class of all RD-injective left R-modules} (see [5, Definition 8.2.13]). Let $\mathrm{Ext}^n_{RD}(-,-)$ denote the nth right derived functor of $\mathrm{Hom}(-,-)$ with respect to {the class of all RD-projective left R-modules} × {the class of all RD-injective left R-modules}. Then, for two left R-modules M and N, $\mathrm{Ext}^n_{RD}(M, N)$ can be computed using a left RD-projective resolution of M or a right RD-injective resolution of N.

For any family $\{M_i\}$ of left R-modules, it is easy to check that the natural map $\mathrm{Ext}^n_{RD}(\oplus M_i, N) \to$

$\prod \operatorname{Ext}^n_{RD}(M_i, N)$ is an isomorphism for any left R-module N and $n \geq 0$. Moreover, we have the following result.

Theorem 3.1 *Let R be a ring such that the class of RD-injective left R-modules is closed under direct sums. If N is a finitely generated left R-module, $\{M_i\}$ is a family of left R-modules, then* $\operatorname{Ext}^n_{RD}(N, \oplus M_i) \cong \oplus \operatorname{Ext}^n_{RD}(N, M_i)$ *for any $n \geq 0$.*

Proof. Every M_i has a right RD-injective resolution

$$0 \to M_i \to E_i^0 \to E_i^1 \to E_i^2 \to \cdots .$$

Then by hypothesis and [22, Proposition 1.2.4],

$$0 \to \oplus M_i \to \oplus E_i^0 \to \oplus E_i^1 \to \oplus E_i^2 \to \cdots$$

is a right RD-injective resolution of $\oplus M_i$. Applying $\operatorname{Hom}(N, -)$, we have the following commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus\operatorname{Hom}(N, E_i^0) & \longrightarrow & \oplus\operatorname{Hom}(N, E_i^1) & \longrightarrow & \oplus\operatorname{Hom}(N, E_i^2) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_2} & & \\
0 & \longrightarrow & \operatorname{Hom}(N, \oplus E_i^0) & \longrightarrow & \operatorname{Hom}(N, \oplus E_i^1) & \longrightarrow & \operatorname{Hom}(N, \oplus E_i^2) & \longrightarrow & \cdots .
\end{array}$$

Since N is finitely generated, every θ_i is an isomorphism by [1, Exercise 16.3]. So $\operatorname{Ext}^n_{RD}(N, \oplus M_i) \cong \oplus \operatorname{Ext}^n_{RD}(N, M_i)$ for any $n \geq 0$ by [17, Exercise 6.7]. □

We now compare the RD-derived functor $\operatorname{Ext}^n_{RD}(-,-)$ with the usual derived functor $\operatorname{Ext}^n(-,-)$. There is a natural transformation $\operatorname{Ext}^n_{RD}(-,-) \to \operatorname{Ext}^n(-,-)$.

Theorem 3.2 *The following are true for any ring R.*

(1) $\operatorname{Ext}^0_{RD}(M, N) \cong \operatorname{Hom}(M, N) \cong \operatorname{Ext}^0(M, N)$ *for all left R-modules M and N.*

(2) $\operatorname{Ext}^1_{RD}(M, N) \to \operatorname{Ext}^1(M, N)$ *is a monomorphism for all left R-modules M and N.*

Proof. Let

$$0 \to N \xrightarrow{\epsilon} D^0 \xrightarrow{d^0} D^1 \xrightarrow{d^1} D^2 \xrightarrow{d^2} \cdots$$

be a right RD-injective resolution of N. Since D^0 can be embedded in an injective left R-module E^0, N admits a right injective resolution

$$0 \to N \xrightarrow{\lambda} E^0 \xrightarrow{e^0} E^1 \xrightarrow{e^1} E^2 \xrightarrow{e^2} \cdots .$$

So we can complete the following commutative diagram uniquely up to homotopy, where τ_0 is a monomorphism:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & N & \xrightarrow{\epsilon} & D^0 & \xrightarrow{d^0} & D^1 & \xrightarrow{d^1} & D^2 & \xrightarrow{d^2} & \cdots \\
 & & \| & & \downarrow{\scriptstyle \tau_0} & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \tau_2} & & \\
0 & \longrightarrow & N & \xrightarrow{\lambda} & E^0 & \xrightarrow{e^0} & E^1 & \xrightarrow{e^1} & E^2 & \xrightarrow{e^2} & \cdots .
\end{array}$$

Applying $\mathrm{Hom}(M,-)$ for any left R-module M, we have the following commutative diagram of complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}(M,D^0) & \xrightarrow{d^0_*} & \mathrm{Hom}(M,D^1) & \xrightarrow{d^1_*} & \mathrm{Hom}(M,D^2) & \xrightarrow{d^2_*} & \cdots \\
 & & \downarrow{\scriptstyle \tau_{0*}} & & \downarrow{\scriptstyle \tau_{1*}} & & \downarrow{\scriptstyle \tau_{2*}} & & \\
0 & \longrightarrow & \mathrm{Hom}(M,E^0) & \xrightarrow{e^0_*} & \mathrm{Hom}(M,E^1) & \xrightarrow{e^1_*} & \mathrm{Hom}(M,E^2) & \xrightarrow{e^2_*} & \cdots .
\end{array}$$

(1) It is clear that $\mathrm{Ext}^0_{RD}(M,N) \cong \mathrm{Hom}(M,N) \cong \mathrm{Ext}^0(M,N)$.

(2) Note that $\mathrm{Ext}^1_{RD}(M,N) = \ker(d^1_*)/\mathrm{im}(d^0_*)$ and $\mathrm{Ext}^n(M,N) = \ker(e^1_*)/\mathrm{im}(e^0_*)$.

Define $\theta : \mathrm{Ext}^1_{RD}(M,N) \to \mathrm{Ext}^n(M,N)$ via $\theta(\overline{\alpha}) = \overline{\tau_{1*}(\alpha)}$ for any $\alpha \in \ker(d^1_*)$.

Let $\theta(\overline{\alpha}) = \overline{\tau_{1*}(\alpha)} = 0$ for some $\alpha \in \ker(d^1_*)$. Then

$$\tau_{1*}(\alpha) = \tau_1\alpha \in \mathrm{im}(e^0_*).$$

So there exists $\beta \in \mathrm{Hom}(M,E^0)$ such that

$$\tau_1\alpha = e^0_*(\beta) = e^0\beta.$$

Since $d^1\alpha = d^1_*(\alpha) = 0$, we have $\alpha(x) \in \ker(d^1) = \mathrm{im}(d^0)$ for any $x \in M$. Thus there exists $y \in D^0$ such that $\alpha(x) = d^0(y)$. Hence

$$e^0\beta(x) = \tau_1\alpha(x) = \tau_1 d^0(y) = e^0\tau_0(y),$$

and so

$$\beta(x) - \tau_0(y) \in \ker(e^0) = \mathrm{im}(\lambda) = \mathrm{im}(\tau_0\epsilon).$$

Therefore there exists $t \in N$ such that

$$\beta(x) - \tau_0(y) = \tau_0\epsilon(t).$$

Thus $\beta(x) = \tau_0(y + \epsilon(t))$. Define $\gamma : M \to D^0$ via

$$\gamma(x) = y + \epsilon(t).$$

Then γ is well defined since τ_0 is a monomorphism. Note that $\alpha = d^0_*(\gamma)$, and so $\overline{\alpha} = 0$. It follows that $\theta : \mathrm{Ext}^1_{RD}(M,N) \to \mathrm{Ext}^1(M,N)$ is a monomorphism. □

In general, $\mathrm{Ext}^1_{RD}(M,N) \to \mathrm{Ext}^1(M,N)$ need not be an epimorphism. In fact, $\mathrm{Ext}^1_{RD}(M,N) \to \mathrm{Ext}^1(M,N)$ is an epimorphism if and only if R is a von Neumann regular ring as shown by the following proposition.

Proposition 3.3 *The following are equivalent for a ring R:*

(1) R is a von Neumann regular ring.

(2) $\mathrm{Ext}^n_{RD}(M,N) \to \mathrm{Ext}^n(M,N)$ is an isomorphism for all left R-modules M and N and $n \geq 1$.

(3) $\mathrm{Ext}^1_{RD}(M,N) \to \mathrm{Ext}^1(M,N)$ is an isomorphism for all left R-modules M and N.

Proof. (1) $\Rightarrow$ (2) By (1) and Corollary 2.14, the class of RD-injective left R-modules coincides with the class of injective left R-modules. So $\mathrm{Ext}^n_{RD}(M,N)\cong \mathrm{Ext}^n(M,N)$ for all left R-modules M and N and $n\geq 1$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let N be any RD-injective left R-module. Then $\mathrm{Ext}^1_{RD}(M,N)=0$ for any left R-module M since there exists a right RD-injective resolution $0\to N\to N\to 0\to 0\to\cdots$. So $\mathrm{Ext}^1(M,N)=0$ by (3). Thus N is injective. Hence R is a von Neumann regular ring by Corollary 2.14. □

Next we introduce the RD-projective and RD-injective dimensions for modules and rings.

Definition 3.4 Let R be a ring. For a left R-module M, let $RD-pd(M) = \inf\{n:$ there exists a left RD-projective resolution $0\to P_n\to\cdots\to P_0\to M\to 0\}$ and call $RD-pd(M)$ the RD-projective dimension of M. If no such sequence exists for any n, set $RD-pd(M)=\infty$.

Put $lRD-PD(R)=\sup\{RD-pd(M)\colon M$ ranges over all left R-modules$\}$ and call $lRD-PD(R)$ the left global RD-projective dimension of the ring R.

Dually, we can define the RD-injective dimension $RD-id(M)$ of a left R-module M, and the left global RD-injective dimension $lRD-ID(R)$ of the ring R.

Proposition 3.5 *The following are equivalent for a left R-module M and an integer $n\geq 0$:*

(1) $RD-pd(M)\leq n$.

(2) $\mathrm{Ext}^{n+j}_{RD}(M,N)=0$ *for all left R-modules N and $j\geq 1$.*

(3) $\mathrm{Ext}^{n+1}_{RD}(M,N)=0$ *for all left R-modules N.*

(4) Every nth RD-projective syzygy of M is RD-projective.

Proof. (1) $\Rightarrow$ (2) By (1), M admits a left RD-projective resolution

$$0\to P_n\to\cdots\to P_0\to M\to 0.$$

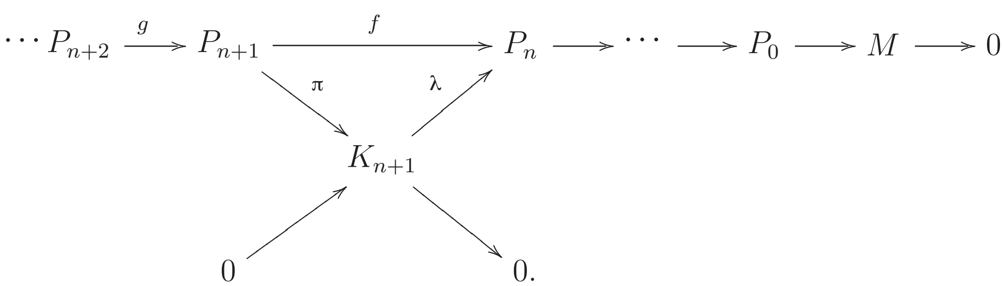
Then $\mathrm{Hom}(P_{n+j},N)=0$ for all left R-modules N and $j\geq 1$. So $\mathrm{Ext}^{n+j}_{RD}(M,N)=0$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) Let

$$\cdots\to P_{n+2}\to P_{n+1}\to P_n\to P_{n-1}\to\cdots\to P_0\to M\to 0$$

be a left RD-projective resolution of M with $K_n=\ker(P_{n-1}\to P_{n-2})$ and $K_{n+1}=\ker(P_n\to P_{n-1})$. Then we have the following exact commutative diagram:



By (3), $\mathrm{Ext}_{RD}^{n+1}(M,K_{n+1})=0$. Thus the sequence

$$\mathrm{Hom}(P_n,K_{n+1})\xrightarrow{f^*}\mathrm{Hom}(P_{n+1},K_{n+1})\xrightarrow{g^*}\mathrm{Hom}(P_{n+2},K_{n+1})$$

is exact. Since $g^*(\pi)=\pi g=0$, $\pi\in\ker(g^*)=\mathrm{im}(f^*)$. Thus there exists $h\in\mathrm{Hom}(P_n,K_{n+1})$ such that $\pi=f^*(h)=hf=h\lambda\pi$, and hence $h\lambda=1$ since π is epic. So the exact sequence $0\to K_{n+1}\xrightarrow{\lambda}P_n\to K_n\to 0$ is split. Therefore K_n is RD-projective.

(4) $\Rightarrow$ (1) is obvious. □

Dually, we have the following proposition.

Proposition 3.6 *The following are equivalent for a left R-module N and an integer $n\geq 0$:*

() $RD-id(N)\leq n$.

(1) $\mathrm{Ext}_{RD}^{n+j}(M,N)=0$ *for all left R-modules M and $j\geq 1$.*

(2) $\mathrm{Ext}_{RD}^{n+1}(M,N)=0$ *for all left R-modules M.*

(3) Every nth RD-injective cosyzygy of N is RD-injective.

Combining Propositions 3.5 with 3.6, we have

Theorem 3.7 *The following are equivalent for a ring R and an integer $n\geq 0$:*

(1) $lRD-PD(R)\leq n$.

(2) $lRD-ID(R)\leq n$.

(3) $\mathrm{Ext}_{RD}^{n+j}(M,N)=0$ *for all left R-modules M, N and $j\geq 1$.*

(4) $\mathrm{Ext}_{RD}^{n+1}(M,N)=0$ *for all left R-modules M and N.*

We list some corollaries of Theorem 3.7 as follows.

Corollary 3.8 *For any ring R, $lRD-PD(R)=lRD-ID(R)$.*

Corollary 3.9 *The following are equivalent for a ring R:*

(1) $lRD-PD(R)=lRD-ID(R)=0$.

(2) Every left R-module is RD-projective.

(3) Every left R-module is RD-injective.

(4) $\mathrm{Ext}_{RD}^{n}(M,N)=0$ *for all left R-modules M, N and $n\geq 1$.*

(5) $\mathrm{Ext}_{RD}^{1}(M,N)=0$ *for all left R-modules M and N.*

(6) Every left R-module RD-exact sequence is split.

Corollary 3.10 *The following are equivalent for a ring R:*

(1) $lRD-PD(R)=lRD-ID(R)\leq 1$.

(2) Every RD-submodule of an RD-projective left R-module is RD-projective.

(3) For any RD-submodule of an RD-injective left R-module M, M/N is RD-injective.

(4) $\mathrm{Ext}^n_{RD}(M,N)=0$ for all left R-modules M, N and $n\geq 2$.

(5) $\mathrm{Ext}^2_{RD}(M,N)=0$ for all left R-modules M and N.

Finally, we discuss the relations between the RD-homological dimensions and other homological dimensions.

Recall that R is *left strongly P-coherent* [15] if every principal left ideal of R is cyclically presented.

Theorem 3.11 *Let R be a left strongly P-coherent ring. Then*

(1) $RD-id(M)=id(M)$ for a divisible left R-module M.

(2) $RD-pd(M)=pd(M)$ for a torsionfree left R-module M.

(3) sup{id(M): M is any divisible left R-module} $\leq lRD-ID(R)$.

(4) sup{pd(M): M is any torsionfree left R-module} $\leq lRD-PD(R)$.

Proof. (1) Let M be a divisible left R-module. By [15, Lemma 4.10] and Proposition 2.3, a right injective resolution of M must be its right RD-injective resolution. So $RD-id(M)\leq id(M)$. Conversely, we may assume $RD-id(M)=m<\infty$. There is an exact sequence

$$0\to M\to E^0\to E^1\to\cdots\to E^{m-1}\to L^m\to 0$$

with each E^i injective. By [15, Lemma 4.10] and Proposition 2.3, the above sequence is an RD-exact sequence. Thus L^m is divisible and RD-injective by Proposition 3.6, and hence is injective by Corollary 2.5 (1). So $id(M)\leq m$. Thus $RD-id(M)=id(M)$.

(2) Let M be a torsionfree left R-module. By [15, Lemma 4.10] and Proposition 2.4, a left projection resolution of M must be its left RD-projective resolution. So $RD-pd(M)\leq pd(M)$.

Conversely, we may assume $RD-pd(M)=n<\infty$. There exists an exact sequence

$$0\to K_n\to P_{n-1}\to\cdots\to P_1\to P_0\to M\to 0,$$

where each P_i is projective. By [15, Lemma 4.10] and Proposition 2.4, the above sequence is an RD-exact sequence. So K_n is torsionfree and RD-projective by Proposition 3.5, and so is projective by Corollary 2.5 (2). Thus $pd(M)\leq n$. Hence $RD-pd(M)=pd(M)$.

(3) follows from (1), (4) holds by (2). □

Observing the following facts:

(1) If R is a von Neumann regular ring, then $lD(R) = lRD - ID(R)$ by Corollary 2.14.

(2) If $lRD - ID(R) = 0$, then $lD(R) = wD(R)$.

In general, we have the following inequalities.

Theorem 3.12 *Let R be a ring. Then*

$$lD(R) \leq lRD - ID(R) \ + sup\{id(M) : M \text{ is any } RD\text{-injective left } R\text{-module}\}$$
$$\leq lRD - ID(R) + wD(R).$$

Proof. By Theorem 2.12, $\sup\{id(M) : M$ is any RD-injective left R-module$\} = \sup\{fd(M) : M$ is any RD-flat right R-module$\} \leq wD(R)$. So the second inequality in the theorem holds.

Next we show that $lD(R) \leq lRD - ID(R) \ + \sup\{id(M) : M$ is any RD-injective left R-module$\}$. We may assume that both $lRD - ID(R)$ and $\sup\{id(M) : M$ is any RD-injective left R-module$\}$ are finite. Let $lRD - ID(R) = m < \infty$ and $\sup\{id(M) : M$ is any RD-injective left R-module$\} = n < \infty$. Suppose M is a left R-module, then M admits a right RD-injective resolution

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^{m-1} \to E^m \to 0.$$

Note that $id(E^i) \leq n$. For every left R-module N, we have

$$\text{Ext}^{n+m+1}(N, M) \cong \text{Ext}^{n+1}(N, E^m) = 0.$$

So $id(M) \leq n + m$. Thus $lD(R) \leq n + m$. □

We conclude this paper with the following

Remark 3.13 (1) Let $R = \mathbb{Z}$. Then $D(R) = RD - ID(R) = wD(R) = 1$.

By [21, 40.5], $\sup\{id(M) : M$ is any divisible left R-module$\} = 0$. So the inequality $\sup\{id(M) : M$ is any divisible left R-module$\} \leq lRD - ID(R)$ in Theorem 3.11 may be strict.

On the other hand, by Corollaries 2.13 and 2.14, $\sup\{id(M) : M$ is any RD-injective left R-module$\} = 1$. Thus the inequality $lD(R) \leq lRD - ID(R) \ + \sup\{id(M) : M$ is any RD-injective left R-module$\}$ in Theorem 3.12 may be strict.

(2) The second inequality in Theorem 3.12 may be also strict. For example, by [10, Corollary, p.439], there exists a left Noetherian domain R with $lD(R) = wD(R) = 2$. Then $\sup\{id(M) : M$ is any RD-injective left R-module$\} = 1$ by Corollary 2.13.

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References

[1] Anderson, F.W. and Fuller, K.R.: Rings and Categories of Modules. Springer-Verlag, New York, 1974.

[2] Couchot, F.: Modules with RD-composition series over a commutative ring. Comm. Algebra 31(7), 3171-3194 (2003).

[3] Couchot, F.: RD-flatness and RD-injectivity. Comm. Algebra 34, 3675-3689 (2006).

[4] Enochs, E.E.: Injective and flat covers, envelopes and resolvents. Israel J. Math. 39, 189-209 (1981).

[5] Enochs, E.E. and Jenda, O.M.G.: Relative Homological Algebra, GEM 30. Walter de Gruyter, Berlin-New York, 2000.

[6] Fuchs, L. and Salce, L.: Modules over Non-noetherian Domains. Math. Surveys and Monographs. Vol. 84. Providence, Amer. Math. Soc. 2001.

[7] Göbel, R. and Trlifaj, J.: Approximations and Endomorphism Algebras of Modules, GEM 41. Walter de Gruyter, Berlin-New York, 2006.

[8] Hattori, A.: A foundation of torsion theory for modules over general rings. Nagoya Math. J. 17, 147-158 (1960).

[9] Holm, H. and Jørgensen, P.: Covers, precovers, and purity. Illinois J. Math. 52, 691-703 (2008).

[10] Jategornkar, A.V.: A counter example in ring theory and homological algebra. J. Algebra 12, 418-440 (1969).

[11] Lam, T.Y.: Lectures on Modules and Rings. Springer-Verlag, New York-Heidelberg-Berlin, 1999.

[12] Maddox, B. Absolutely pure modules. Proc. Amer. Math. Soc. 18, 155-158 (1967).

[13] Mao, L.X.: On P-coherent endomorphism rings. Proc. Indian Acad. Sci. (Math. Sci.) 118 (4), 557-567 (2008).

[14] Mao, L.X. and Ding, N.Q.: On relative injective modules and relative coherent rings. Comm. Algebra 34 (7), 2531-2545 (2006).

[15] Mao, L.X. and Ding, N.Q.: On divisible and torsionfree modules. Comm. Algebra 36(2), 708-731 (2008).

[16] Puninski, G., Prest, M. and Rothmaler, P.: Rings described by various purities. Comm. Algebra 27(5), 2127-2162 (1999).

[17] Rotman, J.J.: An Introduction to Homological Algebra. Academic Press, New York, 1979.

[18] Rutter, Jr. E.A.: Rings with the principal extension property. Comm. Algebra 3(3), 203-212 (1975).

[19] Stenström, B.: Pure submodules. Ark. Mat. 7 (10), 159-171 (1967).

[20] Warfield, Jr., R.B.: Purity and algebraic compactness for modules. Pacific J. Math. 28, 699-719 (1969).

[21] Wisbauer, R.: Foundations of Module and Ring Theory. Gordon and Breach, 1991.

[22] Xu, J.: Flat Covers of Modules. Lecture Notes in Math. 1634, Springer-Verlag, Berlin-Heidelberg-New York, 1996.

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