Turk J Math
35 (2011) , 219 – 248.
© TÜBİTAK
doi:10.3906/mat-0904-18

Existence theory for positive solutions of p-laplacian multi-point BVPs on time scales*

You-Hui Su

Abstract

This paper is concerned with the one-dimensional p-Laplacian multi-point boundary value problem on time scales $\mathbb{T}$:

$$(\varphi_p(u^{\Delta}))^{\nabla} + h(t)f(u) = 0,\ t \in [0,T]_{\mathbb{T}},$$

subject to multi-point boundary conditions

$$u(0) - B_0\left(\textstyle\sum_{i=1}^{m-2} a_i u^{\Delta}(\xi_i)\right) = 0,\ u^{\Delta}(T) = 0,$$

or

$$u^{\Delta}(0) = 0, u(T) + B_1\left(\textstyle\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i')\right) = 0,$$

where $\varphi_p(u)$ is p-Laplacian operator, i.e., $\varphi_p(u) = |u|^{p-2}u,\ p > 1,\ \xi_i, \xi_i' \in [0,T]_{\mathbb{T}},\ m \geq 3$ and satisfy $0 \leq \xi_1 < \xi_2 < \ldots < \xi_{m-2} < \rho(T),\ \sigma(0) < \xi_1' < \xi_2' < \ldots < \xi_{m-2}' \leq T,\ a_i, b_i \in [0,\infty)\ (i = 1,2,\ldots,m-2)$. Some new sufficient conditions are obtained for the existence of at least one positive solution by using Krasnosel'skii's fixed-point theorem and new sufficient conditions are obtained for the existence of twin, triple or arbitrary odd positive solutions by using generalized Avery and Henderson fixed-point theorem and Avery-Peterson fixed-point theorem. Our results include and extend some known results. As applications, two examples are given to illustrate the main results and their differences. These results are new even for the special cases of continuous and discrete equations, as well as in the general time scale setting.

Key Words: Time scales; boundary value problem; positive solutions; p-Laplacian; fixed-point theorem.

1. Introduction

The development of the theory of time scales was initiated by Hilger in his Ph.D thesis in 1988 [17]. An initial motivation to develop and study calculus on time scales was to provide the unification of continuous and discrete calculus. Such a study on time scales lead to deeper understanding of modeling hybrid-type continuous-discrete systems. Since then, the theory of dynamic equations on time scales are considerably active. Now, it

2000 *AMS Mathematics Subject Classification:* 34B15, 39A10.

Supported by the grant of Department of Education Jiangsu Province (09KJD110006).
*Supported by the NSFC (No.10571078).

is still a new subject, and research in this area is rapidly developed. Furthermore, the time scales calculus has tremendous potential in application, for example, in the study of biology, heat transfer, stock market, wound healing and epidemic models [1, 19, 20, 37], etc.

For convenience, we make the blanket assumption that $0, T$ are points in $\mathbb{T}$, for an interval $(0,T)_{\mathbb{T}}$ we always mean $(0,T)\cap\mathbb{T}$. Other types of interval are defined similarly.

Throughout this paper, we denote the p-Laplacian operator by $\varphi_p(u)$, i.e., $\varphi_p(u)=|u|^{p-2}u$ for $p>1$ with $(\varphi_p)^{-1}=\varphi_q$ and $1/p+1/q=1$. In addition, B_0 and B_1 satisfy

$$A'x\le B_i(x)\le Bx,\ x\in\mathbb{R}^+,\ i=0,1, \tag{1.1}$$

here, A' and B are positive real numbers.

Recently, boundary value problems (BVPs) for dynamic equation on time scales have received considerable attention [3, 29, 30, 32, 35, 41]. In particular, there is some attention focused on the study of two-point, three-point BVPs for p-Laplacian dynamic equation on time scales. For two-point BVPs, see [15, 16, 36] and references therein. As far as three-point BVPs, see [12, 18, 31, 38]. However, little work has been done to existence of positive solutions to multi-point BVPs for one-dimensional p-Laplacian dynamic equation on time scales [32, 33, 34].

In the following, we would like to review some results of He [12], He and Li [15], He and Jiang [16], Hong [18], Sun and Li [36] and Wang [38], which motivate us to consider one dimensional p-Laplacian multi-point BVPs on time scales.

For the two-point BVP

$$\begin{gathered}(\varphi_p(u^\Delta))^\Delta+h(t)f(u^\sigma)=0,\ t\in[a,b]_{\mathbb{T}},\\ u(a)-B_0(u^\Delta(a))=0,\ u^\Delta(\sigma(b))=0;\end{gathered}$$

Sun and Li [36] established the existence theory for positive solutions of the above dynamic equation by using some fixed-point theorems.

In terms of the generalized Avery and Henderson fixed point theorem due to Ren et al. [28], He and Jiang [16] considered the dynamic equation on time scales

$$(\varphi_p(u^\Delta))^\nabla+h(t)f(u)=0,\ t\in[0,T]_{\mathbb{T}}, \tag{1.2}$$

with boundary conditions

$$u(0)-B_0(u^\Delta(0))=0,\ u^\Delta(T)=0, \tag{1.3}$$

or

$$u^\Delta(0)=0,\ u(T)+B_1(u^\Delta(T))=0, \tag{1.4}$$

and gave the sufficient condition for the existence of at least *three* positive solutions. Furthermore, He et al. [15] also obtained the existence criteria of at least *three* positive solutions of problem (1.2) and (1.3) or (1.4) by using the five functional fixed point theorem [8].

For the dynamic equation (1.2) satisfying the three-point boundary value conditions

$$u(0)-B_0(u^\Delta(\eta))=0,\ u^\Delta(T)=0, \tag{1.5}$$

or

$$u^{\Delta}(0)=0,\ u(T)+B_1(u^{\Delta}(\eta))=0, \tag{1.6}$$

where $\eta\in(0,\rho(T))_{\mathbb{T}}$, by using the fixed point theorem due to Avery and Henderson [6], He [12] proved that the BVPs (1.2) and (1.5) or (1.6) has at least *two* positive solutions under some suitable assumption. In addition, Hong [18] used the fixed point theorem due to Avery-Peterson [7] and established the existence criteria for at least a *triple* positive solutions of problems (1.2) and (1.5) or (1.6). Wang [38] also gave the existence criteria for at least *three* positive solutions of problems (1.2) and (1.5) or (1.6) by using the Leggett-Williams fixed point theorem [23].

It is also noted that the above mentioned researchers [12, 15, 16, 18, 32, 38] only considered partial results on existence of positive solutions. On the one hand, they failed to further provide comprehensible results of positive solutions of dynamic equations. On the other hand, few literature resources [29, 36] are available concerning the *arbitrary positive solutions* of boundary value problems for p-Laplacian dynamic equations on time scales. Naturally, it is quite necessary to consider the *arbitrary positive solutions* for p-Laplacian dynamic equations *in all respects.*

In this paper, we all-sidedly consider the dynamic equation (1.2) subject to multi-point boundary conditions

$$u(0)-B_0\left(\sum_{i=1}^{m-2}a_iu^{\Delta}(\xi_i)\right)=0,\ u^{\Delta}(T)=0, \tag{1.7}$$

or

$$u^{\Delta}(0)=0,\ u(T)+B_1\left(\sum_{i=1}^{m-2}b_iu^{\Delta}(\xi_i')\right)=0, \tag{1.8}$$

where $\xi_i,\xi_i'\in[0,T]_{\mathbb{T}}$, $m\geq 3$ and satisfy $0\leq\xi_1<\xi_2<\ldots<\xi_{m-2}<\rho(T)$, $\sigma(0)<\xi_1'<\xi_2'<\ldots<\xi_{m-2}'\leq T$, $a_i,b_i\in[0,\infty)$ $(i=1,2,\ldots,m-2)$. Some new and more general results are obtained for the existence of at least *one, two, three or arbitrary odd* positive solutions for the above problems by using Krasnosel'skii's fixed point theorem [21], the generalized Avery and Henderson fixed point theorem [28] and fixed point theorem due to Avery-Peterson [7]. Our results are new even for some special cases of difference equations and differential equations as well as in the general time scale setting. As applications, two examples are given to illustrate the result, in addition, these two examples show the differences of the existence criteria established in Section 4.

In particular, our results include and extend many results of Avery et al. [5] $(p=2)$, Li et al. [24], Liu et al. [25], Lü, et al. [27] and Wang [40] in the case $\mathbb{T}=\mathbb{R}$; Avery et al. [5] $(p=2)$, He [13, 14] and Liu et al. [26] in the case of $\mathbb{T}=\mathbb{Z}$. That is to say, when $\mathbb{T}=\mathbb{R}$, if $T=1$, $B_0(u)$ is nondecreasing odd function, then our results in Section 3 reduce to those of Li et al. [24] and Wang [40]; the results in Section 4 include and extend the results of Liu et al. [25]. If $B_0(u)\equiv 0, T=1$, then the results in [27] is a special case of ours in Section 4. For the case $\mathbb{T}=\mathbb{Z}$, our criteria in Section 3 include and generalize the main results of He [14]. The results in Section 4 improve and generalize the main results of He [13] and Liu et al. [26]. For the general time scale $\mathbb{T}$, the results in Sections 4 improve and generalize some known works of He et al. [16] $(m=3,\xi_1=0,\xi_1'=T$ and $a_1=b_1=1)$, He [12], Hong [18] and Wang [38] (note, the later three, $m=3,\xi_1=\xi_1'=\eta$ and $a_1=b_1=1)$, respectively.

We note that, by a solution u of problems (1.2) and (1.7) or (1.8), we mean $u:[0,T]_{\mathbb{T}}\to\mathbb{R}$ which is delta differentiable, u^{Δ} and $(\varphi_p(u^{\Delta}))^{\nabla}$ are both regulated on $[0,T]_{\mathbb{T}^{\kappa}\cap\mathbb{T}_{\kappa}}$, and u satisfies problems (1.2) and

(1.7) or (1.8). If $(u^{\Delta})^{\nabla} \leq 0$, then we say u is concave on $[0,T]_{\mathbb{T}}$.

The rest of the paper is organized as follows. In Section 2, we first present two lemmas which are needed throughout this paper and then state several fixed point results. In Section 3, by using Krasnosel'skii's fixed point theorem, we obtain the existence of at least *one* or *two* positive solutions of problems (1.2) and (1.7) or (1.8). In Section 4, the existence criteria for at least *three* positive and arbitrary odd positive solutions of problems (1.2) and (1.7) or (1.8) are established. In Section 5, we present two simple examples to illustrate our main results.

For the convenience of statements, now we present some basic definitions concerning the calculus on time scales that one needs to read this manuscript, which can be found in [9] and [10]. One of another excellent sources for dynamical systems on measure chains is the book [22].

Definition 1.1 [4, 9, 10] *A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward and back jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are well defined, respectively, by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. In this definition one put $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is called left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_{\kappa} = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^{\kappa} = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^{\kappa} = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\nu(t) := t - \rho(t)$.*

Definition 1.2 [9, 10] *If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^{\kappa}$, then the delta derivative of f at the point t is defined as the number $f^{\Delta}(t)$ (provided it exists) with the property that for any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that $\left|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| \leq \epsilon\,|\sigma(t) - s|$ for all $s \in U$. If $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_{\kappa}$, then the nabla derivative of f at the point t is defined by the number $f^{\nabla}(t)$ (provided it exists) with the property that for any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that $\left|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)\right| \leq \epsilon\,|\rho(t) - s|$ for all $s \in U$.*

Definition 1.3 [9, 10] *A function $h : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limit exists (finite) at left-dense points in $\mathbb{T}$.*

Throughout this paper, it is assumed that

(S1) $f \circ u : [0,T]_{\mathbb{T}} \to [0,+\infty)$ is rd-continuous and does not vanish identically;

(S2) $h :\in C_{rd}([0,T]_{\mathbb{T}}, [0,+\infty))$ and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$, where $C_{rd}([0,T]_{\mathbb{T}}, [0,+\infty))$ denotes the set of all right dense continuous functions from $[0,T]_{\mathbb{T}}$ to $[0,+\infty)$;

(S3) While discussing problem (1.2) and (1.7), we assume that if $\xi_{m-2} > 0$, then choose $\eta = \xi_{m-2}$, if $\xi_{m-2} = 0$, then let $\eta = \min\{t \in \mathbb{T} : t \geq \frac{T}{2}\}$, and there exists $r \in \mathbb{T}$ such that $\eta < r < T$ holds. While discussing problem (1.2) and (1.8), we assume that if $\xi_1' < T$, then choose $\xi = \xi_1'$, if $\xi_1' = T$, then let $\xi = \max\left\{t \in \mathbb{T} : 0 < t \leq \frac{T}{2}\right\}$, and there exists $l \in \mathbb{T}$ such that $0 < l < \xi < T$ holds.

2. Some Lemmas

Let the Banach space $E = C_{rd}([0,T]_{\mathbb{T}}, \mathbb{R})$ with the norm $\|u\| = \sup_{t\in[0,T]_{\mathbb{T}}} |u|$, and define the cone $P \subset E$ by

$$P = \left\{ u \in E \,\middle|\, u^{\Delta}(T) = 0, u \text{ is concave and nonnegative on } [0,T]_{\mathbb{T}} \right\}.$$

Clearly, $\|u\| = u(T)$ for $u \in P$.

First, integrating (1.2) from t to T, one obtains

$$\varphi_p\left(u^{\Delta}(T)\right) - \varphi_p(u^{\Delta}(t)) = -\int_t^T h(s)f(u)\nabla s.$$

Thus, in view of (1.7), we have

$$u^{\Delta}(t) = \varphi_q\left(\int_t^T h(s)f(u)\nabla s\right), \tag{2.1}$$

which implies

$$u(t) - u(0) = \int_0^t \varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau. \tag{2.2}$$

From boundary condition (1.7) and

$$u^{\Delta}(\xi_i) = \varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right) \text{ for } i = 1, 2, \ldots, m-2,$$

one gets

$$u(t) = B_0\left(\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right)\right) + \int_0^t \varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau. \tag{2.3}$$

Define the operator $A : P \to E$ by

$$Au = B_0\left(\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right)\right) + \int_0^t \varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau. \tag{2.4}$$

Now, we show that $A : P \to P$ is completely continuous.

Lemma 2.1 *$A : P \to P$ is completely continuous.*

Proof. Firstly, it is easy to obtain that $(Au)(t) \geq 0$ for $t \in [0,T]_{\mathbb{T}}$ and $(Au)^{\Delta}(T) = 0$.

In addition, $(Au)^{\Delta}(t) = \varphi_q\left(\int_t^T h(s)f\left(u\right)\nabla s\right) \geq 0,\ t \in [0,T]_{\mathbb{T}}$ is differentiable and nonincreasing in $[0,T]_{\mathbb{T}}$. Moreover, $\varphi_q(x)$ is a monotone increasing continuously differentiable function for $x > 0$.

$$\left(\int_t^T h(s)f\left(u\right)\nabla s\right)^{\nabla} = -h(t)f\left(u\right) \leq 0,\ t \in [0,T]_{\mathbb{T}}.$$

If $\int_t^T h(s)f\left(u\right)\nabla s > 0$, by the chain rule [10, Theorem 1.87, p. 31], we obtain $(Au)^{\Delta\nabla}(t) \leq 0$ for $[0,T]_{\mathbb{T}}$.

If $\int_t^T h(s)f\left(u\right)\nabla s = 0$, we have $(Au)^{\Delta}(t) = \varphi_q\left(\int_t^T h(s)f\left(u\right)\nabla s\right) = 0,\ t \in [0,T]_{\mathbb{T}}$, then $(Au)^{\Delta\nabla}(t) = 0$ for $[0,T]_{\mathbb{T}}$.

So, $A : P \to P$.

Secondly, we show that A maps bounded set into bounded set. Assume $c > 0$ is a constant and $u \in \overline{P}_c = \{x \in P : \|x\| \leq c\}$. Note that the rd-continuity of $f \circ u$ guarantees that there is a $C > 0$ such that $f(u) \leq \varphi_p(C)$:

$$\begin{aligned}
&\|Au\| = (Au)(T) \\
&= B_0 \left(\sum_{i=1}^{m-2} a_i \varphi_q \left(\int_{\xi_i}^T h(s) f(u) \nabla s \right) \right) + \int_0^T \varphi_q \left(\int_\tau^T h(s) f(u) \nabla s \right) \Delta\tau \\
&\leq \left(B \sum_{i=1}^{m-2} a_i + T \right) \varphi_q \left(\int_0^T h(s) \varphi_p(C) \nabla s \right) \leq C \left(B \sum_{i=1}^{m-2} a_i + T \right) \varphi_q \left(\int_0^T h(s) \Delta s \right).
\end{aligned}$$

That is, $A\overline{P}_c$ is uniformly bounded.

Thirdly, for $t_1, t_2 \in [0, T]_{\mathbb{T}}$, we have

$$\begin{aligned}
&|(Au)(t_1) - (Au)(t_2)| = \left| \int_{t_1}^{t_2} \varphi_q \left(\int_\tau^T h(s) f(u) \nabla s \right) \Delta\tau \right| \\
&\leq C \left| \int_{t_1}^{t_2} \varphi_q \left(\int_\tau^T h(s) \nabla s \right) \Delta\tau \right| \leq C |t_1 - t_2| \varphi_q \left(\int_0^T h(s) \nabla s \right) \to 0 \text{ as } t_1 \to t_2.
\end{aligned}$$

The Arzela-Ascoli Theorem on time scales [2] tells us that $A\overline{P}_c$ is relatively compact.

We next claim that $A : \overline{P}_c \to P$ is continuous. Assume that $\{u_n\}_{n=1}^\infty \subset \overline{P}_c$ and $u_n \to u_0$ for $[0, T]_{\mathbb{T}}$. Since $\{(Au_n)(t)\}_{n=1}^\infty$ is uniformly bounded and equicontinuous on $[0, T]_{\mathbb{T}}$, there exists uniformly convergent subsequence in $\{(Au_n)(t)\}_{n=1}^\infty$. Let $\left\{ \left(Au_{n(m)} \right)(t) \right\}_{m=1}^\infty$ be a subsequence which converges to $v(t)$ uniformly on $[0, T]_{\mathbb{T}}$. Observe that

$$Au_n(t) = B_0 \left(\sum_{i=1}^{m-2} a_i \varphi_q \left(\int_{\xi_i}^T h(s) f(u_n(s)) \nabla s \right) \right) + \int_0^t \varphi_q \left(\int_\tau^T h(s) f(u_n(s)) \nabla s \right) \Delta\tau.$$

Inserting $u_{n(m)}$ into the above and then letting $m \to \infty$, we obtain

$$v(t) = B_0 \left(\sum_{i=1}^{m-2} a_i \varphi_q \left(\int_{\xi_i}^T h(s) f(u_0(s)) \nabla s \right) \right) + \int_0^t \varphi_q \left(\int_\tau^T h(s) f(u_0(s)) \nabla s \right) \Delta\tau.$$

Here, we used the Lebesgue's dominated convergence Theorem on time scales [4]. From the definition of A, we know that $v(t) = Au_0(t)$ on $[0, T]_{\mathbb{T}}$. This shows that each subsequence of $\{Au_n(t)\}_{n=1}^\infty$ uniformly converges to $(Au_0)(t)$. Therefore, the sequence $\{(Au_n)(t)\}_{n=1}^\infty$ uniformly converges to $(Au_0)(t)$. This means that A is continuous at $u_0 \in \overline{P}_c$. So, A is continuous on $\overline{P}_c$ since u_0 is arbitrary. Thus, A is completely continuous. The proof is complete. $\square$

Hence, we obtain that every fixed point of A is a solution of the problem (1.2) and (1.7).

Second, define the cone $P_1 \subset E$ by

$$P_1 = \left\{ u \in E \,\middle|\, u^\Delta(0) = 0,\ u \text{ is concave and nonnegative on } [0, T]_{\mathbb{T}} \right\}.$$

Clearly, $\|u\| = u(0)$ for $u \in P_1$. Define the operator $A_1 : P_1 \to E$ by

$$A_1 u = B_1 \left(\sum_{i=1}^{m-2} b_i \varphi_q \left(\int_0^{\xi_i'} h(s) f(u) \nabla s \right) \right) + \int_t^T \varphi_q \left(\int_0^\tau h(s) f(u) \nabla s \right) \Delta\tau. \tag{2.5}$$

It is easy to see that $A_1 : P_1 \to P_1$ is completely continuous and every fixed point of A_1 is a solution of the problem (1.2) and (1.8).

To obtain our main results, we make use of the following two lemmas.

Lemma 2.2 [15] *If $u \in P$, then (i) $u(t) \geq \frac{t}{T}\|u\| = \frac{t}{T}u(T)$ for $t \in [0,T]_{\mathbb{T}}$; (ii) $su(t) \leq tu(s)$ for $s,t \in [0,T]_{\mathbb{T}}$ and $s \leq t$.*

Lemma 2.3 [15] *If $u \in P_1$, then (i) $u(t) \geq \frac{T-t}{T}\|u\| = \frac{T-t}{T}u(0)$ for $t \in [0,T]_{\mathbb{T}}$; (ii) $(T-t)u(s) \leq (T-s)u(t)$ for $s,t \in [0,T]_{\mathbb{T}}$ and $s \leq t$.*

Now, we provide some background material from the theory of cones in Banach spaces [11], and state several fixed point theorems needed later.

Definition 2.4 *A map α is said to be a nonnegative continuous concave (convex) functional on a cone P of a real Banach space E if $\alpha : P \to [0,\infty)$ is continuous and $\alpha(tx+(1-t)y) \geq t\alpha(x)+(1-t)\alpha(y)$ $(\beta(tx+(1-t)y) \leq t\beta(x)+(1-t)\beta(y))$ for all $x,y \in P$ and $t \in [0,1]$.*

We firstly list the Krasnosel'skii's fixed point theorem [11, 21].

Lemma 2.5 [11, 21] *Let P be a cone in a Banach space E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$. If $A : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$.

Then A has a fixed point in $P \cap (\overline{\Omega_2} \backslash \Omega_1)$.

Given a nonnegative continuous functional γ on a cone P of a real Banach space E, we define, for each $d > 0$, the set $P(\gamma, d) = \{x \in P : \gamma(x) < d\}$.

The following fixed-point theorem due to Ren et al. [28], which is motivated by Avery and Henderson's double fixed-point theorem [6].

Lemma 2.6 [28] *Let P be a cone in a real Banach space E. Let α, β and γ be increasing, nonnegative continuous functionals on P such that for some $c > 0$ and $H > 0$, $\gamma(x) \leq \beta(x) \leq \alpha(x)$ and $\|x\| \leq H\gamma(x)$ for all $x \in \overline{P(\gamma,c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ and $A : \overline{P(\gamma,c)} \to P$ is a completely continuous operator such that*

(i) $\gamma(Ax) < c$ *for all* $x \in \partial P(\gamma,c)$;

(ii) $\beta(Ax) > b$ *for all* $x \in \partial P(\beta,b)$;

(iii) $P(\alpha,a) \neq \emptyset$ *and* $\alpha(Ax) < a$ *for* $x \in \partial P(\alpha,a)$.

Then A has at least three fixed points x_1, x_2 and x_3 belonging to $\overline{P(\gamma,c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2) \text{ with } \beta(x_2) < b < \beta(x_3) \text{ and } \gamma(x_3) < c.$$

Lemma 2.7 [29] *Let P be a cone in a real Banach space E. Let α, β and γ be increasing, nonnegative continuous functionals on P such that for some $c > 0$ and $H > 0$, $\gamma(x) \leq \beta(x) \leq \alpha(x)$ and $\|x\| \leq H\gamma(x)$ for all $x \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ and $A : \overline{P(\gamma, c)} \to P$ is a completely continuous operator such that*

(i) $\gamma(Ax) > c$ *for all* $x \in \partial P(\gamma, c)$;

(ii) $\beta(Ax) < b$ *for all* $x \in \partial P(\beta, b)$;

(iii) $P(\alpha, a) \neq \emptyset$ *and* $\alpha(Ax) > a$ *for* $x \in \partial P(\alpha, a)$.

Then A has at least three fixed points x_1, x_2 and x_3 belonging to $\overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2) \text{ with } \beta(x_2) < b < \beta(x_3) \text{ and } \gamma(x_3) < c.$$

Let β and ϕ be nonnegative continuous convex functionals on P, λ be a nonnegative continuous concave functional on P and φ be a nonnegative continuous functional respectively on P. We define the following convex sets:

$$P(\phi, \lambda, b, d) = \{x \in P : b \leq \lambda(x), \phi(x) \leq d\},$$
$$P(\phi, \beta, \lambda, b, c, d) = \{x \in P : b \leq \lambda(x), \beta(x) \leq c, \phi(x) \leq d\},$$

and a closed set $R(\phi, \varphi, a, d) = \{x \in P : a \leq \varphi(x), \phi(x) \leq d\}$.

Finally, we list the fixed point theorem due to Avery-Peterson [7].

Lemma 2.8 [7] *Let P be a cone in a real Banach space E and β, ϕ, λ, φ be defined as above, moreover, φ satisfies $\varphi(\lambda' x) \leq \lambda' \varphi(x)$ for $0 \leq \lambda' \leq 1$ such that, for some positive numbers h and d,*

$$\lambda(x) \leq \varphi(x) \text{ and } \|x\| \leq h\phi(x) \tag{2.6}$$

for all $x \in \overline{P(\phi, d)}$. Suppose $A : \overline{P(\phi, d)} \to \overline{P(\phi, d)}$ is completely continuous and there exist positive real numbers a, b, c, with $a < b$ such that:

(i) $\{x \in P(\phi, \beta, \lambda, b, c, d) : \lambda(x) > b\} \neq \emptyset$ *and* $\lambda(A(x)) > b$ *for* $x \in P(\phi, \beta, \lambda, b, c, d)$;

(ii) $\lambda(A(x)) > b$ *for* $x \in P(\phi, \lambda, b, d)$ *with* $\beta(A(x)) > c$;

(iii) $0 \notin R(\phi, \varphi, a, d)$ *and* $\lambda(A(x)) < a$ *for all* $x \in R(\phi, \varphi, a, d)$ *with* $\varphi(x) = a$.

Then A has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\phi, d)}$ such that

$$\phi(x_i) \leq d \text{ for i=1,2,3, } b < \lambda(x_1),\ a < \varphi(x_2) \text{ and } \lambda(x_2) < b \text{ with } \varphi(x_3) < a.$$

3. Single or twin solutions

For $t \in [0, T]_{\mathbb{T}}$, let

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{\varphi_p(u)} \text{ and } f_\infty = \lim_{u \to \infty} \frac{f(u)}{\varphi_p(u)},$$

here, while considering the problem (1.2) and (1.7), then we assume that $u \in P$; while considering the problem (1.2) and (1.8), then we assume that $u \in P_1$.

Similarly to [39], we define i_0 =number of zeros in the set $\{f_0, f_\infty\}$ and i_∞ =number of infinities in the set $\{f_0, f_\infty\}$. Clearly, $i_0, i_\infty = 0, 1$, or 2 and there exist six possible cases: (i) $i_0 = 1$ and $i_\infty = 1$; (ii) $i_0 = 0$ and $i_\infty = 0$; (iii) $i_0 = 0$ and $i_\infty = 1$; (iv) $i_0 = 0$ and $i_\infty = 2$; (v) $i_0 = 1$ and $i_\infty = 0$; and (vi) $i_0 = 2$ and $i_\infty = 0$. In the following, by using Krasnosel'skii's fixed point theorem in a cone, we study the existence for positive solutions of problem (1.2) and (1.7) or (1.8) under the above six possible cases.

3.1. For the case $i_0 = 1$ and $i_\infty = 1$

In this subsection, we discuss the existence of single positive solution for the problem (1.2) and (1.7) or (1.8) under $i_0 = 1$ and $i_\infty = 1$.

Theorem 3.1 *Problem* (1.2) *and* (1.7) *has at least one positive solution in the case* $i_0 = 1$ *and* $i_\infty = 1$.

Proof. We divide the proof into two cases:

Case (i) $f_0 = 0$ and $f_\infty = \infty$.

In view of $f_0 = 0$, there exists an $H_1 > 0$ such that $f(u) \le \varphi_p(\varepsilon)\varphi_p(u) = \varphi_p(\varepsilon u)$ for $0 < u \le H_1$, where $\varepsilon > 0$, and satisfies $\varepsilon \left(B \sum_{i=1}^{m-2} a_i + T\right) \varphi_q \left(\int_0^T h(s)\Delta s\right) \le 1$.

If $u \in P$ with $\|u\| = H_1$, then, by (1.1), we have

$$\begin{aligned}
&\|Au\| = \sup_{t\in[0,T]_{\mathbb{T}}} |Au| = Au(T)\\
&= B_0 \left(\sum_{i=1}^{m-2} a_i\varphi_q \left(\int_{\xi_i}^T h(s)f(u)\nabla s\right)\right) + \int_0^T \varphi_q \left(\int_\tau^T h(s)f(u)\nabla s\right) \Delta\tau\\
&\le B \sum_{i=1}^{m-2} a_i\varphi_q \left(\int_0^T h(s)f(u)\nabla s\right) + T\varphi_q \left(\int_0^T h(s)f(u)\nabla s\right)\\
&= \left(B \sum_{i=1}^{m-2} a_i + T\right) \varphi_q \left(\int_0^T h(s)f(u)\nabla s\right)\\
&\le \varepsilon \|u\| \left(B \sum_{i=1}^{m-2} a_i + T\right) \varphi_q \left(\int_0^T h(s)\Delta s\right) \le \|u\| .
\end{aligned}$$

Suppose $\Omega_{H_1} = \{u \in E : \|u\| < H_1\}$, then $\|Au\| \le \|u\|$ for $u \in P \cap \partial\Omega_{H_1}$.

From $f_\infty = \infty$, there exists an $H_2' > 0$ such that $f(u) \ge \varphi_p(k)\varphi_p(u) = \varphi_p(ku)$ for $u \ge H_2'$, where $k > 0$, and satisfies the following inequality

$$\frac{k\eta}{T} \left(A' \sum_{i=1}^{m-2} a_i\varphi_q \left(\int_\eta^T h(s)\Delta s\right) + \int_\eta^T \varphi_q \left(\int_\tau^T h(s)\Delta s\right) \nabla\tau\right) \ge 1. \tag{3.1}$$

Set

$$H_2 = \max\left\{2H_1, \tfrac{T}{\eta}H_2'\right\} \text{ and } \Omega_{H_2} = \{u \in E : \|u\| < H_2\} .$$

If $u \in P$ with $\|u\| = H_2$, then, by Lemma 2.2, one has

$$\min_{t\in[\eta,T]_{\mathbb{T}}} u = u(\eta) \ge \tfrac{\eta}{T} \|u\| \ge H_2'. \tag{3.2}$$

For $u \in P \cap \partial\Omega_{H_2}$, in terms of (1.1), (3.1) and (3.2), we get

$$\begin{aligned}
&\|Au\| = \sup_{t\in[0,T]_{\mathbb{T}}} |Au| = Au(T)\\
&\ge A' \sum_{i=1}^{m-2} a_i\varphi_q \left(\int_{\xi_i}^T h(s)f(u)\nabla s\right) + \int_0^T \varphi_q \left(\int_\tau^T h(s)f(u)\nabla s\right) \Delta\tau\\
&> A' \sum_{i=1}^{m-2} a_i\varphi_q \left(\int_\eta^T h(s)\varphi_p(ku)\Delta s\right) + \int_\eta^T \varphi_q \left(\int_\tau^T h(s)\varphi_p(ku)\Delta s\right) \Delta\tau\\
&\ge \tfrac{k\eta}{T} \|u\| \left(A' \sum_{i=1}^{m-2} a_i\varphi_q \left(\int_\eta^T h(s)\Delta s\right) + \int_\eta^T \varphi_q \left(\int_\tau^T h(s)\Delta s\right) \nabla\tau\right) \ge \|u\| .
\end{aligned} \tag{3.3}$$

Thus, by (i) of Lemma 2.5, the problem (1.2) and (1.7) has at least a single positive solution u in $P\cap(\overline{\Omega}_{H_2}\backslash\Omega_{H_1})$ with $H_1 \leq \|u\| \leq H_2$.

Case (ii) $f_0=\infty$ and $f_\infty=0$.

Since $f_0=\infty$, there exists an $H_3>0$ such that $f(u)\geq\varphi_p(m)\varphi_p(u)=\varphi_p(mu)$ for $0<u\leq H_3$, where m is such that

$$\frac{m\eta}{T}\left(A'\sum_{i=1}^{m-2}a_i\varphi_q\left(\int_\eta^T h(s)\Delta s\right)+\int_\eta^T\varphi_q\left(\int_\tau^T h(s)\Delta s\right)\Delta\tau\right)\geq 1. \tag{3.4}$$

If $u\in P$ with $\|u\|=H_3$, then, by (3.2) and (3.4), one has

$$\begin{aligned}
&\|Au\|=\sup_{t\in[0,T]_{\mathbb{T}}}|Au|=Au(T)\\
&\geq A'\sum_{i=1}^{m-2}a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right)+\int_0^T\varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau\\
&> A'\sum_{i=1}^{m-2}a_i\varphi_q\left(\int_\eta^T h(s)\varphi_p(mu)\Delta s\right)+\int_\eta^T\varphi_q\left(\int_\tau^T h(s)\varphi_p(mu)\Delta s\right)\Delta\tau\\
&\geq\frac{m\eta}{T}\|u\|\left(A'\sum_{i=1}^{m-2}a_i\varphi_q\left(\int_\eta^T h(s)\Delta s\right)+\int_\eta^T\varphi_q\left(\int_\tau^T h(s)\Delta s\right)\Delta\tau\right)\geq\|u\|.
\end{aligned}\tag{3.5}$$

If we let $\Omega_{H_3}=\{u\in E:\|u\|<H_3\}$, then $\|Au\|\geq\|u\|$ for $u\in P\cap\partial\Omega_{H_3}$.

Now, we consider $f_\infty=0$. By definition, there exists $H_4'>0$ such that

$$f(u)\leq\varphi_p(\delta)\varphi_p(u)=\varphi_p(\delta u)\text{ for }u\geq H_4', \tag{3.6}$$

where $\delta>0$ satisfies

$$\delta\left(B\sum_{i=1}^{m-2}a_i+T\right)\varphi_q\left(\int_0^T h(s)\Delta s\right)\leq 1. \tag{3.7}$$

Suppose that f is bounded, then $f(u)\leq\varphi_p(K)$ for all $u\in[0,\infty)$ and some constant $K>0$. Pick

$$H_4=\max\left\{2H_3,K\left(B\sum_{i=1}^{m-2}a_i+T\right)\varphi_q\left(\int_0^T h(s)\Delta s\right)\right\}.$$

If $u\in P$ with $\|u\|=H_4$, then

$$\begin{aligned}
&\|Au\|=\sup_{t\in[0,T]_{\mathbb{T}}}|Au|=Au(T)\\
&\leq B\sum_{i=1}^{m-2}a_i\varphi_q\left(\int_0^T h(s)\varphi_p(K)\Delta s\right)+\int_0^T\varphi_q\left(\int_\tau^T h(s)\varphi_p(K)\nabla s\right)\Delta\tau\\
&\leq K\left(B\sum_{i=1}^{m-2}a_i+T\right)\varphi_q\left(\int_0^T h(s)\Delta s\right)\leq H_4=\|u\|.
\end{aligned}$$

Suppose that f is unbounded. From $f\circ u\in C_{rd}([0,T]_{\mathbb{T}},[0,+\infty))$, we have $f(u)\leq C_3$ for $u\in[0,C_4]$, here C_3 and C_4 are arbitrary positive constants. This implies that $f(u)\to+\infty$ if $u\to+\infty$. Hence, it is easy to know that there exists $H_4\geq\max\{2H_3,H_4'\}$ such that $f(u)\leq f(H_4)$ for $u\in[0,H_4]$. If $u\in P$ with $\|u\|=H_4$, then by using (3.6) and (3.7), we have

$$\begin{aligned}
&\|Au\|=Au(T)\\
&\leq B\sum_{i=1}^{m-2}a_i\varphi_q\left(\int_0^T h(s)f(u)\nabla s\right)+\int_0^T\varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau\\
&\leq\left(B\sum_{i=1}^{m-2}a_i+T\right)\varphi_q\left(\int_0^T h(s)f\left(H_4\right)\Delta s\right)\\
&\leq\left(B\sum_{i=1}^{m-2}a_i+T\right)\varphi_q\left(\int_0^T h(s)\varphi_p(\delta H_4)\Delta s\right)\\
&\leq\delta H_4\left(B\sum_{i=1}^{m-2}a_i+T\right)\varphi_q\left(\int_0^T h(s)\Delta s\right)\leq H_4=\|u\|.
\end{aligned}$$

Consequently, in either case, if we take $\Omega_{H_4} = \{u \in E : \|u\| < H_4\}$, then, for $u \in P \cap \partial\Omega_{H_4}$, we have $\|Au\| \le \|u\|$. Thus, the condition (ii) of Lemma 2.5 is satisfied. Consequently, the problem (1.2) and (1.7) has at least a single positive solution u in $P \cap (\overline{\Omega}_{H_4} \backslash \Omega_{H_3})$ with $H_3 \le \|u\| \le H_4$. The proof is complete. □

For the problem (1.2) and (1.8), we have the following result.

Theorem 3.2 *Problem* (1.2) *and* (1.8) *has at least one positive solution in the case* $i_0 = 1$ *and* $i_\infty = 1$.

Proof. According to the operator $A_1 : P_1 \to E$ defined by (2.5), we can prove it by using the similar way of proving Theorem 3.1. □

3.2. For the case $i_0 = 0$ and $i_\infty = 0$

In this subsection, we discuss the existence for the positive solutions of problems (1.2) and (1.7) or (1.8) under $i_0 = 0$ and $i_\infty = 0$.

First, we shall state and prove the following main result of problem (1.2) and (1.7).

Theorem 3.3 *Suppose that the following conditions hold:*

(i) *there exists constant* $p' > 0$ *such that* $f(u) \le \varphi_p(p'\Lambda_1)$ *for* $0 \le u \le p'$, *where*

$$\Lambda_1 = \left(\left(B\sum_{i=1}^{m-2} a_i + T\right)\varphi_q\left(\int_0^T h(s)\nabla s\right)\right)^{-1};$$

(ii) *there exists constant* $q' > 0$ *such that* $f(u) \ge \varphi_p(q'\Lambda_2)$ *for* $u \in \left[\frac{\eta}{T}q', q'\right]$, *where*

$$\Lambda_2 = \left(\left(A'\sum_{i=1}^{m-2} a_i + T - \eta\right)\varphi_q\left(\int_\eta^T h(s)\nabla s\right)\right)^{-1},$$

furthermore, $p' \ne q'$.

Then problem (1.2) *and* (1.7) *has at least one positive solution* u *such that* $\|u\|$ *lies between* p' *and* q'.

Proof. Without loss of generality, we may assume that $p' < q'$.

Let $\Omega_{p'} = \{u \in E : \|u\| < p'\}$. For any $u \in P \cap \partial\Omega_{p'}$, in view of condition (i) , we have

$$\begin{aligned} \|Au\| &= \sup_{t\in[0,T]_{\mathbb{T}}} |Au| = Au(T) \\ &\le B\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right) + \int_0^T \varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau \\ &\le \left(B\sum_{i=1}^{m-2} a_i + T\right)\varphi_q\left(\int_0^T h(s)\varphi_p(p'\Lambda_1)\Delta s\right) \\ &= p'\Lambda_1\left(B\sum_{i=1}^{m-2} a_i + T\right)\varphi_q\left(\int_0^T h(s)\Delta s\right) = p', \end{aligned} \tag{3.8}$$

which yields

$$\|Au\| \le \|u\| \text{ for } u \in P \cap \partial\Omega_{p'}. \tag{3.9}$$

Now, set $\Omega_{q'} = \{u \in E : \|u\| < q'\}$. For $u \in P \cap \partial\Omega_{q'}$, Lemma 2.2 implies that $\frac{\eta}{T}q' \le u \le q'$ for $t \in [\eta, T]_{\mathbb{T}}$. Hence, by condition (ii) we get

$$\begin{aligned}
&\|Au\| = \sup_{t\in[0,T]_{\mathbb{T}}} |Au| = Au(T)\\
&\ge A' \textstyle\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right) + \int_0^T \varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau\\
&> A' \textstyle\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\eta}^T h(s)f(u)\nabla s\right) + \int_\eta^T \varphi_q\left(\int_\eta^T h(s)f(u)\nabla s\right)\Delta\tau\\
&> q'\Lambda_2\left(A' \textstyle\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\eta}^T h(s)\nabla s\right) + (T-\eta)\varphi_q\left(\int_\eta^T h(s)\nabla s\right)\right) = q'.
\end{aligned}$$

So, if we take $\Omega_{q'} = \{u \in E : \|u\| < q'\}$, then

$$\|Au\| \ge \|u\|,\ u \in P \cap \partial\Omega_{q'}. \tag{3.10}$$

Consequently, in view of $p' < q'$, (3.9) and (3.10), it follows from Lemma 2.5 that problem (1.2) and (1.7) has a positive solution u in $P \cap (\overline{\Omega}_{q'} \backslash \Omega_{p'})$. The proof is complete. □

Now, in terms of the operator $A_1 : P_1 \to E$ defined by (2.5) and the method similar to proving Theorem 3.3, we consider the problem (1.2) and (1.8) and have the following result.

Theorem 3.4 *Suppose that the following conditions hold:*

(i) *there exists constant $p_1' > 0$ such that $f(u) \le \varphi_p(p_1'\Lambda_1')$ for $0 \le u \le p_1'$, where*

$$\Lambda_1' = \left(\left(B \textstyle\sum_{i=1}^{m-2} b_i + T\right)\varphi_q\left(\int_0^T h(s)\Delta s\right)\right)^{-1};$$

(ii) *there exists constant $q_1' > 0$ such that $f(u) \ge \varphi_p(q_1'\Lambda_2')$ for $u \in \left[\frac{T-\xi}{T}q_1', q_1'\right]$, where*

$$\Lambda_2' = \left(\left(A' \textstyle\sum_{i=1}^{m-2} b_i + T - \xi\right)\varphi_q\left(\int_0^\xi h(s)\nabla s\right)\right)^{-1},$$

furthermore, $p_1' \ne q_1'$.

Then problem (1.2) *and* (1.8) *has at least one positive solution u such that $\|u\|$ lies between p_1' and q_1'.*

3.3. For the case $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$

In this subsection, under the conditions $i_0 = 1$ and $i_\infty = 0$ or $i_0 = 0$ and $i_\infty = 1$, we discuss the existence of positive solutions of problem (1.2) and (1.7) or (1.8).

First, we consider the existence on positive solutions of problem (1.2) and (1.7).

Theorem 3.5 *Suppose that $f_0 \in [0, \varphi_p(\Lambda_1))$ and $f_\infty \in \left(\varphi_p\left(\frac{T}{\eta}\Lambda_2\right), \infty\right)$ hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Proof. It is easy to see that under the assumptions, the conditions (i) and (ii) in Theorem 3.3 are satisfied. So the proof is easy and we omit it here. □

Theorem 3.6 *Suppose that* $f_0 \in \left(\varphi_p\left(\frac{T}{\eta}\Lambda_2\right), \infty\right)$ *and* $f_\infty \in [0, \varphi_p(\Lambda_1))$ *hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Proof. Firstly, let $\varepsilon_1 = f_0 - \varphi_p\left(\frac{T}{\eta}\Lambda_2\right) > 0$, there exists a sufficiently small positive number q', which satisfies

$$\frac{f(u)}{\varphi_p(u)} \geq f_0 - \varepsilon_1 = \varphi_p\left(\frac{T}{\eta}\Lambda_2\right) \text{ for } u \in (0, q'].$$

Thus, if $u \in \left[\frac{\eta}{T}q', q'\right]$, then we have

$$f(u) \geq \varphi_p\left(\frac{T}{\eta}\Lambda_2\right)\varphi_p(u) \geq \varphi_p(\Lambda_2 q'),$$

which implies that the condition (ii) in Theorem 3.3 holds.

Next, for $\varepsilon_2 = \varphi_p(\Lambda_1) - f_\infty > 0$, there exists a sufficiently large $p''(> q')$, which satisfies

$$\frac{f(u)}{\varphi_p(u)} \leq f_\infty + \varepsilon_2 = \varphi_p(\Lambda_1) \text{ for } u \in [p'', \infty). \tag{3.11}$$

We consider two cases:

Case (i) Assume that f is bounded, that is, $f(u) \leq \varphi_p(K_1)$ for $u \in [0, \infty)$ and some constant $K_1 > 0$. If we take sufficiently large p' such that $p' \geq \max\{K_1/\Lambda_1, p''\}$, then

$$f(u) \leq \varphi_p(K_1) \leq \varphi_p(\Lambda_1 p') \text{ for } u \in [0, p'].$$

Consequently, from the above inequality, the condition (i) of Theorem 3.3 is true.

Case (ii) Assume that f is unbounded.

From $f \circ u \in C_{rd}([0,T]_{\mathbb{T}}, [0,\infty))$, we have $p' > p''$ such that $f(u) \leq f(p')$ for $u \in [0, p']$. Since $p' > p''$, by (3.11), we get $f(p') \leq \varphi_p(\Lambda_1 p')$, hence

$$f(u) \leq f(p') \leq \varphi_p(\Lambda_1 p') \text{ for } u \in [0, p'].$$

Thus, the condition (i) of Theorem 3.3 is fulfilled. The proof is complete. $\square$

From Theorems 3.5 and 3.6, we have the following two results.

Corollary 3.7 *Suppose that* $f_0 = 0$ *and the condition* (ii) *in Theorem* 3.3 *hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Corollary 3.8 *Suppose that* $f_\infty = 0$ *and the condition* (ii) *in Theorem* 3.3 *hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Theorem 3.9 *Suppose that* $f_0 \in (0, \varphi_p(\Lambda_1))$ *and* $f_\infty = \infty$ *hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Proof. First, in view of $f_\infty = \infty$, then by inequality (3.3), we have $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_{H_2}$. Next, by $f_0 \in (0, \varphi_p(\Lambda_1))$, for $\varepsilon_3 = \varphi_p(\Lambda_1) - f_0 > 0$, there exists a sufficiently small $p' \in (0, H_2)$ such that

$$f(u) \leq (f_0 + \varepsilon_3)\varphi_p(u) = \varphi_p(\Lambda_1 u) \leq \varphi_p(\Lambda_1 p') \text{ for } u \in [0, p'],$$

which implies (i) of Theorem 3.3 holds, that is, (3.8) is true, hence, we obtain $\|Au\| \leq \|u\|$ for $u \in P \cap \partial\Omega_{p'}$. The result is obtained and the proof is complete. □

Theorem 3.10 *Suppose that* $f_0 = \infty$ *and* $f_\infty \in (0, \varphi_p(\Lambda_1))$ *hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Proof. On the one hand, since $f_0 = \infty$, by the inequality (3.5), one gets $\|Au\| \geq \|u\|$, $u \in P \cap \partial\Omega_{H_3}$. On the other hand, since $f_\infty \in (0, \varphi_p(\Lambda_1))$, from the technique similar to the second part of the proof in Theorem 3.6, one obtains that the condition (i) of Theorem 3.3 is satisfied, that is, inequality (3.8) holds, one has $\|Au\| \leq \|u\|$, $u \in P \cap \partial\Omega_{p'}$, where $p' > H_3$. Hence, problem (1.2) and (1.7) has at least one positive solution, the proof is complete. □

From Theorems 3.9 and 3.10, respectively, it is easy to obtain the following two corollaries.

Corollary 3.11 *Assume that* $f_\infty = \infty$ *and the condition* (i) *in Theorem* 3.3 *hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Corollary 3.12 *Assume that* $f_0 = \infty$ *and the condition* (i) *in Theorem* 3.3 *hold. Then problem* (1.2) *and* (1.7) *has at least one positive solution.*

Now, in view of the operator $A_1 : P_1 \to E$ defined by (2.5), we consider the problem (1.2) and (1.8) and obtain the following results. the methods are similar to those of proving theorems 3.3, 3.4 by a slight modifications and due to the limited space we omit the proof. In the following, Λ_1' and Λ_2' defined as in Theorem 3.4.

Theorem 3.13 *Suppose that* $f_0 \in [0, \varphi_p(\Lambda_1'))$ *and* $f_\infty \in \left(\varphi_p\left(\frac{T}{T-\xi}\Lambda_2'\right), \infty\right)$ *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

Theorem 3.14 *Suppose that* $f_0 \in \left(\varphi_p\left(\frac{T}{T-\xi}\Lambda_2'\right), \infty\right)$ *and* $f_\infty \in [0, \varphi_p(\Lambda_1'))$ *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

Corollary 3.15 *Suppose that* $f_0 = 0$ *and the condition* (ii) *in Theorem* 3.4 *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

Corollary 3.16 *Suppose that* $f_\infty = 0$ *and the condition* (ii) *in Theorem* 3.4 *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

Theorem 3.17 *Suppose that* $f_0 \in (0, \varphi_p(\Lambda_1'))$ *and* $f_\infty = \infty$ *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

Theorem 3.18 *Suppose that* $f_0 = \infty$ *and* $f_\infty \in (0, \varphi_p(\Lambda_1'))$ *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

Corollary 3.19 *Assume that* $f_\infty = \infty$ *and the condition* (i) *in Theorem* 3.4 *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

Corollary 3.20 *Assume that* $f_0 = \infty$ *and the condition* (i) *in Theorem* 3.4 *hold. Then problem* (1.2) *and* (1.8) *has at least one positive solution.*

3.4. For the case $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$

In this subsection, under $i_0 = 0$ and $i_\infty = 2$ or $i_0 = 2$ and $i_\infty = 0$, we study the existence of multiple positive solutions for problems (1.2) and (1.7) or (1.8).

We first consider the problem (1.2) and (1.7). By combining the proof of Theorems 3.1 and 3.3, it is easy to prove the following two theorems.

Theorem 3.21 *Suppose that* $i_0 = 0$ *and* $i_\infty = 2$ *and the condition* (i) *of Theorem* 3.3 *hold, then problem* (1.2) *and* (1.7) *has at least two positive solutions* $u_1,\ u_2 \in P$ *such that* $0 < \|u_1\| < p' < \|u_2\|$.

Theorem 3.22 *Suppose that* $i_0 = 2$ *and* $i_\infty = 0$ *and the condition* (ii) *of Theorem* 3.3 *hold, then problem* (1.2) *and* (1.7) *has at least two positive solutions* $u_1,\ u_2 \in P$ *such that* $0 < \|u_1\| < q' < \|u_2\|$.

Now, we consider the existence of solution to problem (1.2) and (1.8). According to the completely continuous operator A_1, we can get the following results by using the same reasoning as the proof of Theorems 3.21, 3.22, respectively.

Theorem 3.23 *Suppose that* $i_0 = 0$ *and* $i_\infty = 2$ *and the condition* (i) *of Theorem* 3.4 *hold, then problem* (1.2) *and* (1.8) *has at least two positive solutions* $u_1,\ u_2 \in P$ *such that* $0 < \|u_1\| < p_1' < \|u_2\|$.

Theorem 3.24 *Suppose that* $i_0 = 2$ *and* $i_\infty = 0$ *and the condition* (ii) *of Theorem* 3.4 *hold, then problem* (1.2) *and* (1.8) *has at least two positive solutions* $u_1,\ u_2 \in P$ *such that* $0 < \|u_1\| < q_1' < \|u_2\|$.

4. Triple solutions

In the previous section, we have obtained some results for existence of positive solutions of problems (1.2) and (1.7) or (1.8). In this section, we will further discuss the existence of positive solutions of problems (1.2) and (1.7) or (1.8) by using two different methods. We will give a comparison of this two different methods in the latter part of this section.

For the notational convenience, we denote

$$
\begin{aligned}
&M_\xi=\left(B\sum_{i=1}^{m-2}a_i+\eta\right)\varphi_q\left(\int_0^T h(s)\nabla s\right),N_\xi=\left(A'\sum_{i=1}^{m-2}a_i+\eta\right)\varphi_q\left(\int_\eta^T h(s)\nabla s\right),\\
&L_\xi=\left(B\sum_{i=1}^{m-2}a_i+r\right)\varphi_q\left(\int_0^T h(s)\nabla s\right),M'_\xi=\left(B\sum_{i=1}^{m-2}b_i+T-\xi\right)\varphi_q\left(\int_0^T h(s)\nabla s\right),\\
&W_\xi=\left(B\sum_{i=1}^{m-2}a_i+T\right)\varphi_q\left(\int_0^T h(s)\nabla s\right),N'_\xi=\left(A'\sum_{i=1}^{m-2}b_i+T-\xi\right)\varphi_q\left(\int_0^\xi h(s)\nabla s\right),\\
&L'_\xi=\left(B\sum_{i=1}^{m-2}b_i+T-l\right)\varphi_q\left(\int_0^T h(s)\nabla s\right),Q_\xi=\left(B\sum_{i=1}^{m-2}b_i+T\right)\varphi_q\left(\int_0^T h(s)\nabla s\right).
\end{aligned}
$$

4.1. The generalized Avery and Henderson fixed-point theorem

In this subsection, in view of the generalized Avery and Henderson fixed-point theorem due to Ren et al. [28], the existence criteria for at least *three* positive and arbitrary odd positive solutions of problems (1.2) and (1.7) or (1.8) are established.

For $u\in P$, we define the nonnegative, increasing, continuous functionals γ_2, β_2 and α_2 by

$$\gamma_2(u)=\max_{t\in[0,\eta]_\mathbb{T}}u=u(\eta),\ \beta_2(u)=\min_{t\in[\eta,T]_\mathbb{T}}u=u(\eta),\ \alpha_2(u)=\max_{t\in[0,r]_\mathbb{T}}u=u(r).$$

It is obvious that $\gamma_2(u)=\beta_2(u)\leq\alpha_2(u)$ for each $u\in P$. By Lemma 2.2, one obtains $\|u\|\leq\frac{T}{\eta}u(\eta)=\frac{T}{\eta}\gamma_2(u)$ for all $u\in P$.

We now present the results in this subsection.

Theorem 4.1 *Suppose that there are positive numbers a',b',c' such that $a'<\frac{r}{T}b'<\frac{rN_\xi}{TM_\xi}c'$. In addition, $f(u)$ satisfies the following conditions:*

(i) $f(u)<\varphi_p\left(\frac{c'}{M_\xi}\right)$ *for* $0\leq u\leq\frac{T}{\eta}c'$;

(ii) $f(u)>\varphi_p\left(\frac{b'}{N_\xi}\right)$ *for* $b'\leq u\leq\frac{T}{\eta}b'$;

(iii) $f(u)<\varphi_p\left(\frac{a'}{L_\xi}\right)$ *for* $0\leq u\leq\frac{T}{r}a'$.

Then problem (1.2) *and* (1.7) *has at least three positive solutions u_1, u_2 and u_3 such that*

$$0<\max_{t\in[0,r]_\mathbb{T}}u_1<a'<\max_{t\in[0,r]_\mathbb{T}}u_2 \text{ with}$$
$$\min_{t\in[\eta,T]_\mathbb{T}}u_2<b'<\min_{t\in[\eta,T]_\mathbb{T}}u_3 \text{ and } \max_{t\in[0,\eta]_\mathbb{T}}u_3<c'.$$

Proof. By the definition of completely continuous operator A and its properties, it suffices to show that all the conditions of Lemma 2.6 hold with respect to A. It is easy to obtain that $A:\overline{P(\gamma_2,c)}\to P$.

First, we verify that if $u\in\partial P(\gamma_2,c')$, then $\gamma_2(Au)<c'$.

If $u\in\partial P(\gamma_2,c')$, then $\gamma_2(u)=\max_{t\in[0,\eta]_\mathbb{T}}u=u(\eta)=c'$. Lemma 2.2 implies that $\|u\|\leq\frac{T}{\eta}u(\eta)=\frac{T}{\eta}c'$, we have $0\leq u\leq\frac{T}{\eta}c'$, $t\in[0,T]_\mathbb{T}$.

Thus, by the condition (i), one has

$$
\begin{aligned}
&\gamma_2(Au)=Au(\eta)\\
&\leq B\sum_{i=1}^{m-2}a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right)+\int_0^\eta\varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau\\
&\leq\left(B\sum_{i=1}^{m-2}a_i+\eta\right)\varphi_q\left(\int_0^T h(s)f(u)\nabla s\right)<\frac{c'}{M_\xi}\left(B\sum_{i=1}^{m-2}a_i+\eta\right)\varphi_q\left(\int_0^T h(s)\nabla s\right)=c'.
\end{aligned}
$$

Second, we show that $\beta_2(Au) > b'$ for $u \in \partial P(\beta_2, b')$.

If we choose $u \in \partial P(\beta_2, b')$, then $\beta_2(u) = \min_{t\in[\eta,T]_{\mathbb{T}}} u = u(\eta) = b'$. In view of Lemma 2.2, we have $\|u\| \leq \frac{T}{\eta}u(\eta) = \frac{T}{\eta}b'$. So $b' \leq u \leq \frac{T}{\eta}b'$, $t \in [\eta, T]_{\mathbb{T}}$. Using the condition (ii), we get

$$\begin{aligned}&\beta_2(Au) = (Au)(\eta) \geq \left(A'\textstyle\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_\eta^T h(s)f(u)\nabla s\right)\\&> \left(A'\textstyle\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_\eta^T h(s)\varphi_p\left(\frac{b'}{N_\xi}\right)\nabla s\right)\\&= \frac{b'}{N_\xi}\left(A'\textstyle\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_\eta^T h(s)\nabla s\right) = b'.\end{aligned}$$

Finally, we prove that $P(\alpha_2, a') \neq \emptyset$ and $\alpha_2(Au) < a'$ for all $u \in \partial P(\alpha_2, a')$.

In fact, the constant function $\frac{a'}{2} \in P(\alpha_2, a')$. Moreover, for $u \in \partial P(\alpha_2, a')$, we have $\alpha_2(u) = \max_{t\in[0,r]_{\mathbb{T}}} u = u(r) = a'$, which implies $0 \leq u \leq a'$ for $t \in [0, r]_{\mathbb{T}}$. In view of Lemma 2.2, we have $u \leq \|u\| \leq \frac{T}{r}u(r) = \frac{T}{r}r'$. Hence $0 \leq u \leq \frac{T}{r}a'$, $t \in [0, T]_{\mathbb{T}}$. By using assumption (iii), one has

$$\begin{aligned}&\alpha_2(Au) = (Au)(r)\\&< \left(B\textstyle\sum_{i=1}^{m-2} a_i + r\right)\varphi_q\left(\int_0^T h(s)\varphi_p\left(\frac{a'}{L_\xi}\right)\nabla s\right)\\&= \frac{a'}{L_\xi}\left(B\textstyle\sum_{i=1}^{m-2} a_i + r\right)\varphi_q\left(\int_0^T h(s)\nabla s\right) = a'.\end{aligned}$$

Thus, all the conditions in Lemma 2.6 are satisfied. From (S1) and (S2), we have, the solutions of problem (1.2) and (1.7) does not vanish identically on any closed subinterval of $[0, T]_{\mathbb{T}}$. Consequently, problem (1.2) and (1.7) has at least three positive solutions u_1, u_2 and u_3 belonging to $\overline{P(\gamma_2, c')}$, and satisfying

$$\begin{gathered}0 < \max_{t\in[0,r]_{\mathbb{T}}} u_1 < a' < \max_{t\in[0,r]_{\mathbb{T}}} u_2 \text{ with}\\ \min_{t\in[\eta,T]_{\mathbb{T}}} u_2 < b' < \min_{t\in[\eta,T]_{\mathbb{T}}} u_3 \text{ and } \max_{t\in[0,\eta]_{\mathbb{T}}} u_3 < c'.\end{gathered}$$

The proof is complete. □

From Theorem 4.1, we see that, when assumptions (i), (ii) and (iii) are imposed appropriately on f, we can prove the existence of an arbitrary odd number of positive solutions for the problem (1.2) and (1.7).

Theorem 4.2 *Suppose that there are positive numbers* $a'_{s_i}, b'_{s_i}, c'_{s_i}$ *such that*

$$a'_{s_1} < \frac{r}{T}b'_{s_1} < \frac{rN_\xi}{TM_\xi}c'_{s_1} < a'_{s_2} < \frac{r}{T}b'_{s_2} < \frac{rN_\xi}{TM_\xi}c'_{s_2} < a'_{s_3} < \ldots < a'_{s_n}, \ n \in \mathbb{N},$$

here $i = 1, 2, \ldots, n$. *In addition,* $f(u)$ *satisfies the following conditions:*

(i) $f(u) < \varphi_p\left(\frac{c'_{s_i}}{M_\xi}\right)$ *for* $0 \leq u \leq \frac{T}{\eta}c'_{s_i}$;

(ii) $f(u) > \varphi_p\left(\frac{b'_{s_i}}{N_\xi}\right)$ *for* $b'_{s_i} \leq u \leq \frac{T}{\eta}b'_{s_i}$;

(iii) $f(u) < \varphi_p\left(\frac{a'_{s_i}}{L_\xi}\right)$ *for* $0 \leq u \leq \frac{T}{r}a'_{s_i}$.

Then problem (1.2) *and* (1.7) *has at least* $2n + 1$ *positive solutions.*

Proof. When $i=1$, it is clear that Theorem 4.1 holds. Then we can obtain at least three positive solutions u_1, u_2 and u_3 satisfying

$$0 \le \max_{t\in[0,r]_{\mathbb{T}}} u_1 < a'_{s_1} < \max_{t\in[0,r]_{\mathbb{T}}} u_2 \text{ with}$$
$$\min_{t\in[\eta,T]_{\mathbb{T}}} u_2 < b'_{s_1} < \min_{t\in[\eta,T]_{\mathbb{T}}} u_3 \text{ and } \max_{t\in[0,\eta]_{\mathbb{T}}} u_3 < c'_{s_1}.$$

Following this way, we finish the proof by induction. The proof is complete. $\square$

Denote $L_\theta = \left(A'\sum_{i=1}^{m-2} a_i + r\right)\varphi_q\left(\int_r^T h(s)\nabla s\right)$. Using Lemma 2.7, it is easy to have the following results.

Theorem 4.3 *Suppose that there are positive numbers a', b', c' such that $a' < \frac{L_\theta}{M_\xi}b' < \frac{\eta L_\theta}{TM_\xi}c'$. In addition, $f(u)$ satisfies the following conditions:*

(i) $f(u) > \varphi_p\left(\frac{c'}{N_\xi}\right)$ *for* $c' \le u \le \frac{T}{\eta}c'$;

(ii) $f(u) < \varphi_p\left(\frac{b'}{M_\xi}\right)$ *for* $0 \le u \le \frac{T}{\eta}b'$;

(iii) $f(u) > \varphi_p\left(\frac{a'}{L_\theta}\right)$ *for* $a' \le u \le \frac{T}{r}a'$.

Then problem (1.2) *and* (1.7) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$0 < \max_{t\in[0,r]_{\mathbb{T}}} u_1 < a' < \max_{t\in[0,r]_{\mathbb{T}}} u_2 \textit{ with}$$
$$\mathit{min}_{t\in[\eta,T]_{\mathbb{T}}} u_2 < b' < \min_{t\in[\eta,T]_{\mathbb{T}}} u_3 \textit{ and } \max_{t\in[0,\eta]_{\mathbb{T}}} u_3 < c'.$$

From Theorem 4.3, we can obtain Theorem 4.4 and Corollary 4.5.

Theorem 4.4 *Suppose that there are positive numbers $a'_{\lambda_i}, b'_{\lambda_i}, c'_{\lambda_i}$ such that*

$$a'_{\lambda_1} < \frac{L_\theta}{M_\xi}b'_{\lambda_1} < \frac{\eta L_\theta}{TN_\xi}c'_{\lambda_1} < a'_{\lambda_2} < \frac{L_\theta}{M_\xi}b'_{\lambda_2} < \frac{\eta L_\theta}{TN_\xi}c'_{\lambda_2} < a'_{\lambda_3} < \ldots < a'_{\lambda_n},\ n\in\mathbb{N},$$

here $i = 1, 2, \ldots, n$. In addition, $f(u)$ satisfies the following conditions:

(i) $f(u) > \varphi_p\left(\frac{c'_{\lambda_i}}{N_\xi}\right)$ *for* $c'_{\lambda_i} \le u \le \frac{T}{\eta}c'_{\lambda_i}$;

(ii) $f(u) < \varphi_p\left(\frac{b'_{\lambda_i}}{M_\xi}\right)$ *for* $0 \le u \le \frac{T}{\eta}b'_{\lambda_i}$;

(iii) $f(u) > \varphi_p\left(\frac{a'_{\lambda_i}}{L_\theta}\right)$ *for* $a'_{\lambda_i} \le u \le \frac{T}{r}a'_{\lambda_i}$.

Then problem (1.2) *and* (1.7) *has at least* $2n+1$ *positive solutions.*

Corollary 4.5 *Assume that f satisfies conditions*

(i) $f_0 = \infty$, $f_\infty = \infty$;

(ii) *there exists* $c_0 > 0$ *such that* $f(u) < \varphi_p\left(\frac{\eta}{M_\xi T}c_0\right)$ *for* $0 \le u \le c_0$.

Then problem (1.2) *and* (1.7) *has at least three positive solutions.*

Proof. First, by the condition (ii), let $b' = \frac{\eta}{T}c_0$, one gets

$$f(u) < \varphi_p\left(\frac{b'}{M_\xi}\right) \text{ for } 0 \le u \le \frac{T}{\eta}b',$$

which implies that (ii) of Theorem 4.3 holds.

Second, choose K_3 sufficiently large to satisfy

$$K_3 L_\theta = K_3\left(A'\sum_{i=1}^{m-2} a_i + r\right)\varphi_q\left(\int_r^T h(s)\nabla s\right) > 1. \tag{4.1}$$

Since $f_0 = \infty$, there exists $r_1 > 0$ sufficiently small such that,

$$f(u) \ge \varphi_p(K_3)\varphi_p(u) = \varphi_p(K_3 u) \text{ for } 0 \le u \le r_1. \tag{4.2}$$

Without loss of generality, suppose $r_1 \le \frac{L_\theta T}{M_\xi r}b'$. Choose $a' > 0$ such that $a' < \frac{r}{T}r_1$. For $a' \le u \le \frac{T}{r}a'$, we have $u \le r_1$ and $a' < \frac{L_\theta}{M_\xi}b'$. Thus, by (4.1) and (4.2), we have

$$f(u) \ge \varphi_p(K_3 u) \ge \varphi_p(K_3 a') > \varphi_p\left(\frac{a'}{L_\theta}\right) \text{ for } a' \le u \le \frac{T}{r}a',$$

this implies that (iii) of Theorem 4.3 is true.

Third, choose K_2 sufficiently large such that

$$K_2 N_\xi = K_2\left(A'\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_\eta^T h(s)\nabla s\right) > 1.$$

Since $f_\infty = \infty$, there exists $r_2 > 0$ sufficiently large such that,

$$f(u) \ge \varphi_p(K_2)\varphi_p(u) = \varphi_p(K_2 u) \text{ for } u \ge r_2.$$

Without loss of generality, suppose $r_2 > \frac{T}{\eta}b'$. Choose $c' = r_2$. Then

$$f(u) \ge \varphi_p(K_2 u) \ge \varphi_p(K_2 c') > \varphi_p\left(\frac{c'}{N_\xi}\right) \text{ for } c' \le u \le \frac{T}{\eta}c',$$

which means that (i) of Theorem 4.3 holds.

From above analysis, we get $0 < a' < \frac{L_\theta}{M_\xi}b' < \frac{\eta L_\theta}{T M_\xi}c'$, then, all conditions in Theorem 4.3 are satisfied. Hence, problem (1.2) and (1.7) has at least three positive solutions. □

In terms of Theorem 4.1, we also have the following corollary.

Corollary 4.6 *Assume that* f *satisfies conditions*

(i) $f_0 = 0$, $f_\infty = 0$;

(ii) *there exists* $c_0 > 0$ *such that* $f(u) > \varphi_p\left(\frac{\eta}{N_\xi T}c_0\right)$ *for* $\frac{\eta}{T}c_0 \le u \le c_0$.

Then problem (1.2) *and* (1.7) *has at least three positive solutions.*

In the following, we consider the problem (1.2) and (1.8).

For $u \in P_1$, we define the nonnegative, increasing, continuous functionals γ_3, β_3 and α_3 by

$$\gamma_3(u) = \max_{t\in[\xi,T]_{\mathbb{T}}} u = u(\xi),\ \beta_3(u) = \min_{t\in[0,\xi]_{\mathbb{T}}} u = u(\xi), \alpha_3(u) = \max_{t\in[l,T]_{\mathbb{T}}} u = u(l).$$

In view of completely continuous operator A_1 defined on P_1, it is easy to obtain the following results by using the similar techniques to those of considering (1.2) and (1.7).

Theorem 4.7 *Suppose that there are positive numbers a', b', c' such that $a' < \frac{T-l}{T} b' < \frac{(T-l)N'_\xi}{TM'_\xi} c'$. In addition, $f(u)$ satisfies the following conditions:*

(i) $f(u) < \varphi_p\left(\frac{c'}{M'_\xi}\right)$ *for* $0 \le u \le \frac{T}{T-\xi} c'$;

(ii) $f(u) > \varphi_p\left(\frac{b'}{N'_\xi}\right)$ *for* $b' \le u \le \frac{T}{T-\xi} b'$;

(iii) $f(u) < \varphi_p\left(\frac{a'}{L'_\xi}\right)$ *for* $0 \le u \le \frac{T}{T-l} a'$.

Then problem (1.2) *and* (1.8) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$0 < \max_{t\in[l,T]_{\mathbb{T}}} u_1 < a' < \max_{t\in[l,T]_{\mathbb{T}}} u_2 \ \textit{with}$$
$$\min_{t\in[0,\xi]_{\mathbb{T}}} u_2 < b' < \min_{t\in[0,\xi]_{\mathbb{T}}} u_3 \ \textit{and} \ \max_{t\in[\xi,T]_{\mathbb{T}}} u_3 < c'.$$

Theorem 4.8 *Suppose that there are positive numbers $a'_{h_i}, b'_{h_i}, c'_{h_i}$ such that*

$$a'_{h_1} < \tfrac{T-l}{T} b'_{h_1} < \tfrac{(T-l)N'_\xi}{TM'_\xi} c'_{h_1} < a'_{h_2} < \tfrac{T-l}{T} b'_{h_2} < \tfrac{(T-l)N'_\xi}{TM'_\xi} c'_{h_2} < a'_{h_3} < \ldots < a'_{h_n},\ n \in \mathbb{N},$$

here $i = 1, 2, \ldots, n$. In addition, $f(u)$ satisfies the following conditions:

(i) $f(u) < \varphi_p\left(\frac{c'_{h_i}}{M'_\xi}\right)$ *for* $0 \le u \le \frac{T}{T-\xi} c'_{h_i}$;

(ii) $f(u) > \varphi_p\left(\frac{b'_{h_i}}{N'_\xi}\right)$ *for* $b'_{h_i} \le u \le \frac{T}{T-\xi} b'_{h_i}$;

(iii) $f(u) < \varphi_p\left(\frac{a'_{h_i}}{L'_\xi}\right)$ *for* $0 \le u \le \frac{T}{T-l} a'_{h_i}$;

Then problem (1.2) *and* (1.8) *has at least* $2n+1$ *positive solutions.*

Let $L'_\theta = \left(A' \sum_{i=1}^{m-2} b_i + T - l\right) \varphi_q\left(\int_0^l h(s)\nabla s\right)$, we also have the following theorem.

Theorem 4.9 *Suppose that there are positive numbers a', b', c' such that $a' < \frac{L'_\theta}{M'_\xi} b' < \frac{(T-\xi)L'_\theta}{TM'_\xi} c'$. In addition, $f(u)$ satisfies the following conditions:*

(i) $f(u) > \varphi_p\left(\frac{c'}{N'_\xi}\right)$ *for* $c' \le u \le \frac{T}{T-\xi} c'$;

(ii) $f(u) < \varphi_p\left(\frac{b'}{M'_\xi}\right)$ *for* $0 \le u \le \frac{T}{T-\xi} b'$;

(iii) $f(u) > \varphi_p\left(\frac{a'}{L'_\theta}\right)$ *for* $a' \le u \le \frac{T}{T-l}a'$.

Then problem (1.2) *and* (1.8) *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$0 < \max_{t\in[l,T]_{\mathbb{T}}} u_1 < a' < \max_{t\in[l,T]_{\mathbb{T}}} u_2 \text{ with}$$
$$\min_{t\in[0,\xi]_{\mathbb{T}}} u_2 < b' < \min_{t\in[0,\xi]_{\mathbb{T}}} u_3 \text{ and } \max_{t\in[\xi,T]_{\mathbb{T}}} u_3 < c'.$$

Theorem 4.10 *Suppose that there are positive numbers* $a'_{\theta_i}, b'_{\theta_i}, c'_{\theta_i}$ *such that*

$$a'_{\theta_1} < \frac{L'_\theta}{M'_\xi}b'_{\theta_1} < \frac{(T-\xi)L'_\theta}{TM'_\xi}c'_{\theta_1} < a'_{\theta_2} < \frac{L'_\theta}{M'_\xi}b'_{\theta_2} < \frac{(T-\xi)L'_\theta}{TM'_\xi}c'_{\theta_2} < a'_{\theta_3} < \ldots < a'_{\theta_n},\ n \in \mathbb{N},$$

here $i = 1, 2, \ldots, n$. *In addition,* $f(u)$ *satisfies the following conditions:*

(i) $f(u) > \varphi_p\left(\frac{c'_{\theta_i}}{N'_\xi}\right)$ *for* $c'_{\theta_i} \le u \le \frac{T}{T-\xi}c'_{\theta_i}$;

(ii) $f(u) < \varphi_p\left(\frac{b'_{\theta_i}}{M'_\xi}\right)$ *for* $0 \le u \le \frac{T}{T-\xi}b'_{\theta_i}$;

(iii) $f(u) > \varphi_p\left(\frac{a'_{\theta_i}}{L'_\theta}\right)$ *for* $a'_{\theta_i} \le u \le \frac{T}{T-l}a'_{\theta_i}$.

Then problem (1.2) *and* (1.8) *has at least* $2n+1$ *positive solutions.*

Corollary 4.11 *Assume that* f *satisfies conditions*

(i) $f_0 = 0,\ f_\infty = 0$;

(ii) *there exists* $c_0 > 0$ *such that*

$$f(u) > \varphi_p\left(\frac{T-\xi}{N'_\xi T}c_0\right) \text{ for } \frac{T-\xi}{T}c_0 \le u \le c_0.$$

Then problem (1.2) *and* (1.8) *has at least three positive solutions.*

Corollary 4.12 *Assume that* f *satisfies conditions*

(i) $f_0 = \infty,\ f_\infty = \infty$;

(ii) *there exists* $c_0 > 0$ *such that* $f(u) < \varphi_p\left(\frac{T-\xi}{M_\xi T}c_0\right)$ *for* $0 \le u \le c_0$.

Then problem (1.2) *and* (1.8) *has at least three positive solutions.*

4.2. The Avery-Peterson fixed point theorem

In this subsection, the existence criteria for at least *three* positive and arbitrary odd positive solutions of problems (1.2) and (1.7) or (1.8) are established by the Avery-Peterson fixed point theorem [7].

Define the nonnegative continuous convex functionals ϕ and β, nonnegative continuous concave functional λ, and nonnegative continuous functional φ respectively on P by

$$\phi(u) = \max_{t\in[0,T]_{\mathbb{T}}} u = u(T),\ \beta(u) = \max_{t\in[r,T]_{\mathbb{T}^\kappa}} |u^\Delta| = |u^\Delta(r)|,$$
$$\lambda(u) = \varphi(u) = \min_{t\in[\eta,T]_{\mathbb{T}}} u = u(\eta).$$

Now, we list and prove the results in this subsection.

Theorem 4.13 *Suppose that there exist constants* a^*, b^*, d^* *such that* $0 < a^* < \frac{\eta}{T} b^* < \frac{\eta N_\xi}{T W_\xi} d^*$. *In addition, suppose that* $W_\xi > \varphi_q\left(\int_\eta^T h(s)\nabla s\right)$ *holds,* f *satisfies the following conditions:*

(i) $f(u) \le \varphi_p\left(\frac{d^*}{W_\xi}\right),\ 0 \le u \le d^*$;

(ii) $f(u) > \varphi_p\left(\frac{b^*}{N_\xi}\right),\ b^* \le u \le d^*$;

(iii) $f(u) < \varphi_p\left(\frac{a^*}{M_\xi}\right),\ 0 \le u \le \frac{T}{\eta} a^*$.

Then problem (1.2) *and* (1.7) *has at least three positive solutions* $u_1,\ u_2,\ u_3$ *such that*

$$\|x_i\| \le d^* \text{ for } i=1,2,3,\ b^* < \min_{t\in[\eta,T]_{\mathbb{T}}} u_1,\ a^* < \min_{t\in[\eta,T]_{\mathbb{T}}} u_2 \\ \text{ and } \min_{t\in[\eta,T]_{\mathbb{T}}} u_2 < b^* \text{ with } \min_{t\in[\eta,T]_{\mathbb{T}}} u_3 < a^*.$$

Proof. By the definition of completely continuous operator A and its properties, it suffices to show that all the conditions of Lemma 2.8 hold with respect to A.

For all $u \in P$, $\lambda(u) = \varphi(u) = u(\eta)$ and $\|u\| = u(T) = \phi(u)$. Hence, the condition (2.6) is satisfied.

First, we show that $A : \overline{P(\phi, d^*)} \to \overline{P(\phi, d^*)}$.

For any $u \in \overline{P(\phi, d^*)}$, in view of $\phi(u) = \|u\| \le d^*$ and the assumption (i), one has

$$\begin{aligned}
&\|Au\| = Au(T) \\
&\le B\sum_{i=1}^{m-2} a_i \varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right) + \int_0^T \varphi_q\left(\int_0^T h(s)f(u)\nabla s\right)\Delta\tau \\
&\le \left(B\sum_{i=1}^{m-2} a_i + T\right)\varphi_q\left(\int_0^T h(s)\varphi_p\left(\frac{d^*}{W_\xi}\right)\nabla s\right) \\
&= \frac{d^*}{W_\xi}\left(B\sum_{i=1}^{m-2} a_i + T\right)\varphi_q\left(\int_0^T h(s)\Delta s\right) = d^*.
\end{aligned}$$

From above analysis, it remains to show that (i)–(iii) of Lemma 2.8 hold.

Second, we verify that condition (i) of Lemma 2.8 holds, let $u \equiv kb^*$ with $k = \frac{W_\xi}{N_\xi}$. From the definitions of N_ξ, W_ξ and $\beta(u)$, respectively, it is easy to see that $u = kb^* > b^*$ and $\beta(u) = 0 < kb^*$. In addition, in view of $b^* < \frac{N_\xi}{W_\xi} d^*$, we have $\phi(u) = kb^* < d^*$. Thus

$$\{u \in P(\phi, \beta, \lambda, b^*, kb^*, d^*) : \lambda(x) > b^*\} \ne \emptyset.$$

For any $u \in P(\phi, \beta, \lambda, b^*, kb^*, d^*)$, then we get $b^* \le u \le d^*$ for all $t \in [\eta, T]_{\mathbb{T}}$, hence, by the assumption (ii), we have

$$\begin{aligned}
&\lambda(Au) = Au(\eta) \\
&> \left(A'\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_\eta^T h(s)\varphi_p\left(\frac{b^*}{N_\xi}\right)\nabla s\right) \\
&= \frac{b^*}{N_\xi}\left(A'\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_\eta^T h(s)\Delta s\right) = b^*.
\end{aligned}$$

Third, we prove that the condition (ii) of Lemma 2.8 holds. For any $u \in P(\phi, \lambda, b^*, d^*)$ with $\beta(Au) > kb^*$, that is, $\beta(Au) = \left|(Au)^\Delta(r)\right| = \varphi_q\left(\int_r^T h(s)f(u(s))\nabla s\right) > kb^*$. So, in view of $k = \frac{W_\xi}{N_\xi}$ and $W_\xi >$

$\varphi_q\left(\int_\eta^T h(s)\nabla s\right)$, one has

$$\begin{aligned}&\lambda(Au) = Au(\eta)\\ &\geq A'\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right) + \int_0^\eta \varphi_q\left(\int_\tau^T h(s)f(u)\nabla s\right)\Delta\tau\\ &> \left(A'\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_r^T h(s)f(u)\nabla s\right) > \left(A'\sum_{i=1}^{m-2} a_i + \eta\right)kb^* > b^*.\end{aligned}$$

Finally, we check condition (iii) of Lemma 2.8. Clearly, since $\varphi(0) = 0 < a^*$, we have $0 \notin R(\phi, \varphi, a^*, d^*)$. If $u \in R(\phi, \varphi, a^*, d^*)$ with $\varphi(u) = \min_{t\in[\eta,T]_{\mathbb{T}}} u = u(\eta) = a^*$, then, Lemma 2.2 implies that $\|u\| \leq \frac{T}{\eta}u(\eta) = \frac{T}{\eta}a^*$. This yields $0 \leq u \leq \frac{T}{\eta}a^*$ for all $t \in [0,T]_{\mathbb{T}}$. Hence, by assumption (iii), we have

$$\begin{aligned}&\lambda(Au) = Au(\eta)\\ &\leq B\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^T h(s)f(u)\nabla s\right) + \int_0^\eta \varphi_q\left(\int_0^T h(s)f(u)\nabla s\right)\Delta\tau\\ &< \left(B\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_0^T h(s)\varphi_p(\tfrac{a^*}{M_\xi})\nabla s\right)\\ &= \tfrac{a^*}{M_\xi}\left(B\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_0^T h(s)\Delta s\right) = a^*.\end{aligned}$$

Consequently, all the conditions of Lemma 2.8 are satisfied. The proof is completed.

We remark that the condition (i) in Theorem 4.13 can be replaced by the following condition (i'):

$$\lim_{u\to\infty}\frac{f(u)}{\varphi_p(u)} \leq \varphi_p\left(\frac{1}{W_\xi}\right),$$

which is a special case of (i).

Corollary 4.14 *If the condition* (i) *in Theorem* 4.13 *is replaced by* (i'), *then the conclusion of Theorem* 4.13 *also holds.*

Proof. By Theorem 4.13, we only need to prove that (i') implies that (i) holds, that is, if (i') holds, then there is a number $d^* \geq \max\left\{\frac{a^*TW_\xi}{\eta N_\xi}, \frac{W_\xi}{N_\xi}b^*\right\}$ such that $f(u) \leq \varphi_p\left(\frac{d^*}{W_\xi}\right)$ for $u \in [0, d^*]$.

Suppose on the contrary that for any $d^* \geq \max\left\{\frac{a^*TW_\xi}{\eta N_\xi}, \frac{W_\xi}{N_\xi}b^*\right\}$, there exists $u_c \in [0, d^*]$ such that $f(u_c) > \varphi_p\left(\frac{d^*}{W_\xi}\right)$. Hence, if we choose $c_n' > \max\left\{\frac{a^*TW_\xi}{\eta N_\xi}, \frac{W_\xi}{N_\xi}b^*\right\}$ $(n = 1, 2, \ldots)$ with $c_n' \to \infty$, then there exist $u_n \in [0, c_n']$ such that

$$f(u_n) > \varphi_p\left(\frac{c_n'}{W_\xi}\right), \tag{4.3}$$

and so

$$\lim_{n\to\infty} f(u_n) = \infty. \tag{4.4}$$

Since the condition (i') holds, then there exists $\tau > 0$ such that

$$f(u) \leq \varphi_p\left(\frac{u}{W_\xi}\right), u > \tau. \tag{4.5}$$

Hence, we have $u_n \leq \tau$. Otherwise, if $u_n > \tau$, then it follows from (4.5) that

$$f(u_n) \leq \varphi_p\left(\frac{u_n}{W_\xi}\right) \leq \varphi_p\left(\frac{c_n'}{W_\xi}\right),$$

which contradicts (4.3).

Let $W=\max_{u\in[0,\tau]_{\mathbb{T}}} f(u)$, then $f(u_n)\le W(n=1,2,\ldots)$, which also contradicts (4.4). The proof is complete. □

Theorem 4.15 *Suppose that there exist constants a_i^*, b_i^*, d_i^* such that*

$$0<a_1^*<\frac{\eta}{T}b_1^*<\frac{\eta N_\xi}{TW_\xi}d_1^*<a_2^*<\frac{\eta}{T}b_2^*<\frac{\eta N_\xi}{TW_\xi}d_2^*<a_3^*<\ldots<a_n^*,\ n\in\mathbb{N},$$

here $i=1,2,\ldots,n$. In addition, suppose that $W_\xi>\varphi_q\left(\int_\eta^T h(s)\nabla s\right)$ holds, f satisfies the following conditions:

(i) $f(u)<\varphi_p\left(\frac{d_i^*}{W_\xi}\right),\ 0\le u\le d_i^*$;

(ii) $f(u)>\varphi_p\left(\frac{b_i^*}{N_\xi}\right),\ b_i^*\le u\le d_i^*$;

(iii) $f(u)<\varphi_p\left(\frac{a_i^*}{M_\xi}\right),\ 0\le u\le \frac{T}{\eta}a_i^*$.

Then problem (1.2) and (1.7) has at least $2n+1$ positive solutions.

Proof. Similar to the proof of Theorem 4.2; we omit it here. □

In the following, we deal with problem (1.2) and (1.8), we define the nonnegative continuous convex functionals ϕ_1 and β_1, nonnegative continuous concave functional λ_1, and nonnegative continuous functional φ_1 respectively, on P_1 by

$$\phi_1(u)=\max\nolimits_{t\in[0,T]_{\mathbb{T}}} u=\|u\|,\ \beta_1(u)=\min\nolimits_{t\in[0,l]_{\mathbb{T}^\kappa}}|u^\Delta|=|u^\Delta(l)|,$$
$$\lambda_1(u)=\varphi_1(u)=\min\nolimits_{t\in[0,\xi]_{\mathbb{T}}} u=u(\xi).$$

Again, we use Lemma 2.8 to study the existence of solutions of problem (1.2) and (1.8). In view of operator A_1 defined on P_1, similar to those techniques of considering problem (1.2) and (1.7), we have the following results of problem (1.2) and (1.8).

Theorem 4.16 *Assume that there exist constant a_s^*, b_s^*, d_s^* such that $0<a_s^*<\frac{T-\xi}{T}b_s^*<\frac{(T-\xi)N'_\xi}{TQ_\xi}d_s^*$, In addition, suppose that $Q_\xi>\varphi_q\left(\int_0^\xi h(s)\nabla s\right)$ holds, f satisfies the following conditions:*

(i) $f(u)\le\varphi_p\left(\frac{d_s^*}{Q_\xi}\right),\ 0\le u\le d_s^*$;

(ii) $f(u)>\varphi_p\left(\frac{b_s^*}{N'_\xi}\right),\ b_s^*\le u\le d_s^*$;

(iii) $f(u)<\varphi_p\left(\frac{a_s^*}{M'_\xi}\right),\ 0\le u\le \frac{Ta_s^*}{T-\xi}$.

Then problem (1.2) and (1.8) has at least three positive solutions $u_1,\ u_2,\ u_3$ such that

$$\|x_i\|\le d_s^* \text{ for i=1,2,3, } b_s^*<\min\nolimits_{t\in[l,T]_{\mathbb{T}}} u_1,\ a_s^*<\min\nolimits_{t\in[0,\xi]_{\mathbb{T}}} u_2$$
$$\text{and } \min\nolimits_{t\in[l,T]_{\mathbb{T}}} u_2<b_s^* \text{ with } \min\nolimits_{t\in[0,\xi]_{\mathbb{T}}} u_3<a_s^*.$$

□

Note that the condition (i) in Theorem 4.16 can be replaced by the following condition (i"):

$$\lim_{u\to\infty} \frac{f(u)}{\varphi_p(u)} \le \varphi_p\left(\frac{1}{Q_\xi}\right),$$

which is a special case of (i).

Corollary 4.17 *If the condition* (i) *in Theorem* 4.16 *is replaced by* (i"), *then the conclusion of Theorem* 4.16 *also holds.*

Theorem 4.18 *Suppose that there exist constants* $a^*_{s_i}, b^*_{s_i}, d^*_{s_i}$ *such that*

$$0 < a^*_{s_1} < \frac{T-\xi}{T} b^*_{s_1} < \frac{(T-\xi)N'_\xi}{TQ_\xi} d^*_{s_1} < a^*_{s_2} < \frac{T-\xi}{T} b^*_{s_2} < \frac{(T-\xi)N'_\xi}{TQ_\xi} d^*_{s_2} < a^*_{s_3} < \ldots < a^*_{s_n},\ n \in \mathbb{N},$$

here $i - 1, 2, \ldots, n$. *In addition, suppose that* $Q_\xi > \varphi_q\left(\int_0^\xi h(s)\nabla s\right)$ *holds,* f *satisfies the following conditions:*

(i) $f(u) < \varphi_p\left(\frac{d^*_{s_i}}{Q_\xi}\right)$, $0 \le u \le d^*_{s_i}$;

(ii) $f(u) > \varphi_p\left(\frac{b^*_{s_i}}{N'_\xi}\right)$, $b^*_{s_i} \le u \le d^*_{s_i}$;

(iii) $f(u) < \varphi_p\left(\frac{a^*_{s_i}}{M'_\xi}\right)$, $0 \le u \le \frac{Ta^*_{s_i}}{T-\xi}$.

Then problem (1.2) *and* (1.8) *has at least* $2n + 1$ *positive solutions.*

Recall the methods mentioned above. We note that these two methods have their own advantages respectively. In the following, we show the differences of these two methods from two aspects.

From the viewpoint of the obtained solution position and local properties, by using method one, we only get some local properties of solutions, however, the position of solutions is not determined. For method two, we not only get some local properties of solutions but also give the position of all solutions, with regard to some subsets of the cone. In addition, by using these two different methods, the local properties of obtained solutions are obviously different. Hence, it is convenient for us to comprehensively comprehend the solutions of the models by using these two different techniques.

From the viewpoint of the satisfied conditions, we take Theorem 4.1 and Theorem 4.13 as examples to illustrate. Under the same parameters conditions, the $f(u)$ of (i) in Theorem 4.1 has the wider range than the $f(u)$ of (i) in Theorem 4.13, and the region of $f(u)$ in (iii) of Theorem 4.13 is wider than that of Theorem 4.1.

5. Examples

In this section, we present two simple examples to illustrate our result. In addition, these two examples show the differences of the two methods in Section 4.

Example 5.1 Let

$$\mathbb{T} = \left\{\left(\tfrac{1}{3}\right)^{\mathbb{N}_0}\right\} \cup \left\{0, \tfrac{1}{10}, \tfrac{3}{20}, \tfrac{157}{1000}, \tfrac{1}{5}, \tfrac{23}{100}, \tfrac{3}{10}, \tfrac{31}{100}, \tfrac{33}{100}\right\} \cup \left[\tfrac{1}{3}, \tfrac{1}{2}\right],$$

here $\mathbb{N}_0 = \{1, 2, 3, \ldots, \}$.

Consider the following boundary value problem with $p = 5$ and $k \in \mathbb{N}_0$.

$$\begin{array}{c} \left(\varphi_p\left(u^{\Delta}\right)\right)^{\nabla} + \sum_{k=0}^{7} t^k(\rho(t))^{7-k} f(u) = 0,\ t \in \left[0, \frac{1}{3}\right]_{\mathbb{T}}, \\ u(0) - 10^{-3}\left(u^{\Delta}\left(\frac{1}{81}\right) + u^{\Delta}\left(\frac{1}{5}\right)\right) = 0,\ u^{\Delta}(\frac{1}{3}) = 0, \end{array} \tag{5.1}$$

where

$$f(u) = \begin{cases} 2 \times 10^7, & 0 \le u \le 3, \\ 5.4054 \times 10^{11} u - 1.6216 \times 10^{12}, & 3 \le u \le 40, \\ 2 \times 10^{13}, & 40 \le u \le 80, \\ 1.5748 \times 10^{11} u + 7.4016 \times 10^{12}, & 80 \le u \le 334, \\ 6 \times 10^{13}, & u \ge 334. \end{cases}$$

If $h(t) = \sum_{k=0}^{7} t^k(\rho(t))^{7-k}$, then by Exercise 1.23 in [10], we have $(t^8)^{\nabla} = \sum_{k=0}^{7} t^k(\rho(t))^{7-k}$.

It is obvious that $\xi_2 = \eta = \frac{1}{5}$, $A = B = 10^{-3}$ and $a_1 = a_2 = 1$. Choose $r = \frac{23}{100}$, direct calculation shows that $N_\xi = \left(2 \times 10^{-3} + \frac{1}{5}\right)\left(\int_{\frac{1}{5}}^{\frac{1}{3}} \sum_{k=0}^{7} t^k(\rho(t))^{7-k} \nabla t\right)^{\frac{1}{4}} \approx 0.02235$, by a similar way, we have $M_\xi \approx 2.2444 \times 10^{-2}, L_\xi \approx 2.5778 \times 10^{-2}$ and $W_\xi \approx 3.7259 \times 10^{-2}$.

If we take $a' = 2,\ b' = 40,\ c' = 200$, then $0 < a' < \frac{r}{T} b' < \frac{r N_\xi}{T M_\xi} c'$,

$$\begin{array}{l} f(u) = 2 \times 10^{13} > 1.0089 \times 10^{13} = \varphi_5\left(\frac{b'}{M_\xi}\right) \text{ for } 40 \le u \le \frac{Tb'}{\eta} = 66.667, \\ f(u) < 6.4122 \times 10^{15} \approx \varphi_5\left(\frac{c'}{N_\xi}\right) \text{ for } 0 \le u \le \frac{Tc'}{\eta} = 400, \\ f(u) = 2 \times 10^7 < 3.6235 \times 10^7 \approx \varphi_5\left(\frac{a'}{L_\xi}\right) \text{ for } 0 \le u \le \frac{Ta'}{r} = 2.8986. \end{array}$$

Therefore, all the conditions of Theorem 4.1 are satisfied. By Theorem 4.1, we see that the boundary value problem (5.1) has at least three positive solutions $u_1,\ u_2$ and u_3 such that

$$\begin{array}{c} 0 < \max_{t \in \left[0, \frac{23}{100}\right]_{\mathbb{T}}} u_1 < 2 < \max_{t \in \left[0, \frac{23}{100}\right]_{\mathbb{T}}} u_2 \text{ with} \\ \min_{t \in \left[\frac{1}{5}, \frac{1}{3}\right]_{\mathbb{T}}} u_2 < 40 < \min_{t \in \left[\frac{1}{5}, \frac{1}{3}\right]_{\mathbb{T}}} u_3 \text{ and } \max_{t \in [0, \frac{1}{5}]_{\mathbb{T}}} u_3 < 200. \end{array}$$

However, $\varphi_q\left(\int_{\frac{1}{5}}^{\frac{1}{3}} h(s) \Delta s\right) \approx 0.1106 > W_\xi \approx 3.7259 \times 10^{-2}$, hence, the existence of positive solutions of boundary value problem (5.1) is not obtained by using Theorem 4.13.

Example 5.2 Let

$$\mathbb{T} = \left\{2 - \left(\tfrac{1}{3}\right)^{\mathbb{N}_0}\right\} \cup \left\{0, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, 1, \tfrac{5}{4}, \tfrac{3}{2}, \tfrac{7}{4}, 2, \tfrac{21}{10}, \tfrac{23}{10}, \tfrac{12}{5}\right\} \cup \left[\tfrac{1}{10}, \tfrac{1}{9}\right].$$

Consider the following boundary value problem

$$\begin{array}{c} \left(\varphi_p\left(u^{\Delta}\right)\right)^{\nabla} + \sum_{k=0}^{7} t^k(\rho(t))^{7-k} f(u) = 0,\ t \in [0, 2]_{\mathbb{T}}, \\ u(0) - 2\left(u^{\Delta}\left(\frac{1}{4}\right) + u^{\Delta}(1)\right) = 0,\ u^{\Delta}(2) = 0. \end{array} \tag{5.2}$$

Let ε be an arbitrary small positive number, a^*, b^* and d^* be arbitrary positive numbers with $a^* < b^* < d^*$, and

$$f(u) = \begin{cases} \max\left\{\varphi_p\left(\frac{a^*}{20}\right) - \varepsilon, \varphi_p\left(\frac{a^*}{22}\right) + \varepsilon\right\}, & 0 \le u \le 2a^*, \\ k(u), & 2a^* \le u \le b^*, \\ \varphi_p\left(\frac{b^*}{19.98}\right) + \varepsilon, & b^* \le u \le d^*, \\ r(u), & d^* \le u \le 2d^*, \\ \varphi_p\left(\frac{b^*}{20}\right) - \varepsilon, & u \ge 2d^*, \end{cases}$$

here $k(u)$ and $p(u)$ satisfy $k\left(2a^*\right) = \max\left\{\varphi_p\left(\frac{a^*}{20}\right) - \varepsilon, \varphi_p\left(\frac{a^*}{22}\right) + \varepsilon\right\}$, $k(b^*) = \varphi_p\left(\frac{b^*}{19.98}\right) + \varepsilon$, $r(d^*) = \varphi_p\left(\frac{b^*}{19.98}\right) + \varepsilon$, $r(2d^*) = \varphi_p\left(\frac{b^*}{20}\right) - \varepsilon$, $(k^{\Delta}(u))^{\Delta} = 0$, $(r^{\Delta}(u))^{\Delta} = 0$.

It is obvious that $\xi_2 = \eta = 1$, $A = B = 2$ and $a_1 = a_2 = 1$. Choose $r = \frac{3}{2}$, direct calculation shows that $N_\xi = 5\left(\int_1^2 \sum_{k=0}^{7} t^k(\rho(t))^{7-k}\nabla t\right)^{\frac{1}{4}} \approx 19.98$, by a similar way, we have $M_\xi = 20, L_\xi = 22$, $W_\xi = 24$ and $W_\xi = 24 > 4 \approx \varphi_q\left(\int_1^2 h(t)\Delta t\right)$.

If we take a^*, b^* and d^* satisfy $0 < a^* < \frac{1}{2}b^* < \frac{N_\xi}{2W_\xi}d^* = \frac{19.98}{48}d^*$.

It is obvious that (i), (ii) and (iii) in Theorem 4.13 are satisfied. By Theorem 4.13, we see that the boundary value problem (5.2) has at least three positive solutions u_1, u_2 and u_3 such that

$$\begin{gathered}\|x_i\| \le d^* \text{ for i=1,2,3, } b^* < \min\nolimits_{t\in[1,2]_{\mathbb{T}}} u_1,\ a^* < \min\nolimits_{t\in[1,2]_{\mathbb{T}}} u_2 \\ \text{and } \min\nolimits_{t\in[1,2]_{\mathbb{T}}} u_2 < b^* \text{ with } \min\nolimits_{t\in[1,2]_{\mathbb{T}}} u_3 < a^*.\end{gathered}$$

However, for arbitrary positive numbers a^*, b^*, d^* with $a^* < b^* < d^*$, the condition (iii) of Theorem 4.1 is not satisfied. Therefore, Theorem 4.1 is not fit to study the boundary value problem (5.2).

Acknowledgment

The author is very grateful to the anonymous referees for their helpful comments and suggestions.

References

[1] Agarwal, R.P., Bohner, M. and Li, W.T.: Nonoscilation and oscillation theory for functional differential equations, Pure Appl. Math. S. 267, Marcel Dekker, 2004.

[2] Agarwal, R.P., Bohner,M. and Rehak, P.: Half-linear dynamic equations, Nonlinear Anal. Appl.: to V. Lakshmikantham on his 80th birthday, vol. 1, Kluwer Acad. Publ. Dordrecht, 1-57 (2003).

[3] Anderson, D.R.: Solutions to second-order three-point problems on time scales, J. Differ. Equations Appl. 8, 673-688 (2002).

[4] Aulbach, B. and Neidhart, L.: Integration on measure chains, In proceedings of the sixth international conference on difference equations, CRC, Boca Raton, FL, 239-252 (2004).

[5] Avery, R.I., Chyan, C.J. and Henderson, J.: Twin solutions of boundary value problems for ordinary differential equations and finite difference equations, Comput. Math. Appl. 42, 695-704 (2001).

[6] Avery, R.I. and Henderson, J.: Two positive fixed points of nonlinear operator on ordered Banach spaces, Comm. Appl. Nonlinear Anal. 8, 27-36 (2001).

[7] Avery, R.I. and Peterson, A.: Three positive fixed points of nonlinear operators on ordered Banach spaces, Comput. Math. Appl. 42, 313-422 (2001).

[8] Avery, R.I.: A generalization of the Leggett-Williams fixed point theorem, MSR Hot-Line. 2, 9-14 (1999).

[9] Bohner, M. and Peterson, A.: Advances in dynamic equations on time scales, Birkhauser Boston, 2003.

[10] Bohner, M. and Peterson, A.: Dynamic equations on time scales: an introduction with applications, Birkhauser Boston, MA, 2001.

[11] Guo, D. and Lakshmikantham, V.: Nonlinear problems in abstract cones, Academic press, San Diego, CA, 1988.

[12] He, Z.: Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales, J. Comput. Appl. Math. 182, 304-315 (2005).

[13] He, Z.: Double positive solutions of three-point boundary value problems for p-Laplacian difference equations, Z. Anal. Anwendungen 24, 305-315 (2005).

[14] He, Z.: On the existence of positive solutions of p-Laplacian difference equation, J. Comput. Appl. Math. 161, 193-201 (2003).

[15] He, Z. and Li L.: Triple positive solutions for the one-dimensional p-Laplacian dynamic equations on time scales, Math. Comput. Modelling 45, 68-79 (2007).

[16] He, Z. and Jiang, X.: Multiple positive solutions of boundary value problems for p-Laplacian dynamic equations on time scales, J. Math. Anal. Appl. 321, 911-920 (2006).

[17] Hilger, S.: Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. Thesis, Universität Würzburg, (1988).

[18] Hong, S.: Triple positive solutions of three-point boundary value problem for p-Laplacian dynamic equations on time scales, J. Comput. Appl. Math. 206, 967-976 (2007).

[19] Jamieson, V. and Spedding, V.: Taming nature's numbers, the global science and technology weekly, New Scientist, 2404, 28-31 (2003).

[20] Jones M.A., Song,B. and Thomas, D.M.: Controlling wound healing through debridement, Math. Comput. Modelling 40, 1057-1064 (2004).

[21] Krasnosel'skii, M.: Positive solutions of operator equations, Noordhoff, Groningen (1964).

[22] Lakshmikantham, V., Sivasundaram, S. and Kaymakcalan, B.: Dynamic systems on measure chains, Kluwer Academic Publishers, Boston, 1996.

[23] Leggett, R. and Williams, L.: Multiple positive fixed points of nonlinear operators on ordered Banach spaces, Indiana Univ. Math. J. 28, 673-688 (1979).

[24] Li, C. and Ge, W.: Positive solutions of one-dimensional p-Laplacian singular Sturm-Liouville boundary value problems, Math. Appl. 15, 13-17 (2002).

[25] Liu, Y. and Ge, W.: Multiple positive solutions to a three-point boundary value problems with p-Laplacian, J. Math. Anal. Appl. 277, 293-302 (2003).

[26] Liu, Y. and Ge, W.: Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian, J. Math. Anal. Appl. 278, 551-561 (2003).

[27] Lü, H., O'Regan, D. and Zhong, C.: Multiple positive solutions for the one-dimensional singular p-Laplacian, Appl. Math. Comput. 133, 407-422 (2002).

[28] Ren, J.L., Ge, W. and Ren, B.X.: Existence of three solutions for quasi-nonlinear boundary value problems, Acta. Math. Appl. Sinica 21, 353-358 (2005).

[29] Su, Y.H.: Multiple positive pseudo-symmetric solutions of p-Laplacian dynamic equations on time scales, Math. Comput. Modelling 49, 1664–1681 (2009).

[30] Su, Y.H., Li, W.T.: Existence of positive solutions to a singular p-Laplacian dynamic equations with sign changing nonlinearity, Acta. Math. Sinica, Chinese Series, 52, 181-196 (2009).

[31] Su, Y.H., Li, W.T. and Sun, H.R.: Triple positive pseudo-symmetric solutions of three-point BVPs for p-Laplacian dynamic equations on time scales, Nonlinear Anal. 68, 1442-1452 (2008).

[32] Su, Y.H. and Li,W.T.: Triple positive solutions of m-point BVPs for p-Laplacian dynamic equations on time scales, Nonlinear Anal. 69, 3811-3820 (2008).

[33] Su,Y.H., Li, W.T. and Sun, H.R.: Positive solutions of singular p-Laplacian dynamic equations with sign changing nonlinearity, Appl. Math. Comput. 200, 352-368 (2008).

[34] Su,Y.H., Li,W.T. and Sun, H.R.: Positive solutions of singular p-Laplacian BVPs with sign changing nonlinearity on time scales, Math. Comput. Modelling 48, 845-858 (2008).

[35] Sun, J.P.: Existence of solution and positive solution of BVP for nonlinear third-order dynamic equation, Nonlinear Anal. 64, 629-636 (2006).

[36] Sun, H.R. and Li, W.T.: Existence theory for positive solutions to one-dimensional p-Laplacian boundary value problems on time scales, J. Differential Equations 240, 217-248 (2007).

[37] Thomas, D.M., Vandemuelebroeke, L. and Yamaguchi, K.: A mathematical evolution model for phytoremediation of metals, Discrete Contin. Dyn. Syst. Ser. B 5, 411-422 (2005).

[38] Wang, D.B.: Triple positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales, Nonlinear Anal. 68, 2172-2180 (2008).

[39] Wang, H.: Positive periodic solutions of functional differential equations, J. Differential Equations 202, 354-366 (2004).

[40] Wang, J.: The existence of positive solutions for the one-dimensional p-Laplacian, Proc. Amer. Math. Soc. 125, 2275-2283 (1997).

[41] Yaslan, I.: Existence of positive solutions for nonlinear three-point problems on time scales, J. Comput. Appl. Math. 206, 888-897 (2007).

You-Hui SU
Mathematics and Physical Sciences Technology,
Xuzhou Institute of Technology,
Xuzhou, Jiangsu, 221111, CHINA
e-mail: suyouhui@xzit.edu.cn

Received: 09.04.2009