Turk J Math
35 (2011) , 249 – 258.
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doi:10.3906/mat-0910-98

Universal inequalities and bounds for weighted eigenvalues of the Schrödinger operator on the Heisenberg group

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Abstract

For a bounded domain Ω in the Heisenberg group $\mathbb{H}^n$, we investigate the Dirichlet weighted eigenvalue problem of the Schrödinger operator $-\Delta_{\mathbb{H}^n}+V$, where $\Delta_{\mathbb{H}^n}$ is the Kohn Laplacian and V is a nonnegative potential. We establish a Yang-type inequality for eigenvalues of this problem. It contains the sharpest result for $\Delta_{\mathbb{H}^n}$ in [17] of Soufi, Harrel II and Ilias. Some estimates for upper bounds of higher order eigenvalues and the gaps of any two consecutive eigenvalues are also derived. Our results are related to some previous results for the Laplacian Δ and the Schrödinger operator $-\Delta+V$ on a domain in $\mathbb{R}^n$ and other manifolds.

Key Words: Eigenvalue, universal inequality, Heisenberg group, Schrödinger operator, Kohn Laplacian.

1. Introduction

In 1956, Payne, Pólya and Weinberger [16] considered the Dirichlet Laplacian problem (also called the fixed membrance problem)

$$\begin{cases} -\Delta u=\lambda u, & \text{in } \Omega, \\ u|_{\partial\Omega}=0, \end{cases} \tag{1.1}$$

and established an universal inequality for $\Omega\subset\mathbb{R}^2$ which is easily extended to $\Omega\subset\mathbb{R}^n$ as the PPW inequality

$$\lambda_{k+1}-\lambda_k\le\frac{4}{nk}\sum_{r=1}^{k}\lambda_r. \tag{1.2}$$

The work of Payne, Pólya, Weinberger and other mathematics provided us a precious wealth of results, and in some sense, we still walk along the road which is illuminated by them. Hile and Protter [11], Yang [19] and other mathematicians made their contributions in extending the PPW inequality. Namely, in 1980, Hile and Protter [11] proved the HP inequality

$$\sum_{r=1}^{k}\frac{\lambda_r}{\lambda_{k+1}-\lambda_r}\ge\frac{nk}{4}. \tag{1.3}$$

2000 *AMS Mathematics Subject Classification:* Primary 35P15, 53C17; Secondary 58C40.

In 1991, Yang [19] obtained (what is now known as) Yang's first inequality,

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq \frac{4}{n}\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r, \tag{1.4}$$

and the Yang's second inequality

$$\lambda_{k+1} \leq (1+\frac{4}{n})\frac{1}{k}\sum_{r=1}^{k}\lambda_r. \tag{1.5}$$

In 1997, Harrell and Stubbe [10] gave a new proof of Yang's inequalities by using the commutator method. One can find more discussions about the PPW, HP and Yang's inequalities in [1, 2] of Ashbaugh.

Further research work have been done on some other manifolds or some more complicated operators. On the one hand, the inequalities (1.2)–(1.5) for eigenvalues of problem (1.1) on $\Omega \subset \mathbb{R}^n$ have been extended to some Riemannian manifolds (see [3, 4, 5, 7, 9, 14]). On the other hand, some interesting inequalities for eigenvalues of the Schrödinger operator have also been established. In 2002, Ashbaugh [2] considered the Dirichlet weighted eigenvalue problem on $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} -\Delta u + Vu = \lambda\rho u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.6}$$

where V is a nonnegative potential, and ρ is a positive function continuous on $\overline{\Omega}$. He derived the Yang-type inequalities ($\rho_{\max}$ and $\rho_{\min}$ denote the obvious quantities)

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq \frac{4\rho_{\max}}{n\rho_{\min}}\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\lambda_r, \tag{1.7}$$

and

$$\lambda_{k+1} < (1+\frac{4\rho_{\max}}{n\rho_{\min}})\frac{1}{k}\sum_{r=1}^{k}\lambda_r. \tag{1.8}$$

They are independent on the potential V. In fact, as one can see from the commutator method, all the results about eigenvalues of $-\Delta$ are automatically generalizable to $-\Delta+V$ because $[-\Delta, G]=[-\Delta+V, G]$. In 2008, Wang and Xia [18] proved

$$\begin{aligned} \lambda_{k+1} \leq & (1+\frac{2P}{nQ})\frac{1}{k}\sum_{r=1}^{k}\lambda_r - \frac{2V_0}{nQ} \\ & + \left\{ [\frac{2P}{nQ}(\frac{1}{k}\sum_{r=1}^{k}\lambda_r - \frac{V_0}{P})]^2 - (1+\frac{4P}{nQ})\frac{1}{k}\sum_{s=1}^{k}(\lambda_s - \frac{1}{k}\sum_{r=1}^{k}\lambda_r)^2 \right\}^{\frac{1}{2}} \end{aligned} \tag{1.9}$$

on a bounded domain in $\mathbb{R}^n$, where $V_0 = \min\limits_{x\in\overline{\Omega}} V(x)$, $P = \max\limits_{x\in\overline{\Omega}}\rho(x)$ and $Q = \min\limits_{x\in\overline{\Omega}}\rho(x)$. The reader can refer to [18] for more results on a domain in an n-dimensinal unit sphere $S^n(1)$, an n-dimensinal minimal subanifold in $S^m(1)$, a domain in a complex projective space $CP^n(4)$, a complex hypersurface in $CP^{n+1}(4)$, an n-dimensional homogenous space. In 2009, based on the work [8], Soufi, Harrel II and Ilias [17] used the commutator method to derive a series of inequalities for eigenvalues of problem (1.6) on a submanifold of a sphere, a submanifold of a projective space, etc. For example, on a closed Riemannian manifold M or a domain in a Riemannian manifold M, they proved (suppose that $X : M \longrightarrow \mathbb{R}^m$ is an isometric immersion)

$$\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)^2 \leq \frac{4}{n}\sum_{r=1}^{k}(\lambda_{k+1}-\lambda_r)\left[\lambda_r + \int_M (\frac{|h|^2}{4} - V)u_r^2\right], \tag{1.10}$$

where h denotes the mean curvature vector field of X.

Now we turn our attention to the eigenvalue problem of the Kohn Lapacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$. $\Delta_{\mathbb{H}^n}$ is one of the invariant differential operators on the nilpotent group $\mathbb{H}^n$ (see [12] of Jerison, [6] of Folland and Stein). It dates from [13] and is also called sub-Laplacian. Let Λ_r be the r-th eigenvalue of the following Dirichlet eigenvalue problem of the Kohn Lapacian on a bounded domain Ω in $\mathbb{H}^n$:

$$\begin{cases} -\Delta_{\mathbb{H}^n} u = \Lambda u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.11}$$

In 2003, Niu and Zhang [15] proved the inequality

$$\Lambda_{k+1} - \Lambda_k \leq \frac{2}{nk}\sum_{r=1}^{k}\Lambda_r, \tag{1.12}$$

which is related to the PPW inequality (1.2). Still in [17], Soufi, Harrel II and Ilias used the commutator method to establish a sharper inequality

$$\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2 \leq \frac{2}{n}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\Lambda_r, \tag{1.13}$$

which is related to the Yang's first inequality (1.4).

In this paper, we investigate the Dirichlet weighted eigenvalue problem of the Schrödinger operator $-\Delta_{\mathbb{H}^n} + V$ on $\mathbb{H}^n$:

$$\begin{cases} -\Delta_{\mathbb{H}^n} u + Vu = \Lambda\rho u, & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.14}$$

In Theorem 1 of Section 2, we prove the general inequality (2.2) for eigenvalues of problem (1.14) which contains a positive constant γ. We follow different route from [17]. From the proof of Theorem 1, one can easily find the reason why the coefficients in estimates for eigenvalues of problem (1.6) and problem (1.14) are different. Then, we obtain a more explicit inequality (2.22) in Theorem 2. In fact, these two general inequalities are equivalent (see Remark 1). In Section 3, by utilizing the general inequality in Theorem 2, we establish a Yang-type inequality for eigenvalues of problem (1.14) (see Theorem 3). It contains the sharpest result (1.13) for $\Delta_{\mathbb{H}^n}$. In Corollary 1-3, some estimates for upper bound of Λ_{k+1} and the gaps of any two consecutive eigenvalues are also given. Our results are related to (1.4), (1.5), (1.7)–(1.9) and (1.13) for Δ, $-\Delta + V$ and $\Delta_{\mathbb{H}^n}$.

2. Some general inequalities

The $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$ is with coordinates $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and satisfies non-commutative group law given by

$$\big(x,y,t\big)\big(x',y',t'\big) = \big(x+x', y+y', t+t' + \frac{1}{2}(\langle x', y\rangle_{\mathbb{R}^n} - \langle x, y'\rangle_{\mathbb{R}^n})\big),$$

where $\big(x,y,t\big)$ and $\big(x',y',t'\big) \in \mathbb{R}^n$, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. The Lie algebra $\mathcal{H}^n$ of $\mathbb{H}^n$ has a basis

$$\{X_j, Y_j, T\}, \quad j = 1, \cdots, n$$

formed by the $2n+1$ left-invariant vector fields

$$X_j = \frac{\partial}{\partial x_j} + \frac{y_j}{2}\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - \frac{x_j}{2}\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

The Kohn Laplacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$ is defined by

$$\Delta_{\mathbb{H}^n} = \sum_{j=1}^{n}(X_j^2 + Y_j^2). \tag{2.1}$$

Theorem 1 *Let Ω be a bounded domain in $\mathbb{H}^n$. Denote by u_r the r-th weighted orthonormal eigenfunction of problem (1.14) corresponding to the eigenvalue Λ_r, $r = 1, 2, \cdots, k$. Then we have*

$$(1-\gamma)\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2 \le \frac{1}{2\gamma}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\int_\Omega \frac{1}{\rho}[(X_i u_r)^2 + (Y_i u_r)^2], \tag{2.2}$$

where the constant $\gamma > 0$.

Proof. The eigenfunction u_r satisfies

$$\begin{cases} -\Delta_{\mathbb{H}^n} u_r + V u_r = \Lambda_r \rho u_r, & \text{in} \quad \Omega, \\ u_r|_{\partial\Omega} = 0, \\ \displaystyle\int_\Omega \rho u_r u_s = \delta_{rs}. \end{cases} \tag{2.3}$$

We define the trial functions

$$\varphi_{rx_i} = x_i u_r - \sum_{s=1}^{k} a_{rsx_i} u_s, \quad \text{for} \quad i = 1, \cdots, n, \ \text{and} \quad r = 1, \cdots, k, \tag{2.4}$$

where

$$a_{rsx_i} = \int_\Omega \rho x_i u_r u_s. \tag{2.5}$$

Then, it is easy to check that for $i = 1, \cdots, n$, and $r, s = 1, \cdots, k$,

$$\int_\Omega \rho \varphi_{rx_i} u_s = 0. \tag{2.6}$$

Hence, it yields to

$$\int_\Omega \rho \varphi_{rx_i} x_i u_r = \int_\Omega \rho \varphi_{rx_i}^2. \tag{2.7}$$

Substituting

$$-\Delta_{\mathbb{H}^n} \varphi_{rx_i} + V\varphi_{rx_i} = -2X_i u_r + x_i \Lambda_r \rho u_r - \sum_{s=1}^{k} a_{rsx_i} \Lambda_s \rho u_s \tag{2.8}$$

into the Rayleigh-Ritz formula

$$\Lambda_{k+1} \leq \frac{\int_\Omega \varphi_{rx_i}(-\Delta_{\mathbb{H}^n}\varphi_{rx_i} + V\varphi_{rx_i})}{\int_\Omega \rho\varphi_{rx_i}^2}, \tag{2.9}$$

we obtain

$$\begin{aligned}(\Lambda_{k+1} - \Lambda_r)\int_\Omega \rho\varphi_{rx_i}^2 \leq & -2\int_\Omega x_i u_r X_i u_r + 2\sum_{s=1}^{k} a_{rsx_i} b_{rsx_i} \\ = & \int_\Omega u_r^2 + 2\sum_{s=1}^{k} a_{rsx_i} b_{rsx_i},\end{aligned} \tag{2.10}$$

where

$$b_{rsx_i} = \int_\Omega u_s X_i u_r = -\int_\Omega u_r X_i u_s = -b_{srx_i}.$$

Using integration by parts, and utilizing (2.8), we have

$$\begin{aligned}\Lambda_r a_{rsx_i} = & \int_\Omega x_i u_s(-\Delta_{\mathbb{H}^n} u_r + V u_r) = -2\int_\Omega u_r X_i u_s + \Lambda_s \int_\Omega \rho x_i u_r u_s \\ = & -2b_{srx_i} + \Lambda_s a_{rsx_i}.\end{aligned} \tag{2.11}$$

It yields to

$$2b_{rsx_i} = (\Lambda_r - \Lambda_s) a_{rsx_i}. \tag{2.12}$$

Substituting (2.12) into (2.10), we have

$$(\Lambda_{k+1} - \Lambda_r)\int_\Omega \rho\varphi_{rx_i}^2 \leq \int_\Omega u_r^2 + \sum_{s=1}^{k}(\Lambda_r - \Lambda_s) a_{rsx_i}^2. \tag{2.13}$$

By direct calculation, we have

$$-2\int_\Omega \varphi_{rx_i} X_i u_r = \int_\Omega u_r^2 + \sum_{s=1}^{k}(\Lambda_r - \Lambda_s) a_{rsx_i}^2. \tag{2.14}$$

Noticing the weight function ρ, it follows from (2.6) and (2.7) that

$$
\begin{aligned}
&- 2(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega \varphi_{rx_i}X_iu_r\\
=&- 2(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega \sqrt{\rho}\varphi_{rx_i}(\frac{1}{\sqrt{\rho}}X_iu_r-\sqrt{\rho}\sum_{s=1}^k b_{rsx_i}u_s)\\
\le&\gamma(\Lambda_{k+1}-\Lambda_r)^3\int_\Omega \rho\varphi_{rx_i}^2+\frac{\Lambda_{k+1}-\Lambda_r}{\gamma}\Big[\int_\Omega\frac{1}{\rho}(X_iu_r)^2-\sum_{s=1}^k b_{rsx_i}^2\Big],
\end{aligned}
\tag{2.15}
$$

where the constants $\gamma>0$. Substituting (2.13) and (2.14) into (2.15), and taking sum on r from 1 to k, we have

$$
\begin{aligned}
&\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2+2\sum_{r,s=1}^k(\Lambda_{k+1}-\Lambda_r)^2a_{rsx_i}b_{rsx_i}\\
\le&\gamma\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2+\frac{1}{\gamma}\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}(X_iu_r)^2\\
&+\gamma\sum_{r,s=1}^k(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s)a_{rsx_i}^2-\frac{1}{\gamma}\sum_{r,s=1}^k(\Lambda_{k+1}-\Lambda_r)b_{rsx_i}^2.
\end{aligned}
\tag{2.16}
$$

On the other hand, we define the trial functions

$$
\varphi_{ry_i}=y_iu_r-\sum_{s=1}^k a_{rsy_i}u_s,\quad \text{for}\quad i=1,\cdots,n,\ \text{and}\quad r=1,\cdots,k, \tag{2.17}
$$

where

$$
a_{rsy_i}=\int_\Omega \rho y_iu_ru_s. \tag{2.18}
$$

Then, similar to the proof of (2.16), we can obtain

$$
\begin{aligned}
&\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2+2\sum_{r,s=1}^k(\Lambda_{k+1}-\Lambda_r)^2a_{rsy_i}b_{rsy_i}\\
\le&\gamma\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2+\frac{1}{\gamma}\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}(Y_iu_r)^2\\
&+\gamma\sum_{r,s=1}^k(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s)a_{rsy_i}^2-\frac{1}{\gamma}\sum_{r,s=1}^k(\Lambda_{k+1}-\Lambda_r)b_{rsy_i}^2,
\end{aligned}
\tag{2.19}
$$

where

$$
b_{rsy_i}=\int_\Omega u_sY_iu_r.
$$

Combining (2.16) and (2.19), and noticing that the following inequalities

$$-\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\Lambda_s)c_{rs}d_{rs}=\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2c_{rs}d_{rs}, \tag{2.20}$$

$$\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2(\Lambda_r-\Lambda_s)c_{rs}^2=-\sum_{r,s=1}^{k}(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\Lambda_s)^2c_{rs}^2 \tag{2.21}$$

hold, where $c_{rs}=c_{sr}$ and $d_{rs}=-d_{sr}$, we can eliminate the unwanted terms to obtain (2.2). □

Taking $\gamma=\frac{1}{2}$ in (2.2), we can get a more explicit general inequality.

Theorem 2 *Under the assumptions of Theorem 1, we have*

$$\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2\leq 2\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}[(X_iu_r)^2+(Y_iu_r)^2]. \tag{2.22}$$

Remark 1 *In fact, inequality (2.22) is equivalent to (2.2). Noticing that*

$$\left[\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2\right]\left\{\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2 - 2\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}[(X_iu_r)^2+(Y_iu_r)^2]\right\}$$

is the discriminant of the quadratic polynomial of γ,

$$\left[\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2\right]\gamma^2-\left[\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2\right]\gamma + \frac{1}{2}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}[(X_iu_r)^2+(Y_iu_r)^2],$$

we can deduce (2.2) from (2.22).

3. Some estimates for eigenvalues of $-\Delta_{\mathbb{H}^n}+V$

In this section, we give some estimates for eigenvalues of problem (1.14).

Theorem 3 *Let* Ω *be a bounded domain in* $\mathbb{H}^n$. *Denote by* Λ_r *the* r*-th eigenvalue of problem (1.14). Set* $V_0=\min_\Omega V$, $\sigma=(\inf_\Omega\rho)^{-1}$, $\tau=(\sup_\Omega\rho)^{-1}$. *Then, we have*

$$\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)^2\leq\frac{2\sigma^2}{n\tau^2}\sum_{r=1}^{k}(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\tau V_0). \tag{3.1}$$

Proof. According to the assumptions of Theorem 3, it is easy to find

$$0<\tau=\tau\int_\Omega \rho u_r^2\le\int_\Omega u_r^2=\int_\Omega \rho u_r^2\frac{1}{\rho}\le\sigma\int_\Omega\rho u_r^2=\sigma, \tag{3.2}$$

and

$$\begin{aligned}\int_\Omega|\nabla_{\mathbb{H}^n}u_r|^2=&\int_\Omega u_r(-\Delta_{\mathbb{H}^n}u_r)=\int_\Omega u_r(-\Delta_{\mathbb{H}^n}u_r+Vu_r)-\int_\Omega Vu_r^2\\ \le&\Lambda_r-\tau V_0.\end{aligned} \tag{3.3}$$

Taking sum on i from 1 to n in (2.22), we have

$$n\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2\int_\Omega u_r^2\le 2\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)\int_\Omega\frac{1}{\rho}|\nabla_{\mathbb{H}^n}u_r|^2. \tag{3.4}$$

Utilizing (3.2) and (3.3), we derive (3.1). □

Remark 2 *It is easy to find that our result (3.1) contains the sharpest result (1.13) for the Kohn Laplacian* $\Delta_{\mathbb{H}^n}$*. Moreover, it is a Yang-type inequality which is related to the sharp results (1.4) and (1.7) for* Δ *and* $-\Delta+V$*.*

Remark 3 *Taking sum on* i *from* 1 *to* n *in (2.2), utilizing (3.2) and (3.3), we have*

$$\begin{aligned}&n\tau\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2\\ \le&\frac{\sigma}{2\gamma}\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\tau V_0)+n\sigma\gamma\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2.\end{aligned} \tag{3.5}$$

Then, putting

$$\gamma=\left[2n\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)^2\right]^{-\frac12}\left[\sum_{r=1}^k(\Lambda_{k+1}-\Lambda_r)(\Lambda_r-\tau V_0)\right]^{\frac12}$$

in (3.5), it also yields to (3.1).

(3.1) is a quadratic inequality. Solving it, we can obtain a more explicit inequality which give an universal upper bound of the $(k+1)$-th eigenvalue Λ_{k+1} in terms of σ,τ, V_0 and the first k eigenvalues.

Corollary 1 *Under the assumptions of Theorem 3, we have*

$$\begin{aligned}\Lambda_{k+1}\le&(1+\frac{\sigma^2}{n\tau^2})\frac1k\sum_{r=1}^k\Lambda_r-\frac{\sigma^2V_0}{n\tau}\\ &+\left\{\left[\frac{\sigma^2}{n\tau^2}(\frac1k\sum_{r=1}^k\Lambda_r-\tau V_0)\right]^2-(1+\frac{2\sigma^2}{n\tau^2})\frac1k\sum_{s=1}^k(\Lambda_s-\frac1k\sum_{r=1}^k\Lambda_r)^2\right\}^{\frac12}.\end{aligned} \tag{3.6}$$

Remark 4 *The inequality (3.6) is related to the sharp result (1.9) for problem (1.6).*

Using the Cauchy-Schwarz inequality, we derive a weaker, but more explicit upper bound of Λ_{k+1} from (3.6):

Corollary 2 *Under the assumptions of Theorem 3, we have*

$$\Lambda_{k+1} \leq (1 + \frac{2\sigma^2}{n\tau^2})\frac{1}{k}\sum_{r=1}^{k}\Lambda_r - \frac{2\sigma^2}{n\tau}V_0. \tag{3.7}$$

At the same time, an explicit estimate on the gaps of any two consecutive eigenvalues of problem (1.14) can be obtained.

Corollary 3 *Under the assumptions of Theorem 3, we have*

$$\begin{aligned}&\Lambda_{k+1} - \Lambda_k \\ \leq& 2\left\{\left[\frac{\sigma^2}{n\tau^2}(\frac{1}{k}\sum_{r=1}^{k}\Lambda_r - \tau V_0)\right]^2 - (1 + \frac{2\sigma^2}{n\tau^2})\frac{1}{k}\sum_{s=1}^{k}(\Lambda_s - \frac{1}{k}\sum_{r=1}^{k}\Lambda_r)^2\right\}^{\frac{1}{2}}.\end{aligned} \tag{3.8}$$

Acknowledgement

This work was supported by NUST Research Funding (2010ZYTS064). The author would like to express his gratitude to the referee for his valuable comments and suggestions.

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Received: 26.10.2009