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A note on weighted $A_p(G)$-modules

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Abstract

Let G be a locally compact abelian group and w be a weight function on G. In this paper, we show that the space $A_{p,w}(G)$ is a Banach module over the Figà-Talamanca Herz algebra $A_p(G)$ and study the multiplier space from $A_p(G)$ to $A_{p,w}(G)$.

Key Words: Multiplier, Order free, Essential, Banach module.

1. Introduction

Let G be a locally compact abelian group with Haar measure, w be a weight function on G and $1 < p < \infty$. The work of R. Spector [15] on $A_p(G)$ has motivated us to be interested in the structure theory of weighted $A_p(G)$ denoted by $A_{p,w}(G)$. We show that $A_{p,w}(G)$ is a Banach $A_p(G)$-module under pointwise multiplication and, as such, a fixed $v \in A_{p,w}(G)$ induces by multiplication an operator T_v from $A_p(G)$ to $A_{p,w}(G)$ defined by $T_v(u) = uv$. Following the work of Friedberg [6] we show that the compact multiplier T_v is trivial if G is a nondiscrete. We also study some multiplier problems from $A_p(G)$ to $A_{p,w}(G)$ spaces.

2. Preliminaries

Let $(A, \|\cdot\|_A)$ be a Banach algebra. A Banach space $(B, \|\cdot\|_B)$ is called a Banach A-module if there exists a continuous algebra representation T of A to $BL(B)$, the algebra of all continuous linear operators from B to B, with $\|T_a\| \leq \|a\|_A$ and $T_{a_1 \bullet a_2} = T_{a_1} \circ T_{a_2}$, where $\bullet$ is the multiplication on A. For $b \in B$, $T_a(b)$ is denoted by $a \bullet b$. Such a module is order-free if 0 is the only $b \in B$ for which $a \bullet b = 0$ for all $a \in A$. A Banach A-module B is called essential if the closed linear span of $A \bullet B$, called the essential part of B and denoted by B_e, coincides with B. If the Banach algebra $(A, \|\cdot\|_A)$ contains a bounded approximate identity, i.e., a bounded net $(u_\alpha)_{\alpha \in I}$ such that $\lim_\alpha \|u_\alpha a - a\|_A = 0$ for all $a \in A$, then a Banach A-module B is essential if and only if $\lim_\alpha \|u_\alpha b - b\|_B = 0$ for all $b \in B$ [3]. If B is a Banach A-module, then

$$\mathrm{Hom}_A(B) = \{ T \in BL(B) \mid \forall a \in A, \forall b \in B, T(a \bullet b) = a \bullet T(b) \}$$

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is the space of all A-module homomorphisms. The elements of $\mathrm{Hom}_A(B)$ are traditionally called the multipliers from B to B.

Let G be a locally compact abelian group with Haar measure dx. A continuous function on G satisfying $w(x) \geq 1$, $w(x+y) \leq w(x)\, w(y)$ for $x, y \in G$ is called a weight function. Then the space $L^p_w(G) = \{ f \mid fw \in L^p(G) \}$, $1 \leq p < \infty$, is a Banach space on G with the norm $\|f\|_{p,w} = \|fw\|_p$, and its dual space is $L^{p'}_{w^{-1}}(G)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, if $1 < p < \infty$, then $L^p_w(G)$ is a reflexive Banach space. If $p = 1$, $L^1_w(G)$ is a Banach algebra on G with respect to convolution and contains a bounded approximate identity. It is called a Beurling algebra on G [12]. We know that $L^p_w(G)$ is an essential, order-free Banach $L^1_w(G)$-module with respect to convolution [11]. We denote by $C_0(G)$ the space of all continuous functions on G vanishing at infinity, and by $C_{0,w}(G)$ the space of functions f on G such that $fw \in C_0(G)$. Throughout we assume that w is even function, i.e., w satisfies $w(x) = w(-x)$ for all $x \in G$.

The following is a classical technique of harmonic analysis.

Proposition 2.1 *Let $f \in L^p_w(G)$ and $g \in L^{p'}_{w^{-1}}(G)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then $f * g \in C_{0,w^{-1}}(G)$ and*

$$\|f * g\|_{\infty,w^{-1}} \leq \|f\|_{p,w} \|g\|_{p',w^{-1}}.$$

By Proposition 2.1, a bilinear map b can be defined from $L^p_w(G) \times L^{p'}_{w^{-1}}(G)$ into $C_{0,w^{-1}}(G)$ by

$$b(f,g) = f^{\sim} * g, \qquad f^{\sim}(x) = f(-x) \qquad f \in L^p_w(G), \quad g \in L^{p'}_{w^{-1}}(G)$$

such that $\|b\| \leq 1$. Then b lifts to a linear map B from $L^p_w(G) \otimes_\gamma L^{p'}_{w^{-1}}(G)$, the projective tensor product of $L^p_w(G)$ and $L^{p'}_{w^{-1}}(G)$ considered as a Banach space, into $C_{0,w^{-1}}(G)$ such that $\|B\| \leq 1$ (see [1], [7] and [14]).

Definition 2.2 The range of B consisting of all functions $v = \sum_{i=1}^{\infty} f_i^{\sim} * g_i$ on G with $f_i \in L^p_w(G)$, $g_i \in L^{p'}_{w^{-1}}(G)$ and $\sum_{i=1}^{\infty} \|f_i\|_{p,w} \|g_i\|_{p',w^{-1}} < \infty$ equipped with the norm

$$\|v\|_{A_{p,w}} = \inf\left\{ \sum_{i=1}^{\infty} \|f_i\|_{p,w} \|g_i\|_{p',w^{-1}} \;\middle|\; v = \sum_{i=1}^{\infty} f_i^{\sim} * g_i \right\}$$

will be denoted by $A_{p,w}(G)$.

The range $A_{p,w}(G)$ is a Banach space of functions on G and can be viewed as a subspace of $C_{0,w^{-1}}(G)$. It can also be identified with the quotient Banach space of $L^p_w(G) \otimes_\gamma L^{p'}_{w^{-1}}(G)$ with K, i.e.,

$$A_{p,w}(G) \cong L^p_w(G) \otimes_\gamma L^{p'}_{w^{-1}}(G) / K,$$

where K is the kernel of the linear form B.

Since $L^p_w(G)$ is a reflexive Banach space and a Banach $L^1_w(G)$-module with respect to convolution, by [11] we have

$$\mathrm{Hom}_{L^1_w(G)}(L^p_w(G)) \cong \left(L^p_w(G) \otimes_\gamma L^{p'}_{w^{-1}}(G) / K \right)^* \cong K^\perp \cong (A_{p,w}(G))^*.$$

Letting

$$\mathrm{Hom}_G\left(L_w^p\left(G\right)\right)=\left\{T\in BL\left(L_w^p\left(G\right)\right)\mid \forall x\in G,\ TL_x=L_xT\right\},$$

we obtain

$$\mathrm{Hom}_G\left(L_w^p\left(G\right)\right)=\mathrm{Hom}_{L_w^1(G)}\left(L_w^p\left(G\right)\right).$$

Indeed, $T\in \mathrm{Hom}_{L_w^1(G)}\left(L_w^p\left(G\right)\right)$, and let $\left(u_\alpha\right)_{\alpha\in I}$ be an approximate identity of $L_w^1\left(G\right)$. Since $L_w^p\left(G\right)$ is an essential Banach $L_w^1\left(G\right)$-module, for $f\in L_w^p\left(G\right)$, we have

$$\begin{aligned} T(L_xf) &= \lim_\alpha T(u_\alpha * L_xf)=\lim_\alpha((L_xu_\alpha)*Tf),\\ L_x(Tf) &= \lim_\alpha L_x(T(u_\alpha * f))=\lim_\alpha((L_xu_\alpha)*Tf). \end{aligned}$$

Hence $T\in \mathrm{Hom}_G\left(L_w^p\left(G\right)\right)$. Since $L_w^p\left(G\right)$ is an order-free Banach $L_w^1\left(G\right)$-module, it is easy to prove the converse. So we obtain

$$\mathrm{Hom}_G\left(L_w^p\left(G\right)\right)=\mathrm{Hom}_{L_w^1(G)}\left(L_w^p\left(G\right)\right)\cong\left(A_{p,w}(G)\right)^*.$$

Let $C_c(G)$ denote the space of continuous functions on G with compact support. Since $C_c\left(G\right)$ is dense in $L_w^p\left(G\right)$ and in $L_{w^{-1}}^{p'}\left(G\right)$, and the convolution of two functions with compact support has compact support, it follows that $C_c\left(G\right)\cap A_{p,w}(G)$ is dense in $A_{p,w}(G)$. Also $L_w^p\left(G\right)$ and $A_{p,w}(G)$ are essential Banach $L_w^1\left(G\right)$-modules under convolution [11]. Hence, for every $v\in A_{p,w}(G)$ and $x\in G$, $x\rightarrow L_xv$ is a continuous function, where $L_xv\left(y\right)=v\left(y-x\right)$ for all $y\in G$.

3. $A_{p,w}(G)$ as a Banach $A_p(G)$-module

It is well known that the Banach algebra $A_p(G)$, called the Figà-Talamanca Herz algebra, has a bounded approximate identity, i.e., there exists $\left(e_\alpha\right)_{\alpha\in I}\subset A_p\left(G\right)$ such that $\left\|e_\alpha\right\|_{A_p}\leq 1$ and $\lim_\alpha\left\|e_\alpha u-u\right\|_{A_p}=0$ for all $u\in A_p\left(G\right)$ (see [4], [5], [8] and [15]).

Theorem 3.1 *$A_{p,w}(G)$ is an essential, order-free Banach $A_p\left(G\right)$-module under pointwise multiplication.*

Proof. Let us consider two functions $u=h^\sim * k$ and $v=f^\sim * g$, where $f,g,h,k\in C_c\left(G\right)$. Then

$$\begin{aligned} \left\|u\right\|_{A_p} &\leq \left\|h\right\|_p\left\|k\right\|_{p'}\\ \left\|v\right\|_{A_{p,w}} &\leq \left\|f\right\|_{p,w}\left\|g\right\|_{p',w^{-1}}\\ u\left(x\right) &= \left(h^\sim * k\right)\left(x\right)\\ &= \int_G h\left(-x-z\right)k\left(-z\right)dz=\int_G h\left(-x-y-z\right)k\left(-y-z\right)dz\\ v\left(x\right) &= \left(f^\sim * g\right)\left(x\right)\\ &= \int_G f^\sim\left(x+y\right)g\left(-y\right)dy=\int_G f\left(-x-y\right)g\left(-y\right)dy \end{aligned}$$

$$\begin{aligned}(uv)(x) &= \int_G \int_G f(-x-y)\, g(-y)\, h(-x-y-z)\, k(-y-z)\, dy dz \\ &= \int_G A_z(x)\, dz,\end{aligned}$$

where $A_z = a_z^{\sim} * b_z$ and $a_z, b_z \in C_c(G)$ are given by $a_z(x) = f(x)\, h(x-z)$, $b_z(x) = g(x)\, k(x-z)$. Since the mapping $z \to A_z$ is continuous from G into $A_{p,w}(G)$, the integral $\int_G A_z(x) dz$ is in $A_{p,w}(G)$. By [12], it suffices to show that

$$\int_G \|A_z\|_{A_{p,w}} dz \leq \|f\|_{p,w} \|g\|_{p',w^{-1}} \|h\|_p \|k\|_{p'}.$$

It follows from the Hölder inequality that

$$\begin{aligned}\int_G \|A_z\|_{A_{p,w}} dz &\leq \int_G \|a_z\|_{p,w} \|b_z\|_{p',w^{-1}} dz \leq \\ &\leq \left(\int_G \|a_z\|_{p,w}^p dz\right)^{\frac{1}{p}} \left(\int_G \|b_z\|_{p',w^{-1}} dz\right)^{\frac{1}{p'}} \\ &= \left(\int_G |f(x)\, w(x)|^p dx \int_G |h(x-z)|^p dz\right)^{\frac{1}{p}} \cdot \\ &\quad \left(\int_G \left|g(x)\, w^{-1}(x)\right|^{p'} dx \int_G |k(x-z)|^{p'} dz\right)^{\frac{1}{p'}} \\ &= \|f\|_{p,w} \|g\|_{p',w^{-1}} \|h\|_p \|k\|_{p'}.\end{aligned}$$

So we obtain that

$$\|uv\|_{A_{p,w}} \leq \|u\|_{A_p} \|v\|_{A_{p,w}},$$

where $u \in A_p(G)$, $v \in A_{p,w}(G)$.

Since $C_c(G) \cap A_{p,w}(G)$ is dense in $A_{p,w}(G)$ and $A_p(G)$ has a bounded approximate identity, it follows that $A_{p,w}(G)$ is an essential Banach module and is order-free, i.e., if $v \in A_{p,w}(G)$ and $uv = 0$ for all $u \in A_p(G)$, then $v = 0$. □

Remark 3.2 Since $A_2(G) \subset A_r(G) \subset A_p(G)$ for $1 < p < r < 2$ or for $2 \leq r < p < \infty$, $A_{p,w}(G)$ is also $A_r(G)$-module for $1 < p < r \leq 2$ or for $2 \leq r < p < \infty$, that is, $A_r(G)\, A_{p,w}(G) \subset A_{p,w}(G)$.

In particular, for $r = 2$, since $A_2(G)$ can be identified with $L^1(\hat{G})^\wedge = \mathcal{F}(L^1(\hat{G}))$, where $\mathcal{F}$ denotes the Fourier transform, $A_{p,w}(G)$ has the structure of an $L^1(\hat{G})$-module, and the action of φ is given by $\varphi\hat{}\, h \in A_{p,w}(G)$ for $\varphi \in L^1(\hat{G})$, $h \in A_{p,w}(G)$.

4. Multipliers from $A_p(G)$ to $A_{p,w}(G)$

Proposition 4.1 $[Hom_{A_p(G)}(A_p(G), A_{p,\omega}(G))]_e \cong A_{p,\omega}(G)$.

Proof. Since $A_{p,\omega}(G)$ is an essential module, it follows from Theorem 4.5 in [13]. □

Proposition 4.2 If G is compact, then $Hom_{A_p(G)}(A_p(G), A_{p,\omega}(G))$ is an essential $A_p(G)$-module.

Proof. Since G is compact $A_{p,\omega}(G) = A_p(G)$ and $Hom_{A_p(G)}(A_p(G), A_p(G)) \cong A_p(G)$ by Theorem 5.2 in [10]. Again, using Theorem 4.5 in [13], we get the result.

Since $A_{p,w}(G)$ is a Banach $A_p(G)$-module, in particular, a fixed $v \in A_{p,w}(G)$ induces a linear operator T_v from $A_p(G)$ into $A_{p,w}(G)$ by means of

$$T_v(u) = uv, \qquad u \in A_p(G).$$

It is easy see that T_v is a multiplier from $A_p(G)$ into $A_{p,w}(G)$. □

Remark 4.3 Using the fact of the inclusion $A_{p,\omega}(G) \subset Hom_{A_p(G)}(A_p(G), A_{p,\omega}(G))$, we have the following result.

Proposition 4.4 $A_{p,\omega}(G)$ is dense in $Hom_{A_p(G)}(A_p(G), A_{p,\omega}(G))$ for the strong operator topology on $A_{p,\omega}(G)$.

Proof. Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity for $A_p(G)$ and let $T \in Hom_{A_p(G)}(A_p(G), A_{p,\omega}(G))$. Put $u_\alpha = T(e_\alpha)$. Since $A_{p,\omega}(G)$ is an essential Banach $A_p(G)$ module for every $u \in A_p(G)$, we can write

$$\|T_{u_\alpha}(u) - T(u)\| = \|u_\alpha u - T(u)\| = \|(Te_\alpha)u - T(u)\| = \|(Tu)e_\alpha - T(u)\| \to 0.$$

for all $u \in A_p(G)$. □

Theorem 4.5 *If G is a nondiscrete locally compact abelian group and $T_v : A_p(G) \to A_{p,w}(G)$ is a compact multiplier, then T_v is trivial.*

Proof. Let V be an open set in G whose closure is compact. As it is well known [9], for every $x \in V$, there exists $u \in A_p(G)$ such that 1) $u(x) = 1$; 2) supp $u \subset V$; 3) $\|u\|_{A_p} = 1$. Now assume that $v \neq 0$. Then there exists an open set V with compact closure and $\delta > 0$ such that $|v(x)| \geq \delta$ for all $x \in V$. Let (V_n) be a sequence of disjoint open sets in V such that the closure of any $V_n (n = 1, 2, ...)$ is compact. Choose a sequence (x_n) such that $x_n \in V_n (n = 1, 2, ...)$. As we already noted above, there exists a sequence (u_n) in $A_p(G)$ such that $u_n(x_m) = \delta_{nm}$ (here, δ_{nm} is the Kronecker symbol) and $\|u_n\|_{A_p} = 1$.

For $n \neq m$ we have

$$\begin{aligned}
\|T_v u_n - T_v u_m\|_{A_{p,w}} &\geq \|vu_n - vu_m\|_{\infty,w^{-1}} \\
&\geq \sup_k \left(|v(x_k)u_n(x_k) - v(x_k)u_m(x_k)| \frac{1}{w(x_k)} \right) \\
&\geq \delta \inf \left\{ \frac{1}{w(x)} \mid x \in \overline{V} \right\} \sup_k |u_n(x_k) - u_m(x_k)| \\
&= \inf \left\{ \frac{1}{w(x)} \mid x \in \overline{V} \right\}.
\end{aligned}$$

This contradicts compactness of T_v. □

Let us note that we have the inclusion

$$\mathrm{Hom}_{A_p(G)}(A_p(G), E) \subset \mathrm{Hom}_{A_p(G)}(A_p(G), A_{p,w}(G))$$

whenever E is a Banach $A_p(G)$-module contained in $A_{p,w}(G)$. Using the method in [2], we get the following theorem.

Theorem 4.6 *Let E be a Banach $A_p(G)$-module contained in $A_{p,w}(G)$. If $T \in \mathrm{Hom}_{A_p(G)}(A_p(G), A_{p,w}(G))$ such that the net $T(e_a)$ converges in E, where $(e_\alpha)_{\alpha \in I}$ is a bounded approximate identity of $A_p(G)$, then T is a multiplier from $A_p(G)$ to E.*

Proof. For $u \in A_p(G)$, consider (x_α) defined by $x_\alpha = e_\alpha u T(e_\alpha)$. Since $T \in \mathrm{Hom}_{A_p(G)}(A_p(G), A_{p,w}(G))$, we have

$$\begin{aligned}
\|x_\alpha - x_\beta\|_E &= \|e_\alpha u T(e_\alpha) - e_\beta u T(e_\beta)\|_E \\
&\leq \|e_\alpha u T(e_\beta) - e_\beta u T(e_\beta)\|_E + \|e_\alpha u T(e_\alpha) - e_\alpha u T(e_\beta)\|_E \\
&\leq \|e_\alpha u - e_\beta u\|_{A_p} \|T(e_\beta)\|_E + \|e_\alpha u\|_{A_p} \|T(e_\alpha) - T(e_\beta)\|_E \to 0,
\end{aligned}$$

which shows that (x_α) is a Cauchy net in E. Hence for $u \in A_p(G)$, the net $x_\alpha = e_\alpha u T(e_\alpha)$ converges to an element x in E. Now we want to show that x is equal to Tu as an element in $A_{p,w}(G)$.

Let $h \in A_p(G)$. Then $hx_\alpha = (he_\alpha)(uT(e_\alpha)) = (he_\alpha)(T(ue_\alpha)) \to hTu$ in $A_{p,w}(G)$. On the other hand, since $x_\alpha \to x$ in E, we have $hx_\alpha \to hx$ in $A_{p,w}(G)$. Hence we have that $x = Tu$. □

It is easy to obtain the following corollary.

Corollary 4.7 *Let E be a Banach $A_p(G)$-module contained in $A_{p,w}(G)$ and let $A = \{ T \in \mathrm{Hom}_{A_p(G)}(A_p(G), A_{p,w}(G)) \mid (T(e_\alpha))$ converges in $E \}$, where $(e_\alpha)_{\alpha \in I}$ is a bounded approximate identity of $A_p(G)$. Then the multiplier space $\mathrm{Hom}_{A_p(G)}(A_p(G), E)$ is identified with the subspace A of $\mathrm{Hom}_{A_p(G)}(A_p(G), A_{p,w}(G))$.*

Similarly, if we consider that $A_{p,w}(G)$ is a Banach $L^1_w(G)$-module under convolution, we get the following theorem.

Theorem 4.8 *Let* $E \subseteq A_{p,w}(G)$ *two Banach* $L_w^1(G)$ *-modules then* $T \in Hom_{L_w^1(G)}(L_w^1(G), A_{p,w}(G))$ *is a multiplier from* $L_w^1(G)$ *to* E *if and only if* $(Te_\alpha) \in E$ *and* $\sup_\alpha \|(Te_\alpha)\|_E < \infty$ *where* $(e_\alpha)_{\alpha \in I}$ *is a bounded approximate identity of* $L_w^1(G)$.

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