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Invariant subspace problem for positive L-weakly and M-weakly compact operators

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Abstract

In this paper, we show that positive L-weakly and M-weakly compact operators on some real Banach lattices have a non-trivial closed invariant subspace. Also, we prove that any positive L-weakly (or M-weakly) compact operator $T : E \to E$ has a non-trivial closed invariant subspace if there exists a Dunford-Pettis operator $S : E \to E$ satisfying $0 \leq T \leq S$, where E is Banach lattice.

Key word and phrases: Invariant subspace, L- and M-weakly compact operator, Polynomially L-weakly (M-weakly) compact operator, Dunford-Pettis operator.

1. Introduction

Bulk of the papers in Banach lattice theory concern the open problem every positive operator on a Banach lattice of dimension at least two has a non-trivial closed invariant subspace. The problem was solved for positive compact operators on Banach lattices [1], [7], [12].

Our objective in this work is to investigate whether or not every positive L-weakly and M-weakly operator on a real Banach lattice does posses a non-trivial closed invariant subspace. First, we will prove that every L-weakly compact operator on a Banach lattice without order continuous norm has a non-trivial closed invariant subspace. Also, we shall show that if E is a Banach lattice such that either E and $E^{'}$ has order continuous norm, then every bounded operator that commutes with a positive L-weakly (M-weakly) compact operator that on E has a non-trivial closed invariant subspace. Furthermore, we will investigate invariant subspaces of polynomially L-weakly (M-weakly) compact operators for Banach lattices without order continuous norm. Next, we will prove that every bounded operator that commutes with a positive M-weakly (or positive L-weakly) compact operator on a Banach lattice E has a non-trivial closed invariant subspace if it is dominated by a Dunford-Pettis operator. Also, we will see that any positive operator on a Banach lattice of which order dual has order continuous norm has a non-trivial closed invariant subspace if it is dominated by a Dunford-Pettis operator.

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2. Main results

Throughout this paper, unless otherwise state, E will denote an infinite dimensional separable real Banach lattice with norm dual $E^{'}$ and all operators on Banach lattices will be assumed to be non-scalar and non-zero. In the rest of this article, by the term "an operator" between two Banach lattices, we shall mean "a linear norm bounded operator". We refer the reader to [5], [14] and [16] for any unexplained terms from Banach lattice theory and for further details on the theory of invariant subspaces see [1], [2], [15].

Recall that a non-empty bounded subset A of Banach lattice E is said to be L-weakly compact if $\|x_n\| \to 0$ for every disjoint sequence (x_n) in the solid hull of A. A bounded linear operator T from a Banach space X into E is called *L-weakly compact* if $T(U_X)$ is L-weakly compact in E, where U_X denotes the closed unit ball of X. A bounded linear operator from E into X is *M-weakly compact* if $\|Tx_n\| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in U_E. That L-weakly compact and M-weakly compact operators are weakly compact operators was shown by P. Meyer-Nieberg [13]. It is known that

$$E^a = \{x \in E : \text{ every monoton sequence in } [0, |x|] \text{ is convergent}\}$$

is the maximal closed order ideal in E on which the induced norm is order continuous and it is known that any L-weakly compact subset is contained in E^a ([14], p. 92 and p. 212). Recall that a Banach lattice E is said to have an *order continuous norm* if $x_\alpha \downarrow 0$ in E implies $\|x_\alpha\| \downarrow 0$.

Proposition 1 *If E is a Banach lattice without order continuous norm, then every L-weakly compact operator $T : E \to E$ has a non-trivial closed invariant ideal.*

Proof. If E does not have order continuous norm, $E^a \neq E$ since E^a has order continuous norm. Take any $x \in E^a$ (we can suppose that $\|x\| \leq 1$ without loss of generality). $T(U_E) \subset E^a$ since T is an L-weakly compact operator and every L-weakly compact subset is contained in E^a. Therefore $T(x) \in E^a$ i.e. , $T(E^a) \subseteq E^a$. If $E^a = \{0\}$ then $T(U_E) = \{0\}$ which implies that $T = 0$. Since we take T different from zero operator, $E^a \neq \{0\}$. Hence E^a is the invariant ideal for T which we are looking for. □

Corollary 1 *Let E be a Banach lattice without order continuous norm, let $E^a \neq \{0\}$ and let $T : E \to E$ be a non-scalar regular operator. If there exists some element $0 < x_0 \in E^a$ and $n \in \mathbb{N}$ such that $T^n x_0 \neq 0$ and $T^n : E \to E$ is an L-weakly compact operator, then every regular operator on E has a non-trivial invariant closed order ideal.*

Proof. Assume that $T^n : E \to E$ is an L-weakly compact operator for some $n \in \mathbb{N}^+$ and $S : E \to E$ is any regular operator. Since in $\mathcal{L}^r(E)$,the space of the regular operators on E, the L-weakly compact regular operators form closed two-sided ideal, $S^m T^n : E \to E$ is an L-weakly compact operator for each $m \in \mathbb{N}^+$. Thus, $S^m T^n(x_0) \in E^a$ for all $m \in \mathbb{N}^+$. Next, choose the closed order ideal W generated by

$$\{T^n x_0, ST^n x_0, S^2 T^n x_0, ..., S^m T^n x_0, ...\}.$$

Hence, $S(W) \subseteq W$, it follows that W is a non-trivial invariant closed order ideal for S. □

The order ideal generated by an element $0 < x \in E$ is precisely

$$I_x = \{y \in E : \exists\ \lambda > 0 \text{ with } |y| \leq \lambda x\}.$$

For every $z \in I_x$,

$$\|z\|_\infty = \inf\{\lambda > 0;\ |z| \leq \lambda x\}$$

defines a norm on I_x. Thus, $(I_x, \|.\|_\infty)$ is an AM-space and, moreover, its closed unit ball is the interval $[-x, x]$, see [4],[5],[14].

Theorem 1 *If $T : E \to E$ is a positive L-weakly compact operator on a Banach lattice E with order continuous norm, then the operator T^2 is compact.*

Proof. If $T : E \to E$ is a positive L-weakly compact operator, then for each $n \in \mathbb{N}^+$ there exists some $0 < u_n \in E$ lying in the order ideal generated by $T(E)$ satisfying

$$\left\|(|Tx| - u_n)^+\right\| < n^{-1}$$

for all $x \in U_E$ ([5], Th. 18.9, p. 313). From the identity $|Tx| = |Tx| \wedge u_n + (|Tx| - u_n)^+$ we see that

$$T\left(U_E^+\right) \subseteq [0, u_n] + n^{-1}U_E \tag{2.1}$$

for all $n \in \mathbb{N}^+$. Let $0 < y = \sum_{n=1}^{\infty} \frac{1}{2^n\|u_n\|}u_n \in E^+$ and let I_y be the order ideal generated by y. The restriction $T|_{I_y} : (I_y, \|.\|_\infty) \to E$ is a positive L-weakly compact operator, and $(I_y, \|.\|_\infty)'$ is an AL-space [([5], Th. 18.11, p. 315). Thus, since I_y is an AM-space, it satisfies Dunford-Pettis property and so $T|_{I_y}$ is a Dunford-Pettis operator by ([5], Theorems 19.4 and 19.6). Moreover we know that I_y' is an AL-space with order continuous norm and E has order contiuous norm by hypothesis, therefore, $T|_{I_y} : I_y \to E$ is compact ([6], Theorem 2.12(2)ii). Let $\alpha_n = 2^n\|u_n\|$ for every $n \in \mathbb{N}^+$. From the inequality $u_n \leq \alpha_n y$ and (2.1) we obtain that

$$T^2\left(U_E^+\right) \subseteq T[0, u_n] + n^{-1}\|T\|U_E \subseteq \alpha_n T[0, y] + n^{-1}\|T\|U_E$$

and

$$T^2\left(U_E^+\right) \subseteq \alpha_n T[0, y] + n^{-1}\|T\|U_E = \alpha_n\, T|_{I_y}[0, y] + n^{-1}\|T\|U_E.$$

Since $\left\{n^{-1}\|T\|U_E\right\}$ is a norm-neighborhood system at zero, $T^2\left(U_E^+\right)$ is a norm-totally bounded set, from which it follows that the operator T^2 is compact. □

Corollary 2 *Every L-weakly compact operator on a Banach lattice has a non-trivial closed invariant subspace.*

As an immediate consequence of Theorem 1, we obtain the following result:

Corollary 3 *If T is a positive M-weakly compact operator on a Banach lattice E such that E' has order continuous norm, then T^2 is compact.*

Proof. It is enough to consider that T' is a positive L-weakly compact operator. $\square$

Theorem 2 *Let E and F be a Banach lattice and $T : E \to F$ be a positive M-weakly compact operator. If F has order continuous norm, then the operator T is compact.*

Proof. If $T : E \to F$ is a positive M-weakly compact operator, then for each $n \in \mathbb{N}^+$ there exists some $0 < u_n \in E$ such that

$$\left\| T\left(|x| - u_n\right)^+ \right\| < n^{-1}$$

holds for all $x \in U_E$ ([5], Th. 18.9, p.313).

Let $y = \sum_{n=1}^{\infty} \frac{1}{2^n \|u_n\|} u_n \in E^+$ and let I_y be the order ideal generated by y. It is clear that y is well defined. Since the operator $T : E \to F$ is M-weakly compact, the operator $T|_{I_y} : (I_y, \|\centerdot\|_\infty) \to F$ is M-weakly compact, and so $T|_{I_y}$ is compact by Theorem 2.12. in [6]. Let α_n be as mentioned in the proof of above theorem. From the inequality $u_n \le \alpha_n y$ and the identity $|x| = |x| \wedge u_n + (|x| - u_n)^+$ we see that

$$T\left(U_E^+\right) \subseteq T\left[0, u_n\right] + n^{-1} U_F \subseteq \alpha_n T\left[0, y\right] + n^{-1} U_F.$$

Thus, we obtain that

$$T\left(U_E^+\right) \subseteq \alpha_n T\left[0, y\right] + n^{-1} U_F = \alpha_n \, T|_{I_y}\left[0, y\right] + n^{-1} U_F.$$

Moreover, $T|_{I_y}[0, y] \subset F$ is a norm-totally bounded set since $T|_{I_y}$ is a compact operator. Hence, $T\left(U_E^+\right)$ is also a norm-totally bounded set because $\left\{n^{-1} U_F\right\}$, $n \in \mathbb{N}^+$ is a norm-neighborhood system at zero. Therefore, T is a compact operator, as desired. $\square$

Since the dual of an L-weakly compact operator is M-weakly compact, we can state the following result:

Corollary 4 *Let $T : E \to F$ be a positive L-weakly compact operator between Banach lattices. If E' has order continuous norm, then T is compact.*

We obtain anohter consequence of Theorem 1 and 2:

Corollary 5 *If E is a Banach lattice such that either E or E' has order contiuous norm, then every bounded operator that commutes with a positive L-weakly (positive M-weakly) compact operator on E has a non-trivial closed invariant subspace.*

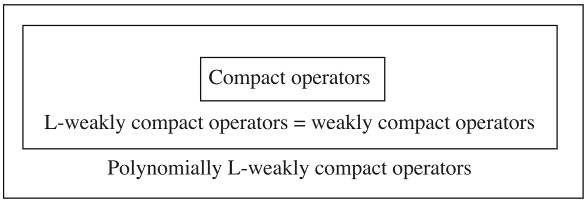
Let T be a bounded operator on a Banach lattice E. T is said to be a *polynomially L-weakly* (*polynomially M-weakly*) *compact operator* whenever there exists a non-zero polynomial p such that $p(T)$ is L-weakly (M-weakly) compact. It is clear that every L-weakly compact operator on a Banach lattice is polynomially L-weakly compact, but sometimes the converse of this statement is false even for AL-spaces. The following example illustrates this point.

Example 1 *Define an operator* $T : L^1[0,1] \to L^1[0,1]$ *by*

$$Tf(x) := \begin{cases} 0 & \text{if} \quad 0 \le x \le 1/2 \\ f\left(x - \frac{1}{2}\right) & \text{if} \quad 1/2 < x \le 1, \end{cases}$$

then T *is not weakly compact, but* T *is polynomially* L*-weakly compact since* $T^2 = 0$.

The following diagram is held for all operators defined on AL-spaces.



Again, we will consider Banach lattices in different properties while we seek invariant subspaces for polynomially L-weakly compact operators.

Theorem 3 *Every polynomially* L*-weakly compact operator on a Banach lattice without order continuous norm has a non-trivial closed invariant subspace.*

Proof. Let E be a Banach lattice without order continuous norm and $T : E \to E$ be a polynomially L-weakly compact operator. Choose a non-zero polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + ... + a_{n-1}t^{n-1} + a_n t^n$ such that $p(t)$ is L-weakly compact.

Assume that $p(T) = 0$ (In this case, there exists a non-zero element x in E such that $p(T)(x) = 0$). Let V denote the non-zero closed subspace generated by the set $\{x, Tx, ..., T^{n-1}x\}$. Since E is infinite dimensional, we have $V \neq E$ and we can see easily that V is T-invariant.

Now suppose that $p(T) \neq 0$. Fix any non-zero vector $x_0 \in E^a$ such that $p(T)(x_0) \neq 0$. Since $p(T)$ is L-weakly compact, we have $p(T)(E^a) \subset E^a$ by Prop.1. Moreover, for each $k = 0, 1, 2...$, $T^k p(T)(E^a) = p(T) T^k (E^a) \subset E^a$ holds because of L-weakly compactness of $p(T) T^k$. Let V be the non-zero closed subspace generated by the set $\{p(T)(x_0), Tp(T)(x_0), T^2 p(T)(x_0), ..., T^k p(T)(x_0), ...\} \subset E^a$. Since E does not have order continuous norm, again we have $V \neq E$, and it can be seen that V is T-invariant. □

Remark 1 *Not every operator* $p(T)$, *where p is a polynomial, is positive when* T *is positive. Thus we cannot say that every polynomially* L*-weakly compact operator on a Banach lattice with order continuous norm has a non-trivial invariant subspace.*

Let X, Y be Banach spaces and let E be a Banach lattice. A linear operator $T : E \to X$ is called AM-*compact* if $T[-x, x]$ is relatively compact for every $x \in E^+$. And, we say that $T : X \to Y$ is a *Dunford-Pettis operator* whenever $x_n \xrightarrow{w} 0$ in X implies $\lim \|Tx_n\| = 0$ [5], [10], [11], [14]. The o-weakly compact operators have been characterized by P.G. Dodds [8]. Recall that a continuous operator $T : E \to X$ is $o-$*weakly compact* whenever $T[0, x]$ is a relatively weakly compact subset of X for each $0 < x \in E$. It is clear that every weakly compact operator is o-weakly compact.

Theorem 4 *Let E be a Banach lattice and $T : E \to E$ be a positive M-weakly compact operator. If there exists a Dunford-Pettis operator $S : E \to E$ such that $0 \leq T \leq S$, then the operator T^3 is compact.*

Proof. Assume that $T : E \to E$ is a positive M-weakly compact operator and $S : E \to E$ is a Dunford-Pettis operator satisfying $0 \leq T \leq S$. Thus, $T^2 : E \to E$ is a Dunford-Pettis operator by ([5], Cor. 19.15, p. 340). Moreover, the operator T is o-weakly compact because it is an M-weakly compact operator ([5], p. 311). Thus, $T[0,x]$ is a relatively weakly compact set for every $x \in E^+$. Therefore, $T^2(T[0,x]) = T^3[0,x]$ is a norm-totally bounded set for every $x \in E^+$ since the Dunford-Pettis operator T^2 carries relatively weakly compact subsets of E onto norm-totally bounded subsets of E ([5], Th. 19.3, p. 334). It follows that the M-weakly compact operator T^3 is AM-compact. This implies that T^3 is a compact operator ([14], Prop. 3.7.4, p.219). □

Corollary 6 *Let E be a Banach lattice and $T : E \to E$ be a positive M-weakly compact operator. If there exists a Dunford-Pettis operator $S : E \to E$ such that $0 \leq T \leq S$, then every bounded operator that commutes with T has a non-trivial closed invariant subspace.*

We can state a similar theorem to the previous theorem for L-weakly compact operators.

Theorem 5 *Let E be a Banach lattice and $T : E \to E$ be a positive L-weakly compact operator. If there exists a Dunford-Pettis operator $S : E \to E$ satisfying $0 \leq T \leq S$, then T^4 is a compact operator.*

Proof. We already known that T^2 is a Dunford-Pettis operator and the operator T is o-weakly compact as mentioned in Theorem 4. Moreover, the Dunford-Pettis operator T^2 carries the relatively weakly compact sets $T[0,x]$ for each $x \in E^+$ onto norm-totally bounded sets $T^3[0,x]$. Since the operator T is a L-weakly compact, for each $\varepsilon > 0$ there exists some $x_\varepsilon \in E^+$ such that

$$T\left(U_E^+\right) \subseteq [0, x_\varepsilon] + \varepsilon U_E.$$

Thus, we obtain that

$$T^4\left(U_E^+\right) \subseteq T^3[0, x_\varepsilon] + \varepsilon \|T\|^3 U_E,$$

and so, $T^4\left(U_E^+\right)$ is a norm-totally bounded set because $\left\{\varepsilon \|T\|^3 U_E\right\}$ is a base at zero. □

Corollary 7 *Let E be a Banach lattice and $T : E \to E$ be a positive L-weakly compact operator. If there exists a Dunford-Pettis operator $S : E \to E$ satisfying $0 \leq T \leq S$, then every bounded operator that commutes with T has a non-trivial closed invariant subspace.*

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