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Domination polynomials of cubic graphs of order 10

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Abstract

Let G be a simple graph of order n. The domination polynomial of G is the polynomial $D(G,x) = \sum_{i=\gamma(G)}^{n} d(G,i)x^i$, where $d(G,i)$ is the number of dominating sets of G of size i, and $\gamma(G)$ is the domination number of G. In this paper we study the domination polynomials of cubic graphs of order 10. As a consequence, we show that the Petersen graph is determined uniquely by its domination polynomial.

Key Words: Domination polynomial, equivalence class, petersen graph, cubic graphs

1. Introduction

Let $G=(V,E)$ be a simple graph. The *order* of G is the number of vertices of G. For any vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a *dominating set* if $N[S] = V$, or equivalently, every vertex in $V \backslash S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A dominating set with cardinality $\gamma(G)$ is called a γ*-set*. The family of all γ-sets of a graph G is denoted by $\Gamma(G)$. For a detailed treatment of these parameters, the reader is referred to [9].

An i*-subset* of $V(G)$ is a subset of $V(G)$ of size i. Let $\mathcal{D}(G,i)$ be the family of dominating sets of a graph G with cardinality i and let $d(G,i) = |\mathcal{D}(G,i)|$. The *domination polynomial* $D(G,x)$ of G is defined as $D(G,x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G,i)x^i$, where $\gamma(G)$ is the domination number of G (see [7]). For more information on this polynomial, refer to [2, 4, 5, 6, 7].

We denote the family of all dominating sets of G with cardinality i and contain a vertex v by $\mathcal{D}_v(G,i)$, and $d_v(G,i) = |\mathcal{D}_v(G,i)|$. Two graphs G and H are said to be *dominating equivalent*, or simply $\mathcal{D}$-equivalent, written $G \sim H$, if $D(G,x) = D(H,x)$. It is evident that the relation $\sim$ of $\mathcal{D}$-equivalent is an equivalence relation on the family $\mathcal{G}$ of graphs, and thus $\mathcal{G}$ is partitioned into equivalence classes, called the $\mathcal{D}$*-equivalence classes*. Given $G \in \mathcal{G}$, let

$$[G] = \{H \in \mathcal{G} : H \sim G\}.$$

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We call $[G]$ the equivalence class determined by G. A graph G is said to be *dominating unique*, or simply $\mathcal{D}$*-unique*, if $[G]=\{G\}$. Similarly to chromatic polynomial and chromaticity of graphs (see [8]), there are two interesting problems on equivalence classes:

(i) Which graphs are $\mathcal{D}$-unique?

(ii) Determine the $\mathcal{D}$-equivalence classes for some families of graphs.

Recently, study of the two above problems have been considered and the $\mathcal{D}$-equivalence classes of some specific graphs, as paths ([2]), cycles ([2, 3]) and complete bipartite graphs ([1]) are determined.

The minimum degree of G is denoted by $\delta(G)$. A graph G is called k*-regular* if all vertices have the same degree k. A *vertex-transitive graph* is a graph G such that for every pair of vertices v and w of G, there exists an automorphism θ such that $\theta(v)=w$. One famous graphs is the Petersen graph which is a symmetric non-planar cubic graph. In the study of domination polynomials, it is interesting to investigate the dominating sets and associated domination polynomial of this graph. We denote the Petersen graph by P.

In this paper, we study the dominating sets and domination polynomials of cubic graphs of order 10. In the next section, we obtain the domination polynomial of the Petersen graph. In Section 3, we list all γ-sets of connected cubic graphs of order 10. This list will be used to study the $\mathcal{D}$-equivalence of these graphs in the last section. In Section 4, we prove that the Petersen graph is $\mathcal{D}$-unique. In the last section, we study the $\mathcal{D}$-equivalence classes of some cubic graphs of order 10.

2. Domination polynomial of the Petersen graph

In this section we shall investigate the domination polynomial of the Petersen graph. First, we state and prove the following lemma.

Lemma 1 *Let G be a vertex transitive graph of order n and $v\in V(G)$. For any $1\le i\le n$, $d(G,i)=\frac{n}{i}d_v(G,i)$.*

Proof. Clearly, if D is a dominating set of G with size i, and $\theta\in Aut(G)$, then $\theta(D)$ is also a dominating set of G with size i. Since G is a vertex transitive graph, then for every two vertices v and w, $d_v(G,i)=d_w(G,i)$. If D is a dominating set of size i, then there are exactly i vertices $v_{j_1},\ldots,v_{j_i}$ such that D counted in $d_{v_{j_r}}(G,i)$, for each $1\le r\le i$. Hence $d(G,i)=\frac{n}{i}d_v(G,i)$, and the proof is complete. □

Theorem 1 ([9], p.48) *If G is a connected graph of order n with $\delta(G)\ge 3$, then $\gamma(G)\le\frac{3n}{8}$.*

We need the following theorem for finding the domination polynomial of the Petersen graph.

Theorem 2 ([2]) *Let G be a graph of order n with domination polynomial $D(G,x)=\sum_{i=1}^{n}d(G,i)x^i$. If $d(G,j)=\binom{n}{j}$ for some j, then $\delta(G)\ge n-j$. More precisely, $\delta(G)=n-l$, where $l=\min\left\{j|d(G,j)=\binom{n}{j}\right\}$, and there are at least $\binom{n}{l-1}-d(G,l-1)$ vertices of degree $\delta(G)$ in G. Furthermore, if for every two vertices* *of degree* $\delta(G)$, *say* u *and* v *we have* $N[u] \neq N[v]$, *then there are exactly* $\binom{n}{l-1} - d(G, l-1)$ *vertices of degree* $\delta(G)$.

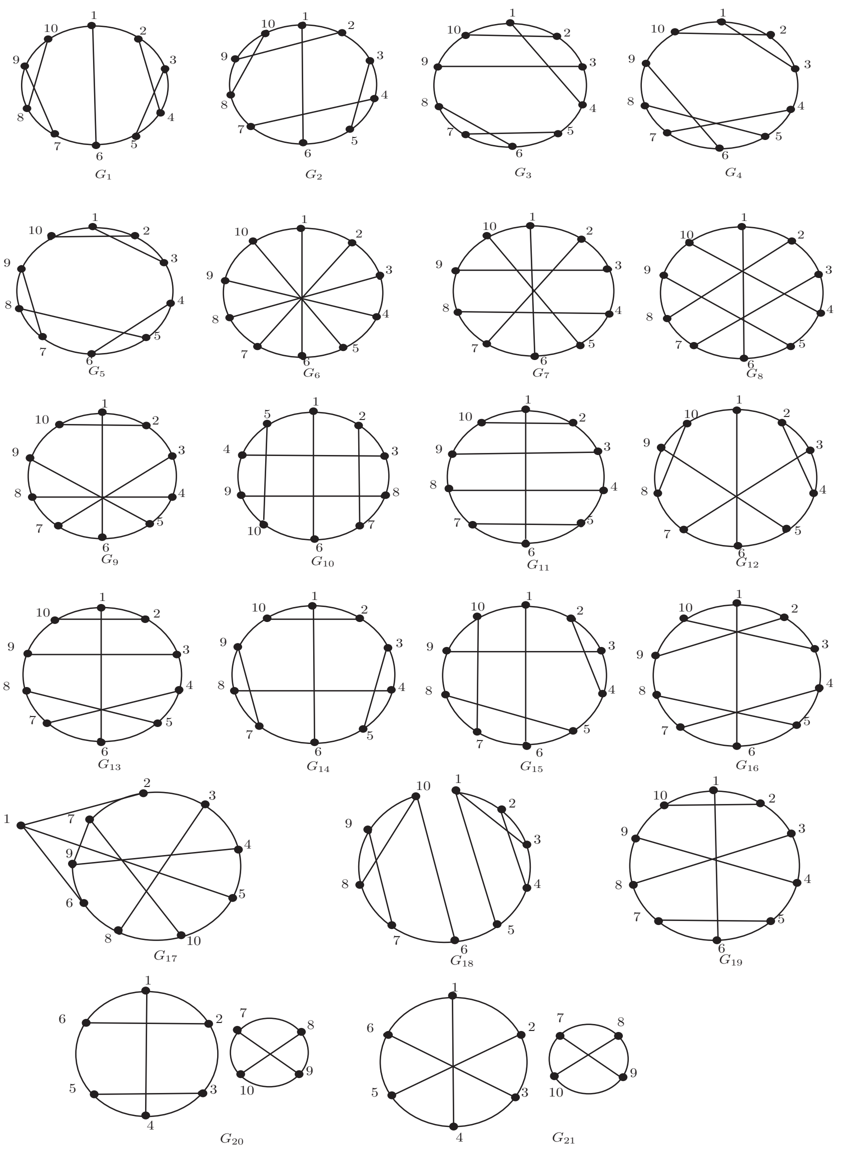

Figure 1. Cubic graphs of order 10.

Indeed, by Theorem 2, we have the following theorem which relates the domination polynomial and the regularity of a graph G.

Theorem 3 *Let* H *be a* k*-regular graph, where for every two vertices* $u, v \in V(H)$, $N[u] \neq N[v]$. *If* $D(G, x) = D(H, x)$, *then* G *is also a* k*-regular graph.*

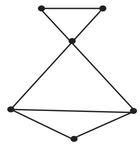

Figure 2. Graph G with $N[u]=N[v]$ for some $u,v\in V(G)$.

Remark. The condition $N[u]\neq N[v]$ in Theorems 2 and 3 is necessary. For example, consider graph in the Figure 2. The domination polynomial of this graph is $x^6+6x^5+15x^4+18x^3+9x^2$. For this graph $l=4$, $\delta(G)=2$, and we have 3 vertices of degree $\delta(G)$, but by Theorem 2, we must have $\binom{6}{3}-d(G,3)=2$ vertices of degree $\delta(G)$.

There are exactly 21 cubic graphs of order 10 given in Figure 1 (see [10]). Using Theorem 1, the domination number of a connected cubic graph of order 10 is 3. Three are just two non-connected cubic graphs of order 10. Clearly, for these graphs, the domination number is also 3. Note that the graph G_{17} is the Petersen graph. For the labeled graph G_{17} in Figure 1, we obtain all dominating sets of size 3 and 4 in the following lemma.

Lemma 2 *For the Petersen graph P, $d(P,3)=10$ and $d(P,4)=75$.*

Proof. First, we list all dominating sets of P of cardinality 3, which are the γ-sets of the labeled Petersen graph (graph G_{17}) given in Figure 1.

$\mathcal{D}(P,3)=\Big\{\{1,3,7\},\{1,4,10\},\{1,8,9\},\{2,4,8\},\{2,5,6\},\{2,9,10\},\{3,5,9\},\{3,6,10\},$

$\{4,6,7\},\{5,7,8\}\Big\}$. Now, we shall compute $d(P,4)$. By Lemma 1, it suffices to obtain the dominating sets of cardinality 4 containing one vertex, say the vertex labeled 1. These dominating sets are listed below:

$\mathcal{D}_1(P,4)=\Big\{\{1,2,3,7\},\{1,2,4,8\},\{1,2,4,10\},\{1,2,5,6\},\{1,2,8,9\},\{1,2,9,10\},$

$\{1,3,4,7\},\{1,3,4,10\},\{1,3,5,7\},\{1,3,5,9\},\{1,3,6,7\},\{1,3,6,10\},\{1,3,7,8\},\{1,3,7,9\},$

$\{1,3,7,10\},\{1,3,8,9\},\{1,3,9,10\},\{1,4,5,10\},\{1,4,6,7\},\{1,4,6,10\},\{1,4,7,8\},$

$\{1,4,7,10\},\{1,4,8,9\},\{1,4,8,10\},\{1,4,9,10\},\{1,5,7,8\},\{1,5,8,9\},\{1,6,8,9\},$

$\{1,7,8,9\},\{1,8,9,10\}\Big\}$.

Therefore by Lemma 1, $d(P,4)=\frac{10\times 30}{4}=75$. □

We need the following lemma.

Lemma 3 *Let G be a cubic graph of order 10. Then the following hold:*

(i) $d(G,i)=\binom{n}{i}$, *for* $i=7,8,9,10$.

(ii) if t and s are the number of subgraphs isomorphic to $K_4\backslash\{e\}$ (e is an edge) and K_4 in G, respectively, then $d(G,6)=\binom{10}{6}-(10-t-3s)$.

(iii) if G has no subgraph isomorphic to graphs given in Figure 3, then $d(G,5)=\binom{10}{5}-60$.

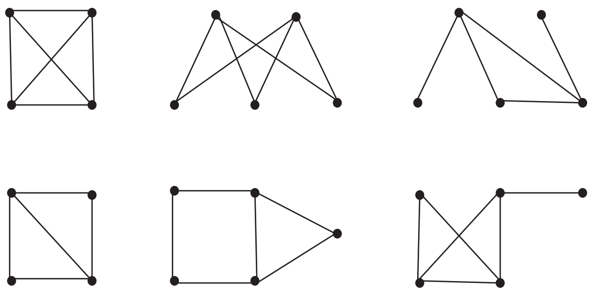

Figure 3. Graphs illustrated in Lemma 3.

Proof. (i) This follows from Theorem 2.

(ii) If G is a cubic graph of order 10, then for every $v \in V(G)$, $V(G)\backslash N[v]$ is not a dominating set. Also, if $S \subset V(G)$, $|S| = 6$ and S is not a dominating set, then $S = V(G)\backslash N[v]$, for some $v \in V(G)$. Note that if G has $K_4\backslash\{e\}$ as a subgraph, then there are two vertices u_1 and u_2 such that $G\backslash N[u_1] = G\backslash N[u_2]$. Also if G has K_4 as its subgraph, then there are four vertices u_i, $1 \leq i \leq 4$ such that $G\backslash N[u_i] = G\backslash N[u_j]$, for $1 \leq i \neq j \leq 4$. Hence we have $d(G,6) = \binom{10}{6} - (10 - t - 3s)$.

(iii) It suffices to determine the number of 5-subsets which are not dominating set. Suppose that $S \subseteq V(G)$, $|S| = 5$, and S is not a dominating set for G. Thus there exists $v \in V(G)$ such that $N[v] \cap S = \emptyset$. Now, note that for every $x \in V(G)$, $V(G)\backslash(N[x] \cup \{y\})$, where $y \in V(G)\backslash N[x]$ is a 5-subset which is not a dominating set for G. Also since none of the graphs given in Figure 3 is a subgraph of G, for every two distinct vertices x and x' and any two arbitrary vertices $y \in V(G)\backslash N[x]$ and $y' \in V(G)\backslash N[x']$, we have $V(G)\backslash(N[x] \cup \{y\}) \neq V(G)\backslash(N[x'] \cup \{y'\})$. This implies that the number of 5-subsets of $V(G)$ which are not dominating sets is $10 \times 6 = 60$. So we have $d(G,5) = \binom{10}{5} - 60$. □

Corollary 1 *For cubic graphs of order* 10*, the following hold:*

(i) *If* $G \in \{G_{20}, G_{21}\}$, *then* $d(G,6) = \binom{10}{6} - 7$.

(ii) *If* $G \in \{G_1, G_{18}\}$, *then* $d(G,6) = \binom{10}{6} - 8$.

(iii) *If* $G \in \{G_3, G_5\}$, *then* $d(G,6) = \binom{10}{6} - 9$.

(iv) *For each* $i, 1 \leq i \leq 21$, *if* $i \notin \{1,3,5,18,20,21\}$, *then* $d(G_i,6) = \binom{10}{6} - 10$.

(v) *If* $G \in \{G_6, G_7, G_8, G_{10}, G_{17}\}$, *then* $d(G,5) = \binom{10}{5} - 60$.

Theorem 4 *The domination polynomial of the Petersen graph* P *is*

$$D(P,x) = x^{10} + \binom{10}{9}x^9 + \binom{10}{8}x^8 + \binom{10}{7}x^7 + \left[\binom{10}{6} - 10\right]x^6 + \left[\binom{10}{5} - 60\right]x^5 + 75x^4 + 10x^3.$$

Proof. The result follows from Lemma 2 and Corollary 1. □

3. γ-Sets of cubic graphs of order 10.

In this section, we present all γ-sets of connected cubic graphs $G_1, G_2, \ldots, G_{18}, G_{19}$ shown in Figure 1. The results here will be useful in studying the $\mathcal{D}$-equivalent of these graphs in the last section.

$\Gamma(G_1) = \Big\{\{1,3,7\},\{1,3,8\},\{1,3,9\},\{1,4,7\},\{1,4,8\},\{1,4,9\},\{1,5,7\},\{1,5,8\},$
$\{1,5,9\},\{2,5,8\},\{2,5,9\},\{2,6,8\},\{2,6,9\},\{2,6,10\},\{3,6,8\},\{3,6,9\},\{3,6,10\},$
$\{3,7,10\},\{4,6,8\},\{4,6,9\},\{4,6,10\},\{4,7,10\}\Big\}$. Therefore, $d(G_1,3) = \big|\Gamma(G_1)\big| = 22$.

$\Gamma(G_2) = \Big\{\{1,3,8\},\{1,4,8\},\{1,4,9\},\{1,4,10\},\{1,5,8\},\{2,5,8\},\{3,6,8\},\{3,6,9\},$
$\{3,6,10\},\{3,7,10\},\{4,6,9\},\{5,6,9\}\Big\}$. Therefore $d(G_2,3) = \big|\Gamma(G_2)\big| = 12$.

$\Gamma(G_3) = \Big\{\{1,3,6\},\{1,3,7\},\{1,4,8\},\{1,5,9\},\{1,6,9\},\{1,7,9\},\{2,3,6\},\{2,3,7\},$
$\{2,4,8\},\{2,5,8\},\{2,5,9\},\{3,6,10\},\{3,7,10\},\{4,6,10\},\{4,7,10\},\{4,8,10\},\{5,9,10\}\Big\}$.
Therefore $d(G_3,3) = \big|\Gamma(G_3)\big| = 17$.

$\Gamma(G_4) = \Big\{\{1,5,6\},\{1,5,8\},\{1,6,7\},\{1,7,8\},\{2,4,9\},\{2,5,6\},\{2,5,8\},\{2,6,7\},$
$\{2,7,8\},\{3,4,9\},\{3,6,9\},\{3,8,9\},\{4,5,10\},\{4,7,10\},\{4,9,10\}\Big\}$. Therefore $d(G_4,3) = \big|\Gamma(G_4)\big| = 15$.

$\Gamma(G_5) = \Big\{\{1,4,7\},\{1,4,8\},\{1,4,9\},\{1,5,7\},\{1,5,8\},\{1,5,9\},\{1,6,7\},\{1,6,8\},$
$\{1,6,9\},\{2,4,7\},\{2,4,8\},\{2,4,9\},\{2,5,7\},\{2,5,8\},\{2,5,9\},\{2,6,7\},\{2,6,8\},\{2,6,9\},$
$\{3,4,9\},\{3,5,9\},\{3,6,9\},\{4,7,10\},\{4,8,10\},\{4,9,10\}\Big\}$. Therefore $d(G_5,3) = \big|\Gamma(G_5)\big| = 24$.

$\Gamma(G_6) = \Big\{\{1,4,7\},\{1,4,8\},\{1,5,8\},\{2,5,8\},\{2,5,9\},\{2,6,9\},\{3,6,9\},\{3,6,10\},$
$\{3,7,10\},\{4,7,10\}\Big\}$. Therefore $d(G_6,3) = \big|\Gamma(G_6)\big| = 10$.

$\Gamma(G_7) = \Big\{\{1,4,8\},\{2,5,8\},\{2,5,9\},\{3,6,9\},\{3,7,10\},\{4,7,10\}\Big\}$.

Therefore $d(G_7,3) = \big|\Gamma(G_7)\big| = 6$

$\Gamma(G_8) = \Big\{\{1,3,9\},\{1,4,8\},\{1,5,7\},\{2,6,10\},\{3,6,9\},\{4,6,8\}\Big\}$.
Therefore $d(G_8,3) = \big|\Gamma(G_8)\big| = 6$.

$\Gamma(G_9) = \Big\{\{1,3,9\},\{1,4,8\},\{1,5,7\},\{2,5,7\},\{2,5,8\},\{2,6,8\},\{3,6,9\},\{4,6,10\},$
$\{4,7,10\},\{5,7,10\}\Big\}$. Therefore $d(G_9,3) = \big|\Gamma(G_9)\big| = 10$.

$\Gamma(G_{10}) = \Big\{\{1,2,9\},\{1,5,8\},\{1,8,9\},\{2,3,10\},\{2,9,10\},\{3,4,6\},\{3,6,10\},\{4,5,7\},$
$\{5,7,8\},\{4,6,7\}\Big\}$. Therefore $d(G_{10},3) = \big|\Gamma(G_{10})\big| = 10$.

$\Gamma(G_{11}) = \Big\{\{1,3,7\},\{1,5,9\},\{2,3,7\},\{2,5,8\},\{2,5,9\},\{2,6,8\},\{2,7,8\},\{3,6,9\},$
$\{3,7,10\},\{4,5,10\},\{4,6,10\},\{4,7,10\}\Big\}$. Therefore $d(G_{11},3) = \big|\Gamma(G_{11})\big| = 12$.

$\Gamma(G_{12}) = \Big\{\{1,3,9\},\{1,4,8\},\{1,5,7\},\{2,5,8\},\{2,6,8\},\{2,6,9\},\{2,6,10\},\{2,7,9\},$
$\{3,5,10\},\{3,6,8\},\{3,6,9\},\{4,6,8\},\{4,6,9\},\{4,6,10\},\{4,7,10\}\Big\}$. Therefore $d(G_{12},3) = \big|\Gamma(G_{12})\big| = 15$.

$\Gamma(G_{13}) = \Big\{\{1,3,8\},\{1,4,8\},\{1,4,9\},\{2,5,8\},\{2,7,8\},\{3,6,9\},\{4,5,10\},\{4,7,10\}\Big\}$.

Therefore $d(G_{13},3) = \big|\Gamma(G_{13})\big| = 8$.

$\Gamma(G_{14}) = \Big\{\{1,3,7\},\{1,3,8\},\{1,3,9\},\{1,4,7\},\{1,4,8\},\{1,4,9\},\{1,5,7\},\{1,5,8\},$
$\{1,5,9\},\{2,3,7\},\{2,4,7\},\{2,5,7\},\{2,5,8\},\{2,5,9\},\{2,6,8\},\{3,6,9\},\{3,7,10\},$
$\{4,6,10\},\{4,7,10\},\{5,7,10\},\{5,8,10\},\{5,9,10\}\Big\}$. Therefore $d(G_{14},3) = \big|\Gamma(G_{14})\big| = 22$.

$\Gamma(G_{15}) = \Big\{\{1,2,8\},\{1,3,8\},\{1,4,8\},\{2,5,10\},\{2,6,9\},\{2,7,8\},\{3,5,10\},\{3,6,7\},$
$\{3,6,9\},\{4,5,10\},\{4,6,9\},\{4,7,10\}\Big\}$. Therefore $\big|\Gamma(G_{15})\big| = 12$.

$\Gamma(G_{16}) = \Big\{\{1,3,8\},\{1,4,8\},\{1,4,9\},\{3,6,8\},\{3,6,9\},\{4,6,9\}\Big\}$. Therefore $d(G_{16},3) = 6$.

$\Gamma(G_{17}) = \Big\{\{1,3,7\},\{1,4,10\},\{1,8,9\},\{2,4,8\},\{2,5,6\},\{2,9,10\},\{3,5,9\},\{3,6,10\},$
$\{4,6,7\},\{5,7,8\}\Big\}$. Therefore, $d(G_{17},3) = 10$.

$\Gamma(G_{18}) = \Big\{\{1,5,8\},\{1,5,9\},\{2,5,8\},\{2,5,9\},\{2,6,7\},\{2,6,8\},\{2,6,9\},\{2,6,10\},$
$\{3,5,8\},\{3,5,9\},\{3,6,7\},\{3,6,8\},\{3,6,9\},\{3,6,10\},\{4,5,8\},\{4,5,9\}\Big\}$. Therefore, $d(G_{18},3) = 16$.

$\Gamma(G_{19}) = \Big\{\{1,4,8\},\{1,5,8\},\{2,4,7\},\{2,5,8\},\{2,5,9\},\{2,6,9\},\{2,7,9\},$
$\{3,5,10\},\{3,6,9\},\{3,6,10\},\{3,7,10\},\{4,7,10\},\{5,8,10\}\Big\}$. Therefore $\big|\Gamma(G_{19})\big| = 13$.

4. $\mathcal{D}$-equivalence class of the Petersen graph

In this section we show that the Petersen graph is $\mathcal{D}$-unique.

Theorem 5 *The Petersen graph P is $\mathcal{D}$-unique.*

Proof. Assume that G is a graph such that $D(G,x) = D(P,x)$. Since for every two vertices $x, y \in V(P)$, $N[x] \neq N[y]$, by Theorem 3, G is a 3-regular graph of order 10. Using the $\Big|\Gamma(G_i)\Big|$ for $i = 1,\ldots,21$ in Section 3 we reject some graphs from $[P]$. Since $d(G_6,3) = d(G_9,3) = d(G_{10},3) = d(G_{17},3) = 10$, we compare the cardinality of the families of dominating sets of these four graphs of size 4.

$\mathcal{D}_1(G_6,4)=\Big\{\{1,2,3,4\},\{1,2,3,10\},\{1,2,4,7\},\{1,2,4,8\},\{1,2,4,9\},\{1,2,5,8\},$
$\{1,2,5,9\},\{1,2,6,9\},\{1,2,9,10\},\{1,3,4,6\},\{1,3,4,7\},\{1,3,4,8\},\{1,3,5,8\},\{1,3,6,8\},$
$\{1,3,6,9\},\{1,3,6,10\},\{1,3,7,10\},\{1,3,8,10\},\{1,4,5,7\},\{1,4,5,8\},\{1,4,6,7\},$
$\{1,4,6,8\},\{1,4,6,9\},\{1,4,7,8\},\{1,4,7,9\},\{1,4,7,10\},\{1,4,8,9\},\{1,4,8,10\},\{1,5,6,8\},$
$\{1,5,7,8\},\{1,5,8,9\},\{1,5,8,10\},\{1,6,8,9\},\{1,8,9,10\}\Big\}.$

Therefore, by Lemma 1, $d(G_6,4)=\frac{34\times 10}{4}=85>d(P,4)=75$.

Now, we obtain the family of all dominating sets of G_{10} of size 4.

$\mathcal{D}_1(G_{10},4)=\Big\{\{1,2,3,9\},\{1,2,3,10\},\{1,2,4,9\},\{1,2,5,8\},\{1,2,5,9\},\{1,2,6,9\},$
$\{1,2,7,9\},\{1,2,8,9\},\{1,2,9,10\},\{1,3,4,6\},\{1,3,5,8\},\{1,3,6,8\},\{1,3,6,9\},$
$\{1,3,6,10\},\{1,3,7,9\},\{1,3,7,10\},\{1,3,8,9\},\{1,3,8,10\},\{1,4,5,7\},\{1,4,5,8\},\{1,4,6,7\},$
$\{1,4,6,8\},\{1,4,6,9\},\{1,4,7,9\},\{1,4,7,10\},\{1,4,8,9\},\{1,4,8,10\},\{1,5,6,8\},\{1,5,7,8\},$
$\{1,5,8,9\},\{1,5,8,10\},\{1,6,8,9\},\{1,7,8,9\},\{1,8,9,10\}\Big\}.$

Therefore, by Lemma 1, $d(G_{10},4)=\frac{34\times 10}{4}=85>d(P,4)=75$.

Now, for graph G_9 we have, $\mathcal{D}(G_9,4)=$
$\Big\{\{1,2,3,9\},\{1,2,4,8\},\{1,2,5,7\},\{1,2,5,8\},\{1,2,6,8\},\{1,2,8,9\},\{1,3,4,5\},\{1,3,4,8\},$
$\{1,3,4,9\},\{1,3,4,10\},\{1,3,5,7\},\{1,3,5,8\},\{1,3,5,9\},\{1,3,6,8\},\{1,3,6,9\},\{1,3,7,9\},$
$\{1,3,8,9\},\{1,3,9,10\},\{1,4,5,6\},\{1,4,5,7\},\{1,4,5,8\},\{1,4,6,8\},\{1,4,6,9\},\{1,4,6,10\},$
$\{1,4,7,8\},\{1,4,7,9\},\{1,4,7,10\},\{1,4,8,9\},\{1,4,8,10\},\{1,5,6,7\},\{1,5,7,8\},\{1,5,7,9\},$
$\{1,5,7,10\},\{1,6,7,8\},\{1,7,8,9\}\{2,3,4,5\},\{2,3,5,7\},\{2,3,5,8\},\{2,3,5,9\},\{2,3,6,8\},$
$\{2,3,6,9\},\{2,3,7,9\},\{2,4,5,6\},\{2,4,5,7\},\{2,4,5,8\},\{2,4,6,8\},\{2,4,6,9\},\{2,4,6,10\},$
$\{2,4,7,8\},\{2,4,7,9\},\{2,4,7,10\},\{2,5,6,7\},\{2,5,6,8\},\{2,5,6,9\},\{2,5,7,8\},\{2,5,7,9\},$
$\{2,5,7,10\},\{2,5,8,9\},\{2,5,8,10\},\{2,6,7,8\},\{2,6,8,9\},\{2,6,8,10\},\{2,7,8,9\},$
$\{3,4,5,10\},\{3,4,6,10\},\{3,4,6,10\},\{3,4,7,10\},\{3,5,6,9\},\{3,5,7,10\},\{3,5,8,10\},$
$\{3,5,9,10\},\{3,6,7,10\},\{3,6,8,10\},\{3,6,7,9\},\{3,6,8,9\},\{3,6,9,10\},\{3,7,9,10\},$
$\{4,5,6,10\},\{4,5,7,10\},\{4,5,8,10\},\{4,6,7,10\},\{4,6,8,10\},\{4,6,9,10\},\{4,7,8,10\},$
$\{4,7,9,10\},\{5,6,7,10\},\{5,7,8,10\},\{5,7,9,10\},\{6,7,8,10\},\{7,8,9,10\}\Big\}.$

Therefore $d(G_9,4)=90>d(P,4)$.

Hence $[P]=\{P\}$, and so the Petersen graph is $\mathcal{D}$-unique. □

By the arguments in the proof of Theorem 5, we have the following corollary.

Corollary 2 (i) *The graph G_9 is $\mathcal{D}$-unique,*

(ii) $[G_6] = \Big\{G_6, G_{10}\Big\}$ *with the following domination polynomial:*

$$x^{10} + \binom{10}{9}x^9 + \binom{10}{8}x^8 + \binom{10}{7}x^7 + (\binom{10}{6} - 10)x^6 + (\binom{10}{5} - 60)x^5 + 85x^4 + 10x^3.$$

5. $\mathcal{D}$-equivalence class of cubic graphs of order 10

In this section, we shall study the $\mathcal{D}$-equivalence classes of other cubic graphs of order 10.

We need the following theorem.

Theorem 6 ([7]) *If a graph G has m components $G_1, \ldots, G_m$, then $D(G, x) = D(G_1, x) \cdots D(G_m, x)$.*

Corollary 3 *Two graphs G_{20} and G_{21} are $\mathcal{D}$-equivalent, with the following domination polynomial:*

$$D(G_{20}, x) = D(G_{21}, x) = x^{10} + 10x^9 + 45x^8 + 120x^7 + 203x^6 + 216x^5 + 134x^4 + 36x^3.$$

Proof. Two graphs G_{20} and G_{21} are disconnected with two components. In other words, $G_{20} = H \cup K_4$ and $G_{21} = H' \cup K_4$, where H and H' are graphs with 6 vertices. It is not hard to see that

$$D(H, x) = D(H', x) = x^6 + 6x^5 + 15x^4 + 20x^3 + 9x^2.$$

On the other hand, $D(K_4, x) = x^4 + 4x^3 + 6x^2 + 4x$. By Theorem 6, we have the result. □

Theorem 7 *The graphs G_{12}, G_{13}, G_{14}, G_{16}, and G_{19} in Figure 1 are $\mathcal{D}$-unique.*

Proof. Using γ-sets in Section 3, $|\Gamma(G_{12})| = 15$, $|\Gamma(G_{13})| = 7$, $|\Gamma(G_{14})| = 22$, and $|\Gamma(G_{19})| = 13$. By comparing these numbers with the cardinality of γ-sets of other 3-regular graphs, we have the result. Now, we consider graph G_{16}. Since $d(G_7, 3) = d(G_8, 3) = d(G_{16}, 3) = 6$, we shall obtain $d(G_i, 4)$ for $i = 7, 8, 16$.

$\mathcal{D}_1(G_7, 4) = \Big\{\{1,2,3,4\}, \{1,2,4,8\}, \{1,2,4,9\}, \{1,2,4,10\}, \{1,2,5,8\}, \{1,2,5,9\}, \{1,2,6,8\}, \{1,2,8,10\}, \{1,3,4,6\}, \{1,3,4,7\}, \{1,3,4,8\}, \{1,3,5,7\}, \{1,3,5,8\}, \{1,3,6,7\}, \{1,3,6,8\}, \{1,3,6,9\}, \{1,3,7,10\}, \{1,3,8,10\}, \{1,4,5,8\}, \{1,4,6,8\}, \{1,4,6,9\}, \{1,4,6,10\}, \{1,4,7,8\}, \{1,4,7,9\}, \{1,4,7,10\}, \{1,4,8,9\}, \{1,4,8,10\}, \{1,5,6,9\}, \{1,5,7,9\}, \{1,5,8,9\}, \{1,6,8,9\}, \{1,8,9,10\}\Big\}$. Therefore, by Lemma 1, $d(G_7, 4) = \frac{32\times 5}{2} = 80$.

$\mathcal{D}_1(G_8, 4) = \Big\{\{1,2,3,5\}, \{1,2,3,9\}, \{1,2,4,8\}, \{1,2,5,6\}, \{1,2,5,7\}, \{1,2,5,8\}, \{1,2,6,10\}, \{1,3,4,8\}, \{1,3,4,9\}, \{1,3,5,7\}, \{1,3,5,8\}, \{1,3,5,9\}, \{1,3,6,8\}, \{1,3,6,9\}, \{1,3,7,9\}, \{1,3,8,9\}, \{1,3,9,10\}, \{1,4,5,7\}, \{1,4,5,8\}, \{1,4,6,8\}, \{1,4,6,9\}, \{1,4,7,8\}, \{1,4,7,9\}, \{1,4,7,10\}, \{1,4,8,9\}, \{1,4,8,10\}, \{1,5,6,7\}, \{1,5,7,8\}, \{1,5,7,9\}, \{1,5,7,10\}, \{1,6,7,10\}, \{1,7,9,10\}\Big\}$. Therefore, by Lemma 1, $d(G_8, 4) = \frac{32\times 5}{2} = 80$.

Now, we obtain $d(G_{16}, 4)$.

$\mathcal{D}_1(G_{16}, 4) = \Big\{\{1,2,3,8\}, \{1,2,4,5\}, \{1,2,4,7\}, \{1,2,4,8\}, \{1,2,4,9\}, \{1,2,5,6\},$

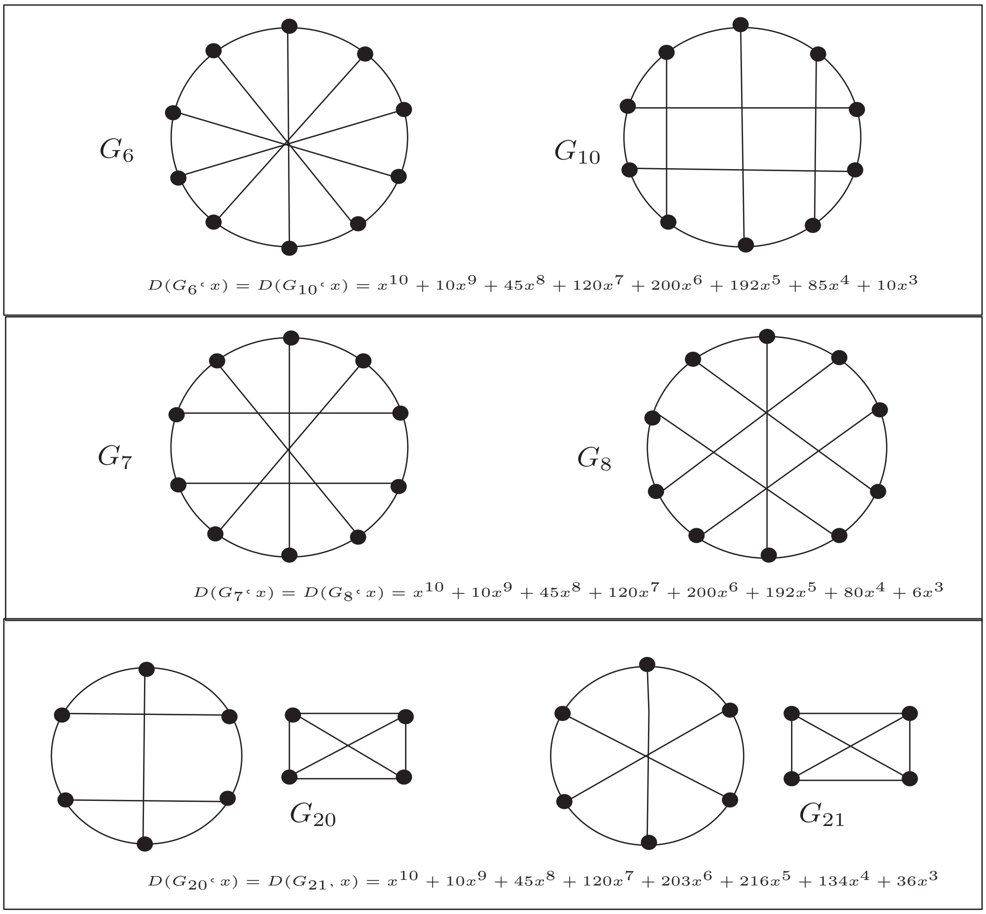

Figure 4. Cubic graphs of order 10 with identical domination polynomial.

$\{1,2,5,7\}, \{1,2,5,8\}, \{1,2,6,7\}, \{1,2,7,8\}, \{1,3,4,8\}, \{1,3,4,9\}, \{1,3,5,8\}, \{1,3,6,8\}, \{1,3,6,9\}, \{1,3,7,8\}, \{1,3,8,9\}, \{1,3,8,10\}, \{1,4,5,8\}, \{1,4,5,9\}, \{1,4,5,10\}, \{1,4,6,8\}, \{1,4,6,9\}, \{1,4,7,8\}, \{1,4,7,9\}, \{1,4,7,10\}, \{1,4,8,9\}, \{1,4,8,10\}, \{1,4,9,10\}, \{1,5,6,10\}, \{1,5,7,10\}, \{1,5,8,10\}, \{1,6,7,10\}, \{1,7,8,10\}\Big\}$. Therefore, by Lemma 1, $d(G_{16},4) = \frac{34\times 5}{2} = 85$. Hence $[G_{16}] = \{G_{16}\}$. □

By the arguments in the proof of Theorem 8, we have the following corollary.

Corollary 4 *Two graphs G_7 and G_8 are $\mathcal{D}$-equivalent.*

Theorem 8 *The graphs G_2, G_{11}, and G_{15} in Figure 1 are $\mathcal{D}$-unique.*

Proof. Using γ-sets in Section 3, $\big|\Gamma(G_2)\big| = \big|\Gamma(G_{11})\big| = \big|\Gamma(G_{15})\big| = 12$. So we shall obtain $d(G_i,4)$ for $i = 2, 11, 15$.

$\mathcal{D}_1(G_2,4) = \Big\{\{1,2,3,8\}, \{1,2,4,7\}, \{1,2,4,8\}, \{1,2,4,9\}, \{1,2,4,10\}, \{1,2,5,7\},$

$\{1,2,5,8\},\{1,2,6,7\},\{1,3,4,8\},\{1,3,4,9\},\{1,3,4,10\},\{1,3,5,8\},\{1,3,6,8\},\{1,3,6,9\},$
$\{1,3,6,10\},\{1,3,7,8\},\{1,3,7,9\},\{1,3,7,10\},\{1,3,8,9\},\{1,3,8,10\},\{1,4,5,8\},$
$\{1,4,5,9\},\{1,4,5,10\},\{1,4,6,8\},\{1,4,6,9\},\{1,4,6,10\},\{1,4,7,8\},\{1,4,7,9\},$
$\{1,4,7,10\},\{1,4,8,9\},\{1,4,8,10\},\{1,4,9,10\},\{1,5,6,8\},\{1,5,6,9\},\{1,5,6,10\},$
$\{1,5,7,8\},\{1,5,7,9\},\{1,5,7,10\},\{1,5,8,9\},\{1,5,8,10\}\Big\}.$

Therefore, by Lemma 1, $d(G_2,4)=\frac{40\times 10}{4}=100$, and so $[G_2]=\{G_2\}$.

$\mathcal{D}_1(G_{11},4)=\Big\{\{1,2,3,7\},\{1,2,4,8\},\{1,2,5,8\},\{1,2,5,9\},\{1,2,6,8\},\{1,2,7,8\},$
$\{1,3,4,5\},\{1,3,4,6\},\{1,3,4,7\},\{1,3,4,8\},\{1,3,5,7\},\{1,3,5,8\},\{1,3,5,9\},\{1,3,6,7\},$
$\{1,3,6,8\},\{1,3,6,9\},\{1,3,7,8\},\{1,3,7,9\},\{1,3,7,10\},\{1,4,5,8\},\{1,4,5,9\},\{1,4,5,10\},$
$\{1,4,6,8\},\{1,4,6,9\},\{1,4,6,10\},\{1,4,7,8\},\{1,4,7,9\},\{1,4,7,10\},\{1,4,8,9\},\{1,4,8,10\},$
$\{1,5,6,9\},\{1,5,7,9\},\{1,5,8,9\},\{1,5,9,10\},\{1,6,8,9\},\{1,7,8,9\}\Big\}.$

Therefore, by Lemma 1, $d(G_{11},4)=\frac{36\times 10}{4}=90$, and so $[G_{11}]=\{G_{11}\}$.

Now, for graph G_{15} we have, $\mathcal{D}(G_{15},4)=$
$\Big\{\{1,2,3,8\},\{1,2,4,8\},\{1,2,5,8\},\{1,2,5,10\},\{1,2,6,8\},\{1,2,6,9\},\{1,2,7,8\},\{1,2,8,9\},$
$\{1,2,8,10\},\{1,3,4,7\},\{1,3,4,8\},\{1,3,5,6\},\{1,3,5,7\},\{1,3,5,8\},\{1,3,5,10\},\{1,3,6,7\},$
$\{1,3,6,8\},\{1,3,6,9\},\{1,3,7,8\},\{1,3,8,9\},\{1,3,8,10\},\{1,4,5,8\},\{1,4,5,10\},\{1,4,6,8\},$
$\{1,4,6,9\},\{1,4,7,8\},\{1,4,7,9\},\{1,4,7,10\},\{1,4,8,9\},\{1,4,8,10\},\{1,4,9,10\},\{1,5,6,9\},$
$\{1,5,7,9\},\{1,5,8,9\},\{1,5,9,10\},\{2,3,4,7\},\{2,3,5,7\},\{2,3,5,10\},\{2,3,6,7\},\{2,3,6,9\},$
$\{2,3,7,8\},\{2,4,6,9\},\{2,4,7,8\},\{2,4,7,9\},\{2,4,7,10\},\{2,5,6,9\},\{2,5,6,10\},\{2,5,7,8\},$
$\{2,5,7,9\},\{2,5,7,10\},\{2,5,8,9\},\{2,5,8,10\},\{2,5,9,10\},\{2,6,7,8\},\{2,6,7,9\},$
$\{2,6,7,10\},\{2,6,8,9\},\{2,6,8,10\},\{2,6,9,10\},\{2,7,8,9\},\{2,7,8,10\},\{3,4,6,7\},$
$\{3,4,6,9\},\{3,4,7,10\},\{3,5,6,7\},\{3,5,6,9\},\{3,5,6,10\},\{3,5,7,10\},\{3,5,8,10\},$
$\{3,5,9,10\},\{3,6,7,8\},\{3,6,7,9\},\{3,6,7,10\},\{3,6,8,9\},\{3,6,8,10\},\{3,6,9,10\},$
$\{3,7,8,10\},\{4,5,6,9\},\{4,5,6,10\},\{4,5,7,10\},\{4,5,8,10\},\{4,5,9,10\},\{4,6,7,8\},$
$\{4,6,7,9\},\{4,6,7,10\},\{4,6,8,9\},\{4,6,8,10\},\{4,6,9,10\},\{4,7,8,10\},\{4,7,9,10\},$
$\{4,8,9,10\}\Big\}.$

Therefore $d(G_{15},4)=91$, and so $[G_{15}]=\{G_{15}\}$. □

In summary, in this paper we showed that the Petersen graph is $\mathcal{D}$-unique. Also, we proved that the graphs $G_2,G_9,G_{11},G_{12},G_{13},G_{14},G_{15},G_{16},G_{17}$, and G_{19} are $\mathcal{D}$-unique, and $[G_6]=\{G_6,G_{10}\}$, $[G_7]=\{G_7,G_8\}$, $[G_{20}]=\{G_{20},G_{21}\}$ (see Figure 4).

Since there are at least two vertices u and v in G_1, G_3, G_4, G_5, and G_{18}, with $N[u]=N[v]$, we are not able to use Theorem 3. So to obtain the $\mathcal{D}$-equivalence classes of these graphs, we have to consider many other graphs with 10 vertices. Therefore the determination of the $\mathcal{D}$-equivalent of G_1, G_3, G_4, G_5, and G_{18}, remain open, although we think that they are $\mathcal{D}$-unique.

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References

[1] Akbari, S. and Aalipour, G.: *On the $\mathcal{D}$-equivalence class of complete bipartite graphs*, Ars Combin., to appear.

[2] Akbari, S., Alikhani, S. and Peng, Y. H.: *Characterization of graphs using Domination Polynomial*, European J. Combin., Vol 31, 1714-1724 (2010).

[3] Akbari, S. and Oboudi, M. R.: *Cycles are determined by their domination polynomials*, Ars combin., to appear.

[4] Alikhani, S. and Peng, Y. H.: *Dominating sets and Domination Polynomials of Certain Graphs, II*, Opuscula Mathematica, 30, No 1, 37-51 (2010).

[5] Alikhani, S. and Peng, Y. H.: *Dominating sets and domination polynomials of cycles*, Global Journal of Pure and Applied Mathematics, 4, no 2, 151-162 (2008).

[6] Alikhani, S. and Peng, Y. H.: *Dominating sets and Domination Polynomials of Paths*, International Journal of Mathematics and Mathematical Science, Vol 2009, Article ID 542040 (2009).

[7] Alikhani, S. and Peng, Y. H.: *Introduction to Domination polynomial of a graph*, Ars Combin., to appear.

[8] Dong, F. M., Koh, K. M. and Teo, K. L.: *Chromatic polynomials and chromaticity of graphs*, World Scientific Publishing Co. Pte. Ltd 2005.

[9] Haynes, T. W., Hedetniemi, S. T. and Slater, P. J.: *Fundamentals of Domination in Graphs*, Marcel Dekker, NewYork 1998.

[10] Khosrovshahi, G. B., Maysoori, Ch. and Tayfeh-Rezaie, B.: *A Note on 3-Factorizations of K_{10}*, J. Combin. Designs 9, 379-383 (2001).

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