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A Fredholm alternative-like result on power bounded operators

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Abstract

Let X be a complex Banach space and $T : X \to X$ be a power bounded operator, i.e., $\sup_{n\geq 0} \|T^n\| < \infty$. We write $\mathcal{B}(X)$ for the Banach algebra of all bounded linear operators on X. We prove that the space $\mathrm{Ran}(I-T)$ is closed if and only if there exist a projection $\theta \in \mathcal{B}(X)$ and an invertible operator $R \in \mathcal{B}(X)$ such that $I - T = \theta R = R\theta$. This paper also contains some consequences of this result.

1. Introduction

Let X be a complex Banach space. It is well known that for every compact operator $K : X \to K$, the range of the operator $I - K$ is closed. However, we cannot expect this to hold for an arbitrary bounded linear operator $T : X \to X$. So it is natural to ask when the range of the operator $I - T$ is closed. In this paper, we answer this problem for power bounded operators by proving that, for a power bounded operator T, the range of the operator $I - T$ is closed if and only if $I - T$ can be written as a product of two commuting operators θ and R where θ is an idempotent and R is invertible. We also present some consequences of this result and it is essentially self-contained.

2. Main results

Let $T : X \to X$ be a power bounded operator on X. If we renorm X with the norm $|||x||| := \sup_{n\geq 0} \|T^n x\|$, then T becomes a contraction on X with this new norm, that is, $\|T\| \leq 1$. For that reason we will work with a fixed contraction operator T. Clearly all of the results presented below are valid for power bounded operators. We will denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X, and by $\mathcal{B}(X^*)$ the Banach algebra of all bounded linear operators on the dual space X^*. Note that one can identify $\mathcal{B}(X^*)$ with the dual space of the projective tensor space $X^*\hat{\otimes}X$ [1, p. 230, Corollary 2]. So it carries a weak* topology. The natural duality between the spaces $B(X^*)$ and $X^*\hat{\otimes}X$ is given by $\langle B, f \otimes x\rangle = \langle B(f), x\rangle$ for every operator $B \in \mathcal{B}(X^*)$, every functional $f \in X^*$, and every vector $x \in X$.

We start with the following observation which will be used in the proof of our main theorem.

Lemma 2.1 *Let $T \in \mathcal{B}(X)$ and assume that* $\mathrm{Ran}(T)$ *is closed. Then the following are equivalent:*

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1. $\operatorname{Ker}(T^*) = \operatorname{Ker}(T^{*^2})$.

2. $\overline{\operatorname{Ran}(T^2)} = \operatorname{Ran}(T)$.

Proof. (1)$\Rightarrow$(2): Since

$$\operatorname{Ran}(T)^{\perp} = \operatorname{Ker}(T^*) = \operatorname{Ker}(T^{*^2}) = \operatorname{Ran}(T^2)^{\perp}$$

and $\operatorname{Ran}(T)$ is closed, we have $\overline{\operatorname{Ran}(T^2)} = \operatorname{Ran}(T)$ by Hahn-Banach Theorem.

(2)$\Rightarrow$(1): We have

$$\operatorname{Ker}(T^{*2}) = \overline{\operatorname{Ran}(T^2)}^{\perp} \quad \text{and} \quad \operatorname{Ker}(T^*) = \operatorname{Ran}(T)^{\perp}.$$

As $\operatorname{Ran}(T)^{\perp} = \overline{\operatorname{Ran}(T^2)}^{\perp}$, it follows that $\operatorname{Ker}(T^*) = \operatorname{Ker}(T^{*2})$. □

The following lemma is proved in ([3], p. 69) for nonexpansive (not necessarily linear) mappings. One can also find a proof of this result in the monograph [2, Lemma 9.4].

Lemma 2.2 *Let $T \in \mathcal{B}(X)$ be a contraction and $S_\lambda = \lambda I + (1-\lambda)T$ for $0 < \lambda < 1$. Then $\lim_{n\to\infty} \|S_\lambda^{n+1}(x) - S_\lambda^n(x)\| = 0$ for every $x \in X$.*

The following results, which will be needed for the proof of Theorem 2.9, follow without much difficulty from the preceding lemma.

Lemma 2.3 *Let $T \in B(X)$ be a contraction. Then*
$\operatorname{Ker}(I-T) \cap \operatorname{Ran}(I-T) = \{0\}$.

Proof. Let $S = S_{\frac{1}{2}} = \frac{I+T}{2}$. Then the range and the kernel of the operator $I - S$ coincide with those of $I - T$. Let $y \in \operatorname{Ker}(I-S) \cap \operatorname{Ran}(I-S)$. Since $(I-S)(y) = 0$, that is, $S(y) = y$, we have $S^n(y) = y$ for every n. We have $(I-S)(x) = y$ for some $x \in X$, that is, $y = x - S(x)$. By applying the operator S^n to this equality we get $y = S^n x - S^{n+1}x$. By the previous lemma, $\|S^n x - S^{n+1}x\|$ converges to 0 as $n \to \infty$, which implies that $y = 0$. So $\operatorname{Ker}(I-S) \cap \operatorname{Ran}(I-S) = \{0\}$. Thus, $\operatorname{Ker}(I-T) \cap \operatorname{Ran}(I-T) = \{0\}$. □

Lemma 2.4 *Let $T : X \to X$ be a contraction. Then,*
$\operatorname{Ker}(I-T^*) = \operatorname{Ker}((I-T^*)^2)$.

Proof. By Lemma 2.3, we have Ker $(I-T^*)$ $\cap$ Ran $(I-T^*)$ $=$ $\{0\}$. Let $x \in \operatorname{Ker}((I-T^*)^2)$. Then the element $y = (I-T^*)(x)$ is in the intersection of the spaces $\operatorname{Ker}(I-T^*)$ and $\operatorname{Ran}(I-T^*)$, which is trivial. Hence $x \in \operatorname{Ker}(I-T^*)$, which implies that $\operatorname{Ker}(I-T^*)^2 \subseteq \operatorname{Ker}(I-T^*)$. The other inclusion is always true. □

The following lemma will be needed in the proof of Theorem 2.6.

Lemma 2.5 *Let (R_α) be a bounded net in $\mathcal{B}(X^*)$. Then we have the following:*

1. *The net* (R_α) *converges to* $R \in \mathcal{B}(X^*)$ *in the weak** *topology if and only if* $\langle R_\alpha(f), x\rangle$ *converges to* $\langle R(f), x\rangle$ *for every* $x \in X$ *and* $f \in X^*$.

2. *If* (R_α) *converges to* R *in the weak** *topology, then* $(R_\alpha \circ Q)$ *converges to* $R \circ Q$ *in the weak** *topology for every operator* $Q \in \mathcal{B}(X^*)$.

3. *If* (R_α) *converges to* R *in the weak** *topology, then* $(L^* \circ R_\alpha)$ *converges to* $L^* \circ R$ *for every operator* $L \in \mathcal{B}(X)$.

Proof. Assertion (1) follows from the fact that the net (R_α) is bounded and the set of atomic tensors $f \otimes x$ are total in the space $X^* \hat{\otimes} X$. Assertions (2) and (3) follow, respectively, from the identities

$$\langle R_\alpha \circ Q, f \otimes x\rangle = \langle R_\alpha, Q(f) \otimes x\rangle.$$

$$\langle L^* \circ R_\alpha, f \otimes x\rangle = \langle (L^* \circ R_\alpha)(f), x\rangle = \langle R_\alpha(f), L(x)\rangle = \langle R_\alpha, f \otimes L(x)\rangle.$$

□

The next result shows that for a power bounded operator $T \in \mathcal{B}(X)$, the kernel of the operator $I - T^*$ is always complemented in X^*.

Theorem 2.6 *Let* $T \in \mathcal{B}(X)$ *be a contraction. Then there exists a projection* $P \in \mathcal{B}(X^*)$ *whose range is* $\mathrm{Ker}(I - T^*)$ *and whose kernel contains* $\mathrm{Ran}(I - T^*)$.

Proof. Let $S = \frac{I+T}{2}$. Since $\|S\| \leq 1$, the set $\{S^{*^n} : n \geq 0\}$ is bounded. So by Alaoglu theorem, the sequence (S^{*^n}) has a convergent subnet $(S^{*^{n_i}})$ that converges to an operator P in $(B(X^*), w^*)$. By Lemma 2.2, we have $\langle S^{*^{n+1}} f - S^{*^n} f, x\rangle \to 0$ for every $f \in X^*$ and $x \in X$. This, together with the fact that P is the weak* limit of the net $(S^{*^{n_i}})$, implies that

$$P \circ S^* = S^* \circ P = P.$$

Then $S^{*^{n_i}} \circ P = P$ for every n_i. So, passing again to the limit in $(\mathcal{B}(X^*), w^*)$ and using Lemma 2.5, we get $P^2 = P$. This proves that every cluster point of the sequence (S^{*n}) is a projection. As $S^* = \frac{I+T^*}{2}$, we also have

$$T^* \circ P = P \circ T^* = P.$$

So $\mathrm{Ran}(P) \subseteq \mathrm{Ker}(I - T^*)$. On the other hand, for $f \in \mathrm{Ker}\ (\ I - T^*\ )$, we have $T^*(f) = f$, so $S^*(f) = f$. Hence $S^{*^{n_i}} f = f$, which implies that $P(f) = f$. Hence $\mathrm{Ran}(P) = \mathrm{Ker}(I - T^*)$. To prove the inclusion $\mathrm{Ran}(I - T^*) \subseteq \mathrm{Ker}(P)$, let $f \in X^*$ be an arbitrary element and $g = f - T^*(f)$. Then, we have $P(g) = (P \circ T^*)(f) = 0$. Hence $\mathrm{Ran}(I - T^*) \subseteq \mathrm{Ker}(P)$. □

As an important corollary of this theorem we present the following result.

Corollary 2.7 *Let* T *be a power bounded operator and* $S = \frac{I+T}{2}$. *Then*

1. $\overline{\mathrm{Ran}(I - T)} = X$ *if and only if* $S^n(x) \to 0$ *weakly for every* $x \in X$.

2. $\mathrm{Ran}(I - T) = X$ *if and only if* $\|S^n\| \to 0$.

Proof. (1): First assume that $\langle S^n x, f\rangle \to 0$ for every $x \in X$ and $f \in X^*$. Then $\langle x, S^{*n} f\rangle \to 0$ for every $x \in X$ and $f \in X^*$, that is, the sequence (S^{*n}) converges to 0 in the weak* topology of $\mathcal{B}(X^*)$. Then the projection P obtained in Theorem 2.6 is trivial, which in turn implies that $\mathrm{Ker}(I - T^*) = \{0\}$. Thus, $\overline{\mathrm{Ran}(I - T)} = X$.

Conversely, if $\overline{\mathrm{Ran}(I - T)} = X$, then, since every weak* cluster point of the sequence (S^{*n}) is a projection on $\mathrm{Ker}(I - T^*)$, the only weak* cluster point of the sequence (S^{*n}) is 0. This implies that the sequence (S^{*n}) itself converges to 0 in the weak* topology since the sequence (S^{*n}) is bounded.

(2): We first note that by a result by Katznelson-Tzafriri [5, Theorem 1], we have $\sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} \subseteq \{1\}$. Note also that by Lemma 2.3, the operator $(I - T)$ is invertible if and only if it is onto. Thus,

$$\begin{aligned}\mathrm{Ran}(I - T) = X &\Leftrightarrow 1 \notin \sigma(T) \Leftrightarrow 1 \notin \sigma(S)\\ &\Leftrightarrow \sigma(S) \cap \{z \in \mathbb{C} : |z| = 1\} = \emptyset\\ &\Leftrightarrow \|S^n\| \to 0.\end{aligned}$$

□

The following corollary, which is of independent interest, will be needed for the proof of our main theorem.

Corollary 2.8 *Let $T \in \mathcal{B}(X)$ be a contraction and assume that* $\mathrm{Ran}(I - T^*)$ *is closed. Then* $\mathrm{Ran}((I - T^*)^2)$ *is also closed.*

Proof. By Lemma 331 of [4, p.274], it is enough to prove that the space $\mathrm{Ran}(I - T^*) + \mathrm{Ker}(I - T^*)$ is closed. Let $((I - T^*)(f_n) + g_n)$ be a sequence in $\mathrm{Ran}(I - T^*) + \mathrm{Ker}(I - T^*)$ that converges to $f \in X^*$. Let P denote the projection obtained in Theorem 2.6. Since Ran ($I - T^*$) is a subset of $\mathrm{Ker}(P)$, we have $P((I-T)^*(f_n)+g_n) = Pg_n \to Pf$. As $\mathrm{Ker}(I-T^*) = \mathrm{Ran}(P)$, the sequence (g_n) converges to $P(f)$. Thus, the sequence $(I-T)^*(f_n)$ converges to $f-P(f)$ which must be in $\mathrm{Ran}(I-T^*)$, as the space $\mathrm{Ran}(I-T^*)$ is closed. □

We can now prove the main result of this paper.

Theorem 2.9 *Let $T \in \mathcal{B}(X)$ be a contraction. Then* $\mathrm{Ran}(I-T)$ *is closed if and only if there exist a projection $\theta \in \mathcal{B}(X)$ and an invertible operator $R \in \mathcal{B}(X)$ such that $I - T = \theta \circ R = R \circ \theta$.*

Proof. Assume that the space $\mathrm{Ran}(I-T)$ is closed. Then the space $\mathrm{Ran}(I-T^*)$ is closed as well. Therefore, by Corollary 2.8, the space $\mathrm{Ran}((I - T^*)^2)$ is also closed, which in turn implies that the space $\mathrm{Ran}((I - T)^2)$ is closed. Note that $\mathrm{Ker}(I - T^*) = \mathrm{Ker}((I - T^*)^2)$ by Lemma 2.4. Hence, it follows from Lemma 2.1 that the range of the operator $(I-T)^2$ coincides with the range of the operator $I-T$. So for every $x \in X$, there exists $y \in X$ such that $(I - T)(x) = (I - T)^2(y)$. Thus $x - (I - T)y$ is in $\mathrm{Ker}(I - T)$, which, together with Lemma 2.3, proves that $\mathrm{Ran}(I - T) \oplus \mathrm{Ker}(I - T) = X$. Now, define $R : X \to X$ as follows:

$$R(z + y) = (I - T)(z) + y \quad \text{where} \quad z \in \mathrm{Ran}(I - T) \quad \text{and} \quad y \in \mathrm{Ker}(I - T).$$

The mapping R is well-defined, linear, and bounded. We claim that it is invertible. To see that it is onto, let $x = z + y \in X$, where $z \in$ Ran ($I - T$) and $y \in \mathrm{Ker}(I - T)$. Then $z = (I - T)w$ for some $w \in X$.

So $R(w+y)=x$. Now, to see that it is one-to-one, let $x=z+y\in \mathrm{Ker}(R)$, where $z\in \mathrm{Ran}(I-T)$ and $y\in \mathrm{Ker}(I-T)$. Then $(I-T)(z)=-y$. Thus $(I-T)(z)=y=0$ since the only point in the intersection of the spaces $\mathrm{Ker}(I-T)$ and $\mathrm{Ran}(I-T)$ is 0. This also implies that z is in the intersection of the spaces $\mathrm{Ker}(I-T)$ and $\mathrm{Ran}(I-T)$, and so it is 0 as well. Hence $x=z+y=0$. Let θ be the projection with range $\mathrm{Ran}(I-T)$ and kernel $\mathrm{Ker}(I-T)$. Then $I-T=R\circ\theta=\theta\circ R$. The reverse implication is clear. □

The following corollary, which is reminiscent of the Fredholm Alternative, is an immediate consequence of the preceding theorem.

Corollary 2.10 *Let $T\in\mathcal{B}(X)$ be a contraction and assume that* $\mathrm{Ran}(I-T)$ *is closed. Then* $\dim(\mathrm{Ker}(I-T))=\mathrm{codim}(\mathrm{Ran}(I-T))$.

We will prove below an analogue of Theorem 2.9 for the operator algebra $B(X)$. In what follows we will denote by $R_{(I-T)}$ the operator defined on $\mathcal{B}(X)$ by $R_{(I-T)}(A)=A\circ(I-T)$, and we will denote by $\mathcal{B}(X)\circ(I-T)$ its image.

Corollary 2.11 *Let $T\in\mathcal{B}(X)$ be a contraction and assume that* $\mathrm{Ran}(I-T)$ *is closed. Then* $\mathcal{B}(X)=\mathrm{Ker}(R_{(I-T)})\oplus\mathcal{B}(X)\circ(I-T)$.

Proof. By Theorem 2.9, we have $I-T=\theta\circ R$, where θ is a bounded projection and R is an invertible operator in $\mathcal{B}(X)$. Consider the operator

$$\Theta:\mathcal{B}(X)\to\mathcal{B}(X)$$

$$A\mapsto A\circ\theta,$$

which is a bounded projection on $\mathcal{B}(X)$. Using the decomposition $I-T=R\circ\theta$, one can easily see that $\mathrm{Ran}(\Theta)=\mathcal{B}(X)\circ(I-T)$ and $\mathrm{Ker}(\Theta)=\mathrm{Ker}(R_{(I-T)})$. □

Let $A(T)$ be the norm closed subalgebra of $B(X)$ generated by an operator $T\in\mathcal{B}(X)$ and the identity operator I, which is clearly a commutative Banach algebra. The proof of the preceding corollary also shows that the following holds.

Corollary 2.12 *Let $T\in\mathcal{B}(X)$ be a contraction. Then the ideal $A(T)\circ(I-T)$ is closed in $A(T)$ if and only if the range of the operator $I-T$ is closed in X.*

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