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On almost complex structures in the cotangent bundle

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Abstract

E. M. Patterson and K. Yano studied vertical and complete lifts of tensor fields and connections from a manifold M_n to its cotangent bundle $T^*(M_n)$. Afterwards, K. Yano studied the behavior on the cross-section of the lifts of tensor fields and connections on a manifold M_n to $T^*(M_n)$ and proved that when φ defines an integrable almost complex structure on M_n, its complete lift ${}^C\varphi$ is a complex structure. The main result of the present paper is the following theorem: Let φ be an almost complex structure on a Riemannian manifold M_n. Then the complete lift ${}^C\varphi$ of φ, when restricted to the cross-section determined by an almost analytic 1-form ω on M_n, is an almost complex structure.

Key word and phrases: Almost complex structure, cotangent bundle, cross-section, Nijenhuis tensor, analytic tensor field.

1. Preliminaries

Let M_n be an n-dimensional manifold and $T^*(M_n)$ its cotangent bundle. We denote by $\Im_s^r(M_n)$ the set of all tensor fields of type (r,s) on M_n. Similarly, we denote by $\Im_s^r(T^*(M_n))$ the corresponding set on $T^*(M_n)$.

In this section, we shall summarize all the basic definitions and results on cross-section in $T^*(M_n)$ that are needed later. Let M_n be an n-dimensional manifold of class C^∞ and $T^*(M_n)$ its cotangent bundle over M_n. If x^i are local coordinates in a neighborhood U of a point $x \in M_n$, then a covector P at x which is an element of $T^*(M_n)$ is expressible in the form (x^i, p_i), where p_i are components of P with respect to the natural frame ∂_i. We may consider $(x^i, p_i) = (x^i, x^{\bar{\imath}}) = x^J$, $i = 1, ..., n$; $\bar{\imath} = n+1, ..., 2n$; $J = 1, ..., 2n$ as local coordinates in a neighborhood $\pi^{-1}(U)$(π is the natural projection $T^*(M_n)$ onto M_n).

Now, consider $X \in \Im_0^1(M_n)$ and $\theta \in \Im_1^0(M_n)$, then ${}^C X$ (complete lift) and ${}^V\theta$ (vertical lift) have, respectively, components [5, p. 236], [6]

$$ {}^C X = \begin{pmatrix} X^h \\ -p_m \partial_h X^m \end{pmatrix}, \quad {}^V\theta = \begin{pmatrix} 0 \\ \theta_h \end{pmatrix} \tag{1.1} $$

with respect to the coordinates $(x^h, x^{\bar{h}})$ in $T^*(M_n)$, where X^h and θ_h are local components of X and θ.

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For $\varphi \in \Im_1^1(M_n)$, we can define a vector field $\gamma\varphi \in \Im_0^1(T^*(M_n))$ [5, p.232], [6]:

$$\gamma\varphi = \begin{pmatrix} 0 \\ p_j\varphi_i^j \end{pmatrix} \tag{1.2}$$

where φ_i^j are local components of φ in M_n. Clearly, we have $(\gamma\varphi)\ {}^V f = 0$ for any $f \in \Im_0^0(M_n)$, where ${}^V f = f \circ \pi$ is a vertical lift of f. So that $\gamma\varphi$ is a vertical vector field.

Suppose that there is given a 1-form $\omega \in \Im_1^0(M_n)$ whose local expression is $\omega = \omega_i(x)dx^i$. Then the correspondence $x \to \omega_x$, ω_x being the value of ω at $x \in M_n$, determines a mapping $\beta_\omega : M_n \to T^*(M_n)$, such that $\pi \circ \beta_\omega = id_{M_n}$ and n-dimensional submanifold $\beta_\omega(M_n)$ of $T^*(M_n)$ is called the cross-section determined by ω and its parametric representations are as follows:

$$\begin{cases} x^k = x^k, \\ p_k = \omega_k(x^1, ..., x^n), \end{cases} \tag{1.3}$$

with respect to the coordinates (x^k, p_k) in $T^*(M_n)$. Differentiating (1.3) by x^j, we see that n tangent vector fields B_j to $\beta_\omega(M_n)$ have component

$$B_j^K = (\frac{\partial x^K}{\partial x^j}) = \begin{pmatrix} \delta_j^k \\ \partial_j\omega_k \end{pmatrix} \tag{1.4}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^*(M_n)$.

On the other hand, the fibre being represented by

$$\begin{cases} x^k = \text{const.}, \\ p_k = p_k. \end{cases} \tag{1.5}$$

On differentiating (1.5) by p_j, we see that n tangent vector fields $C_{\bar{j}}$ to the fibre have components

$$C_{\bar{j}}^K = (\frac{\partial x^K}{\partial p_j}) = \begin{pmatrix} 0 \\ \delta_k^j \end{pmatrix} \tag{1.6}$$

with respect to the natural frame $\{\partial_k, \partial_{\bar{k}}\}$ in $T^*(M_n)$. $2n$ local vector fields B_j and $C_{\bar{j}}$, being linearly independent, form a frame along the cross-section. We call this the adapted (B, C)-frame along the cross-section [4]. Taking account of (1.1) and (1.2) on the cross-section, we can see that ${}^C X$, ${}^V\theta$ and $\gamma\varphi$ have along $\beta_\omega(M_n)$ components of the form [4], (see also [5])

$${}^C X = \begin{pmatrix} X^j \\ -L_X\omega_j \end{pmatrix}, \ {}^V\theta = \begin{pmatrix} 0 \\ \theta_j \end{pmatrix}, \ \gamma\varphi = \begin{pmatrix} 0 \\ \omega_h\varphi_j^h \end{pmatrix} \tag{1.7}$$

with respect to the adapted (B, C)-frame. Similarly, if $N \in \Im_2^1(M_n)$, then $\gamma N \in \Im_1^1(T^*(M_n))$ is an affinor field along $\beta_\omega(M_n)$ with components [5, p. 232]

$$\gamma N = \begin{pmatrix} 0 & 0 \\ N_{ij}^h\omega_h & 0 \end{pmatrix} \tag{1.8}$$

with respect to the adapted (B, C)-frame, where S_{ij}^h are local components of S in M_n (For applications of γN, see the formula (2.8)).

2. Main results

Let $\varphi \in \Im_1^1(M_n)$ and $\omega \in \Im_1^0(M_n)$. We define an operator

$$\Phi_\varphi : \Im_1^0(M_n) \to \Im_2^0(M_n)$$

associated with φ and applied to the 1-form ω by

$$\begin{aligned}(\Phi_\varphi \omega)(X;Y) &= (L_{\varphi X}\omega - L_X\tilde{\omega})(Y) = \\ &= (\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) + \omega((L_Y\varphi)X),\end{aligned}$$

where $\tilde{\omega}(Y) = (\omega \circ \varphi)(X) = \omega(\varphi Y)$ for any $X, Y \in \Im_0^1(M_n)$.

When φ is an almost complex structure, a 1-form satisfying $\Phi_\varphi \omega = 0$ is said to be almost analytic [5, p. 309].

In a Riemannian connection ∇, the equation of almost analytic 1-form ω:

$$(\varphi X)(\omega(Y)) - X(\omega(\varphi Y)) + \omega((L_Y\varphi)X) = 0$$

may be written as

$$(\nabla_{\varphi X}\omega)(Y) - (\nabla_X\omega)(\varphi Y) - \omega((\nabla_X\varphi)Y) + \omega((\nabla_Y\varphi)X) = 0, \qquad (2.1)$$

which is equivalent to the condition for the almost analyticity. Thus, the equation (2.1) is an expression of the condition for the 1-form ω to be almost analytic in terms a Riemannian connection ∇.

Remark: *A tensor field $\eta \in \Im_2^0(M_n)$ which satisfies*

$$\eta(\varphi X, Y) = \eta(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_n)$ is said to be pure. Applications of this type tensor fields are studied by many authors (for example see [1–3]).

From (2.1), taking the alternation with respect to X and Y, we find that

$$(\nabla_{\varphi X}\omega)(Y) - (\nabla_{\varphi Y}\omega)(X) + (\nabla_Y\omega)(\varphi X) - (\nabla_X\omega)(\varphi Y) = 0,$$

i.e. $(\nabla_X\omega)Y - (\nabla_Y\omega)X = (\wedge\nabla\omega)(X,Y)$ is the pure 2-form with respect to the structure φ for an almost analytic 1-form ω on a Riemannian manifold.

We calculate

$$-\omega((\nabla_X\varphi)Y) + \omega((\nabla_Y\varphi)X) = -\omega((\nabla_X\varphi)Y)$$

$$+(\nabla_X\omega)(\varphi Y) - (\nabla_X\omega)(\varphi Y) + \omega((\nabla_Y\varphi)X)$$

$$+(\nabla_Y\omega)(\varphi X) - (\nabla_Y\omega)(\varphi X) = -(\nabla_X\omega \circ \varphi)Y$$

$$+(\nabla_X\omega)(\varphi Y) + (\nabla_Y\omega \circ \varphi) - (\nabla_Y\omega)(\varphi X). \qquad (2.2)$$

By virtue of (2.2), the equation (2.1) is written as

$$(\nabla_Y \tilde{\omega})X - (\nabla_X \tilde{\omega})Y = (\nabla_Y \omega)(\varphi X) - (\nabla_{\varphi X}\omega)(Y). \tag{2.3}$$

If we substitute φX into X, then the equation (2.3) may also be written as

$$-((\nabla_Y \omega)X - (\nabla_X \omega)Y) = (\nabla_Y \tilde{\omega})\varphi X - (\nabla_{\varphi X}\tilde{\omega})Y$$

or

$$(\nabla_Y \tilde{\tilde{\omega}})X - (\nabla_X \tilde{\tilde{\omega}})Y = (\nabla_Y \tilde{\omega})\varphi X - (\nabla_{\varphi X}\tilde{\omega})Y, \tag{2.4}$$

where $\tilde{\tilde{\omega}} = \tilde{\omega} \circ \varphi$. The equation (2.4) is condition that $\tilde{\omega} \in \Im_1^0(M_n)$ be almost analytic.

From equations (2.3) and (2.4), we have

Theorem 1 *If a 1-form ω on a Riemannian manifold with an almost complex structure φ is almost analytic, then the 1-form $\tilde{\omega} = \omega \circ \varphi$ is also almost analytic.*

We shall now prove the following proposition.

Proposition *In a Riemannian manifold, the condition*

$$\Phi_\varphi \tilde{\omega} = (\Phi_\varphi \omega) \circ \varphi + \omega \circ N_\varphi$$

holds, where N_φ is the Nijenhuis tensor of φ.

Proof. We shall now apply the operator Φ_φ to the 1-form $\tilde{\omega} = \omega \circ \varphi$

$$(\Phi_\varphi \tilde{\omega})(X;Y) = (L_{\varphi X}\tilde{\omega} - L_X(\tilde{\omega} \circ \varphi))(Y) = (L_{\varphi X}(\omega \circ \varphi) - L_X((\omega \circ \varphi) \circ \varphi))(Y)$$

$$= ((L_{\varphi X}\omega) \circ \varphi + \omega \circ (L_{\varphi X}\varphi) - (L_X(\omega \circ \varphi)) \circ \varphi - (\omega \circ \varphi) \circ (L_X \varphi))(Y)$$

$$= (L_{\varphi X}\omega - L_X(\omega \circ \varphi))(\varphi Y) + (\omega \circ (L_{\varphi X}\varphi) - (\omega \circ \varphi) \circ (L_X \varphi))(Y)$$

$$= (L_{\varphi X}\omega - L_X(\omega \circ \varphi))(\varphi Y) + \omega((L_{\varphi X}\varphi)Y) - \omega(\varphi(L_X \varphi)Y)$$

$$= (\Phi_{\varphi X}\omega)(\varphi Y) + \omega([\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y])$$

$$= (\Phi_\varphi \omega)(X; \varphi Y) + \omega(N_\varphi(X,Y)). \tag{2.5}$$

Thus, the proof is complete. □

We note that the 1-form ω in Proposition *is not necessary to be almost analytic*, in general. In particular, if the 1-form ω is almost analytic, then from Theorem 1 and Proposition, we have

Theorem 2 *For an almost analytic 1-form ω on a Riemannian manifold with an almost complex structure φ, we have the following equation.*

$$\omega \circ N_\varphi = 0.$$

Let $\varphi \in \Im_1^1(M_n)$. Then, the complete lift ${}^C\varphi$ of φ along the cross-section ω to $T^*(M_n)$ has local components of the form

$$ {}^C\varphi = \begin{pmatrix} \varphi_i^h & 0 \\ (\partial_i \varphi_h^a - \partial_h \varphi_i^a)\omega_a - \varphi_i^t \partial_t \omega_h + \varphi_h^t \partial_i \omega_t & \varphi_i^h \end{pmatrix} $$

with respect to the adapted (B,C)-frame [4]. We consider that the local vector fields

$$ {}^C X_{(i)} = {}^C(\frac{\partial}{\partial x^i}) = {}^C(\delta_i^h \frac{\partial}{\partial x^h}) = \begin{pmatrix} X^i \\ 0 \end{pmatrix} $$

and

$$ {}^V X^{(\bar{\imath})} = {}^V(dx^i) = {}^V(\delta_h^i dx^h) = \begin{pmatrix} 0 \\ \delta_h^i \end{pmatrix} $$

$i = 1, ..., n;\ \bar{\imath} = n+1, ..., 2n$ span the module of vector fields in $\pi^{-1}(U)$. Hence, any tensor fields is determined in $\pi^{-1}(U)$ by their actions on ${}^C X$ and ${}^V\theta$ for any $X \in \Im_0^1(M_n)$ and $\theta \in \Im_1^0(M_n)$. The complete lift ${}^C\varphi$ has the properties

$$ \begin{cases} {}^C\varphi({}^C X) = {}^C(\varphi(X)) + \gamma(L_X \varphi), \\ {}^C\varphi({}^V\theta) = {}^V(\varphi(\theta)), \end{cases} \tag{2.6} $$

which characterize ${}^C\varphi$, where $\varphi(\theta) \in \Im_1^0(M_n)$.

Theorem 3 *Let M_n be a Riemannian manifold with an almost complex structure φ. Then the complete lift ${}^C\varphi \in \Im_1^1(T^*(M_n))$ of φ, when restricted to the cross-section determined by an almost analytic 1-form ω on M_n, is an almost complex structure.*

Proof. Let $\varphi, \psi \in \Im_1^1(M_n)$ and $N \in \Im_2^1(M_n)$, using (1.7), (1.8) and (2.6), we have

$$ \gamma(\varphi \mp \psi) = \gamma(\varphi) \mp \gamma(\psi), \quad {}^C\varphi(\gamma\psi) = \gamma(\psi \circ \varphi), \quad (\gamma N)({}^C X) = \gamma N_X \tag{2.7} $$

where N_X is the tensor field of type (1,1) on M_n defined by $N_X(Y) = N(X,Y)$ for any $Y \in \Im_0^1(M_n)$.

If $X \in \Im_0^1(M_n)$, then from (2.6) and (2.7), we have

$$ ({}^C\varphi)^2({}^C X) = ({}^C\varphi \circ {}^C\varphi)({}^C X) = {}^C\varphi({}^C\varphi({}^C X)) = {}^C\varphi({}^C(\varphi(X)) $$

$$ +\gamma(L_X\varphi)) = {}^C\varphi({}^C(\varphi(X))) + {}^C\varphi(\gamma(L_X\varphi)) = {}^C(\varphi(\varphi(X))) $$

$$ +\gamma(L_{\varphi X}\varphi) + \gamma((L_X\varphi)\circ\varphi) = {}^C((\varphi\circ\varphi)(X)) + \gamma(L_{\varphi X}\varphi + (L_X\varphi)\circ\varphi) $$

$$ = {}^C(\varphi^2)({}^C X) - \gamma(L_X(\varphi\circ\varphi)) + \gamma(L_{\varphi X}\varphi + (L_X\varphi)\circ\varphi) $$

$$ = {}^C(\varphi^2)({}^C X) + \gamma(L_{\varphi X}\varphi - \varphi(L_X\varphi)) = {}^C(\varphi^2)({}^C X) + \gamma(N_{\varphi,X}) $$

$$= {}^{C}(\varphi^2)({}^{C}X) + (\gamma N_\varphi)({}^{C}X), \tag{2.8}$$

where $N_{\varphi,X}(Y) = (L_{\varphi X}\varphi - \varphi(L_X\varphi))(Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + \varphi^2[X,Y] = N_\varphi(X,Y)$ is nothing but the Nijenhuis tensor constructed by φ and γN_φ has local coordinates of the form $\gamma N_\varphi = \begin{pmatrix} 0 & 0 \\ N_{ij}^h \omega_h & 0 \end{pmatrix}$ (see (1.8)).

Similarly, if $\theta \in \Im_1^0(M_n)$, then by (2.6), we have

$$({}^{C}\varphi)^2({}^{V}\theta) = ({}^{C}\varphi \circ {}^{C}\varphi)({}^{V}\theta) = {}^{C}\varphi({}^{C}\varphi({}^{V}\theta)) = {}^{C}\varphi({}^{V}(\varphi(\theta))$$

$$= {}^{V}(\varphi(\varphi(\theta)) = {}^{V}((\varphi \circ \varphi)(\theta)) = {}^{C}(\varphi^2)({}^{V}\theta) \tag{2.9}$$

By virtue of Theorem 2, we can easily say that $\gamma N_\varphi = 0$. From (2.8), (2.9) and linearity of the complete lift, we have

$$({}^{C}\varphi)^2 = {}^{C}(\varphi^2) = {}^{C}(-I_{M_n}) = -I_{T^*(M_n)}.$$

This completes the proof. □

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