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Probabilities for absolute irreducibility of multivariate polynomials by the polytope method

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Abstract

Motivated by the Dubickas's result in [1], which computes the probability of the irreducible polynomials by Eisenstein's criterion for some families of polynomials in $\mathbb{Z}[x]$, we calculate the probabilities which represent the ratio of absolutely irreducible multivariate polynomials by the polytope method in some families of polynomials over arbitrary fields.

Key Words: Absolute irreducibility, polytopes, multivariate polynomials

1. Introduction

Throughout this study, F^* stands for the set of all nonzero elements for a field F, and $n \geq 2$ is an integer. Moreover, all mentioned polynomials have at least two terms.

We need to give some definitions before introducing the main idea.

Let $\mathbb{R}^n$ denote the real n-dimensional space and S be a subset of $\mathbb{R}^n$. The smallest convex set containing S, denoted by *conv(S)*, is called the *convex hull* of S. If $S = \{a_1, a_2, ..., a_n\}$ is a finite set then we shall denote $conv(S)$ by $conv(a_1, ..., a_n)$. Note that

$$conv(S) = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \{x_1, ..., x_k\} \subseteq S, \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

The convex hull of finitely many points in $\mathbb{R}^n$ is called a ***polytope***. A point of a polytope is called a *vertex* if it is not on the line segment joining any other two different points of the polytope. It is known that a polytope is always the convex hull of its vertices (see [6, Proposition 2.2]).

The principle operation for convex sets in $\mathbb{R}^n$ is defined as follows.

Definition 1.1 For any two sets A and B in $\mathbb{R}^n$, the sum

$$A + B = \{a + b : a \in A, b \in B\}$$

is called *Minkowski sum*, or *vector addition* of A and B.

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A point in $\mathbb{R}^n$ is called *integral* if its coordinates are integers. A polytope in $\mathbb{R}^n$ is called *integral* if all of its vertices are integral. An integral polytope C is called ***integrally decomposable*** if there exist integral polytopes A and B such that $C = A + B$ where both A and B have at least two points. Otherwise, C is called ***integrally indecomposable***.

Definition 1.2 *Let F be any field and consider any polynomial*

$$f(x_1, x_2, ..., x_n) = \sum c_{e_1 e_2 ... e_n} x_1^{e_1} x_2^{e_2} ... x_n^{e_n} \in F[x_1, ..., x_n].$$

We can think an exponent vector $(e_1, e_2, ..., e_n)$ of f as a point in $\mathbb{R}^n$. The ***Newton polytope*** *of f, denoted by P_f, is defined as the convex hull in $\mathbb{R}^n$ of all the points $(e_1, ..., e_n)$ with $c_{e_1 e_2 ... e_n} \neq 0$.*

Recall that a polynomial over a field F is called ***absolutely irreducible*** if it remains irreducible over every algebraic extension of F.

Using Newton polytopes of multivariate polynomials, we can determine infinite families of absolutely irreducible polynomials over an arbitrary field F by the following result due to Ostrowski [5]; *c.f.* [2, Lemma 2.1].

Lemma 1.3 *Let $f, g, h \in F[x_1, ..., x_n]$ with $f \neq 0$ and $f = gh$. Then $P_f = P_g + P_h$.*

As a direct result of Lemma 1.3, we have the following corollary which is an *irreducibility criterion* for multivariate polynomials over arbitrary fields.

Corollary 1.4 *Let F be any field and f a nonzero polynomial in $F[x_1, ..., x_n]$ not divisible by any x_i. If the Newton polytope P_f of f is integrally indecomposable then f is absolutely irreducible over F.*

When P_f is integrally decomposable, depending on the given field, f may be reducible or irreducible. For example, the polynomial $f = x^9 + y^9 + z^9$ has the Newton polytope $P_f = conv((9,0,0), (0,9,0), (0,0,9)) = conv((6,0,0), (0,6,0), (0,0,6)) +$
$conv((3,0,0), (0,3,0), (0,0,3))$. But, while $f = x^9 + y^9 + z^9 = (x + y + z)^9$ over $\mathbb{F}_3$, it is irreducible over $\mathbb{F}_2, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_{11}$, where $\mathbb{F}_m$ represents the finite field with m elements.

In [2], [3] and [4], infinitely many integrally indecomposable polytopes in $\mathbb{R}^n$ are presented and then, being associated to these polytopes, infinite families of absolutely irreducible polynomials are determined over any field F.

Let F be any field. For a multivariate polynomial

$$f(x_1, x_2, ..., x_n) \in F[x_1, x_2, ..., x_n],$$

we say that f is *absolutely irreducible over F by the polytope method* if its Newton polytope P_f is integrally indecomposable.

Definition 1.5 *Let F be any field. We define a relation $\sim$ on the ring of multivariate polynomials $F[x_1, ..., x_n]$ by*

$$f = \sum c_{e_1 e_2 ... e_n} x_1^{e_1} x_2^{e_2} ... x_n^{e_n} \quad \sim \quad g = \sum d_{e_1 e_2 ... e_n} x_1^{e_1} x_2^{e_2} ... x_n^{e_n}$$

if and only if, for each coefficient $c_{e_1e_2...e_n}$ *of* f *and* $d_{e_1e_2...e_n}$ *of* g, *there exists an element* $a_{e_1e_2...e_n} \in F^*$ *such that*

$$c_{e_1e_2...e_n} = a_{e_1e_2...e_n} \; d_{e_1e_2...e_n}.$$

It can be verified directly that $\sim$ is an equivalence relation. For $f \in F[x_1, ..., x_n]$, we shall use the notation $[f]$ to denote the equivalence class of f.

Note that, for $f, g \in F[x_1, ..., x_n]$, if $f \sim g$ and f is absolutely irreducible over F by the polytope method then g is also absolutely irreducible over F by the polytope method.

Let F be an arbitrary field. For any family of multivariate polynomials, all of which have bounded degrees,

$$\mathcal{F} = \{f_i(x_1, ..., x_n) \in F[x_1, ..., x_n] : i \in I\},$$

we shall use the notation $P_{\mathcal{F}}$ to denote the chance that the representative f of a random polynomial class $[f]$ in $\mathcal{F}/\sim$ is absolutely irreducible over F by the polytope method.

Let F be a finite field and $\mathcal{F}$ a finite set of multivariate polynomials over F. We shall use the notation $\widetilde{P}_{\mathcal{F}}$ to indicate the probability of a random polynomial f in $\mathcal{F}$ to be irreducible by the polytope method over F.

The main aim of this paper is to introduce the probabilities $P_{\mathcal{F}}$ and $\widetilde{P}_{\mathcal{F}}$ and calculate them for some families $\mathcal{F}$ of multivariate polynomials.

Remark 1.6 *Let* n *be an arbitrary given positive integer. Consider any family of multivariate polynomials*

$$\mathcal{F} = \{f_i(x_1, ..x_n) \in F[x_1, ..., x_n] : deg(f) \leq n, i \in I\}$$

over any field F, *where* $deg(f)$ *indicates the total degree for a polynomial* f. *Let*

$$\mathcal{F} = \bigcup_{i=1}^{\beta} [f_i]$$

be a disjoint union of the equivalence classes $[f_i]$, *where each* f_i *has* s_i *number of terms. Assume that* α *representatives* $f_1, ..., f_\alpha$ *of the classes* $[f_1], ..., [f_\beta]$ *have integrally indecomposable Newton polytopes. Then we have*

$$P_{\mathcal{F}} = \frac{\alpha}{\beta}.$$

Moreover, if F *is finite with* $|F| = q$ *then we have*

$$\widetilde{P}_{\mathcal{F}} = \frac{(q-1)^{s_1} + ... + (q-1)^{s_\alpha}}{(q-1)^{s_1} + (q-1)^{s_2} + ... + (q-1)^{s_\beta}}.$$

We see that over the finite field $\mathbb{F}_2$, $P_{\mathcal{F}} = \widetilde{P}_{\mathcal{F}}$.

Actually, for the family

$$\mathcal{F} = \left\{ \sum c_{ie_1ie_2...ie_n} x_1^{ie_1} x_2^{ie_2} ... x_n^{ie_n} \in F[x_1, ..., x_n] \; : \; c_{ie_1ie_2...ie_n} \in F^*, i \in I \right\}$$

of polynomials with $deg(f) \leq n$, *we have* $\beta = |\mathcal{F}_1|$, *where* $\mathcal{F}_1$ *is the subset of* $\mathcal{F}$, *obtained by taking only the members of* $\mathcal{F}$ *whose all coefficients are* 1, *given by*

$$\mathcal{F}_1 = \left\{ \sum c_{ie_1 ie_2 \ldots ie_n} x_1^{ie_1} x_2^{ie_2} \ldots x_n^{ie_n} \in F[x_1, \ldots, x_n] \ : \ c_{ie_1 ie_2 \ldots ie_n} = 1, i \in I \right\}.$$

In general, for an arbitrary polynomial $f(x_1, ..x_n) \in F[x_1, ..., x_n]$, it is not easy to determine whether f is absolutely irreducible or not over F by the polytope method. As a result, it is hard to find α for a given family $\mathcal{F}$ of multivariate polynomials. In this paper, we have found α only for some special families of multivariate polynomials.

Proposition 1.7 *Let* F *be a finite field with* $|F| = q$. *Consider a finite set of polynomials*

$$\mathcal{F} = \left\{ \sum c_{ie_1 ie_2 \ldots ie_n} x_1^{ie_1} x_2^{ie_2} \ldots x_n^{ie_n} \in F[x_1, \ldots, x_n] \ : \ c_{ie_1 ie_2 \ldots ie_n} \in F^*, i \in I \right\}$$

such that all polynomials in $\mathcal{F}$ *have the same number of terms. Let* $|\mathcal{F}| = \epsilon$. *If we have* δ *polynomials in* $\mathcal{F}$ *whose Newton polytopes are integrally indecomposable, then*

$$P_{\mathcal{F}} = \widetilde{P}_{\mathcal{F}} = \frac{\delta}{\epsilon}.$$

Proof. Let $[f_1], ..., [f_\beta]$ be the classes of $\mathcal{F}/\sim$ such that $[f_1], ..., [f_\alpha]$ have integrally indecomposable representatives. If any polynomial $f \in \mathcal{F}$ has r number of terms, then we have

$$P_{\mathcal{F}} = \frac{\alpha}{\beta} = \frac{(q-1)^r \alpha}{(q-1)^r \beta} = \widetilde{P}_{\mathcal{F}} = \frac{\delta}{\epsilon}.$$

□

In Section 2, we compute $P_{\mathcal{F}}$ or $\widetilde{P}_{\mathcal{F}}$ for some families $\mathcal{F}$ of multivariate polynomials. We consider the families having Newton polytopes as line segments, triangles, pyramids and bipyramids.

2. Some related examples

We start with the simplest forms of two-term multivariate polynomials with two variables having Newton polytopes as line segments. Then we examine some multivariate polynomials with three variables. Of course, we can give arbitrary number of examples of families of polynomials within any number of variables $x_1, ..., x_n$. In this section, for any field F, we assume that all considered polynomials f in $F[x_1, ..., x_n]$ have at least two terms and are not divisible by any x_i.

Throughout this paper, ϕ denotes the Euler-phi function. Moreover, for positive integers $M \leq N$ and i, $S_{M-N}(i)$ denotes the set

$$S_{M-N}(i) = \{x \in \mathbb{Z} : \quad M \leq x \leq N, \quad gcd(x, i) = 1\}.$$

Example 2.1 *Consider the set of polynomials*

$$S = \{ax^n + by^m : a, b \in F^*, 1 \leq n \leq 3, 1 \leq m \leq 5\}.$$

The number of polynomial classes of S *is* $3 \cdot 5 = 15 = \beta$ *since*

$$S = \bigcup_{1 \leq i \leq 3, 1 \leq j \leq 5} [x^i + y^j].$$

Any polynomial $f = ax^n + by^m$ *in* S *has the Newton polytope of the form* $conv((n,0),(0,m))$ *which is integrally indecomposable, by [2, Corollary 4.3] or [4, Corollary 2.4], if and only if* $gcd(n,m) = 1$.

Consequently, we have $\alpha = \sum_{i=1}^{3} |S_{1-5}(i)| = 12$ *giving*

$$P_S = \frac{12}{15} = \frac{4}{5}.$$

Moreover, if F *is a finite field with* $|F| = q$ *then we have*

$$\widetilde{P}_{\mathcal{F}} = \frac{(q-1)^2 12}{(q-1)^2 15} = \frac{4}{5}.$$

In addition, for the family of polynomials

$$F = \{ax^N + by^m : 1 \leq m \leq N\}$$

we have $P_F = \phi(N)/N$.

More generally, for the set of polynomials

$$T = \{ax^n + by^m : 1 \leq n \leq N, 1 \leq m \leq M, N \leq M\},$$

by a similar argument we have

$$P_T = \frac{\sum_{i=1}^{N} |S_{1-M}(i)|}{NM}.$$

Furthermore, if F *is a finite field then* $P_S = \widetilde{P}_S$, $P_F = \widetilde{P}_F$ *and* $P_T = \widetilde{P}_T$ *by Proposition 1.7.*

Example 2.2 *Let* F *be any field. Consider the family of polynomials*

$$S = \{ax^n + by^m + \sum c_{ij} x^i y^j : a, b \in F^*, c_{ij} \in F, mi + nj = mn\}$$

having the family of Newton polytopes $\{conv((n,0),(0,m))\}$, *where* $1 \leq n \leq 3$ *and* $1 \leq m \leq 2$. *We see that*

$$S = \{ax^3 + by, ax^3 + by^2, ax^2 + by, ax^2 + by^2, ax^2 + by^2 + c_{ij}xy, ax + by, ax + by^2\},$$

where $a, b, c_{ij} \in F^*$. *Therefore, we have* $\alpha = 5$ *and* $\beta = 7$, *i.e.*

$$P_S = \frac{5}{7}.$$

In addition, if F is a finite field with $|F| = q$, then we have

$$\widetilde{P}_{\mathfrak{F}} = \frac{5(q-1)^2}{6(q-1)^2 + (q-1)^3}.$$

Example 2.3 *Consider the set of polynomials*

$$S = \{ax^n + by^m + \sum c_{ij}x^i y^j : a, b \in F^*, c_{ij} \in F, mi + nj = mn\}$$

having the family of Newton polytopes $\{conv((n,0),(0,m))\}$, where N and M are given positive integers such that $1 \le n \le N$, $1 \le m \le M$, with $N \le M$. In order to find β, we form the set

$$K = \{x^n + y^m + \sum cx^i y^j : mi + nj = mn\},$$

where $c \in \{0,1\}$. Elements of K have the family of Newton polytopes

$$N = \{conv((n,0),(0,m))\},$$

which are line segments. Any element $conv((a,0),(0,b))$ of N is integrally indecomposable, by [2, Corollary 4.3] or [4, Corollary 2.4], if and only if $gcd(a,b) = 1$. Therefore, we have

$$P_S = \frac{\sum_{i=1}^{N} \phi(i) + \sum_{i=2}^{N} \phi(i)}{|K|} = \frac{2\sum_{i=1}^{N} \phi(i) - 1}{|K|}$$

$$= \frac{\sum_{i=1}^{N} |S_{1-N}(i)|}{|K|} \quad if \quad N = M,$$

and

$$P_S = \frac{2\sum_{i=1}^{N} \phi(i) + \sum_{i=N+1}^{M} |S_{1-N}(i)| - 1}{|K|}$$

$$= \frac{\sum_{i=1}^{N} |S_{1-M}(i)|}{|K|} \quad if \quad N < M.$$

For example, if $N = M = 3$ then we have

$$\begin{aligned} S \;=\; & \{ax + by, ax + by^2, ax + by^3, ax^2 + by, ax^2 + by^2, ax^2 + by^2 + cxy, ax^2 + by^3, \\ & ax^3 + by, ax^3 + by^2, ax^3 + by^3, ax^3 + by^3 + cxy^2, ax^3 + by^3 + cx^2y, \\ & ax^3 + by^3 + cxy^2 + dx^2y : a, b, c, d \in F^*\} \end{aligned}$$

and

$$\begin{aligned} K \;=\; & \{x + y, x + y^2, x + y^3, x^2 + y, x^2 + y^2, x^2 + y^2 + xy, x^2 + y^3, x^3 + y, \\ & x^3 + y^2, x^3 + y^3, x^3 + y^3 + xy^2, x^3 + y^3 + x^2y, x^3 + y^3 + xy^2 + x^2y\}. \end{aligned}$$

Consequently, we get

$$P_S = \frac{\sum_{i=1}^{3} |S_{1-3}(i)|}{|K|} = \frac{7}{13}.$$

Moreover, if F is a finite field with $|F| = q$, then we have

$$\widetilde{P}_S = \frac{7(q-1)^2}{7(q-1)^2 + 2(q-1)^2 + 3(q-1)^3 + (q-1)^4}.$$

Example 2.4 *Consider the family of polynomials*

$$S = \{ax^u + by^v z^w + \sum c_{ijk} x^i y^j z^k : a, b \in F^*, c_{ijk} \in F\}$$

having the family of Newton polytopes

$$N = \{conv((u,0,0),(0,v,w))\},$$

where $1 \le u \le A,\ 1 \le v \le B,\ 1 \le w \le C$ for some positive integers A, B, C. A polynomial

$$f = a_1 x^{u_1} + b_1 y^{v_1} z^{w_1} + \sum c_{ijk} x^i y^j z^k$$

in this family has the Newton polytope $conv((u_1,0,0),(0,v_1,w_1))$ which is integrally indecomposable, by [2, Corollary 4.3] or [4, Corollary 2.4], if and only if

$$gcd(u_1, v_1, w_1) = 1.$$

Hence, we see that

$$P_S \ge \frac{\sum_{i=1}^{A} |S_{1-B}(i)| + \sum_{i=1}^{A} |S_{1-C}(i)| + \sum_{i=1}^{B} |S_{1-C}(i)|}{|K|}.$$

Actually, we have

$$P_S = \frac{r + \sum_{i=1}^{A} \Big(|S_{1-B}(i)| + |S_{1-C}(i)| \Big) + \sum_{i=1}^{B} |S_{1-C}(i)|}{|K|},$$

where r is the cardinality of the set

$$T = \{(e_1,e_2,e_3) : 1 \le e_1 \le A, 1 \le e_2 \le B, 1 \le e_3 \le C, gcd(e_1,e_2,e_3) = 1\},$$

and K is the set given by

$$K = \{x^u + y^v z^w + \sum c x^i y^j z^k : 1 \le u \le A, 1 \le v \le B, 1 \le w \le C\},$$

where $c \in \{0,1\}$ and the point (i,j,k) lies on the line segment from the point $(u,0,0)$ to the point $(0,v,w)$.

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Example 2.5 *Consider the set of polynomials*

$$S = \{ax^n + by^m + cx^u y^v + \sum c_{ij} x^i y^j : a, b, c \in F^*, c_{ij} \in F\}$$

having the family of Newton polytopes

$$F = \{conv((n,0),(0,m),(u,v))\},$$

where $1 \leq n \leq N, 1 \leq m \leq M$ *and N,M,A,B,C,D are given positive integers satisfying*

$$A \leq u \leq B, \quad C \leq v \leq D, \quad Mu + Nv > MN.$$

Without loss of generality, assume that $N \leq M < C \leq A \leq B \leq D$.

A polynomial

$$f = a_1 x^{e_1} + b_1 y^{e_2} + c_1 x^{e_3} y^{e_4} + \sum c_{ij} x^i y^j$$

in the set S *has triangular Newton polytope* $conv((e_1,0),(0,e_2)(e_3,e_4))$ *which is integrally indecomposable, by [2, Corollary 4.5] or [4, Proposition 2.6], if and only if*

$$gcd(e_3 - e_1, e_4, e_3, e_4 - e_2) = gcd(e_1, e_2, e_3, e_4) = 1.$$

Moreover, we observe that $\beta = |K| \geq NM(B-A+1)(D-C+1)$ *for the set*

$$K = \{x^n + y^m + x^u y^v + \sum c x^i y^j : c \in \{0,1\}, (i,j) \in conv((n,0),(0,m),(u,v))\}.$$

As a result, we have

$$P_S \geq \left[r + \sum_{i=1}^{N} \Big(|S_{1-M}(i)| + |S_{A-B}(i)| + |S_{C-D}(i)| \Big) + \sum_{i=1}^{M} \Big(|S_{A-B}(i)| + |S_{C-D}(i)| \Big) + \sum_{i=A}^{B} |S_{C-D}(i)| \right] \Big/ \Big[|K| \Big],$$

where r *is the cardinality of the set of triple and quad relatively prime exponents in the related intervals. That is, e.g., we call an exponent* (e_1, e_2, e_3) *triple relatively prime if* $gcd(e_1, e_2, e_3) = 1$.

Example 2.6 *Consider the set of polynomials*

$$S = \{ax^n + by^m + cz^l + dx^u y^v z^w : a,b,c,d \in F^*\}$$

which have the set of Newton polytopes

$$F = \{conv((n,0,0),(0,m,0),(0,0,l),(u,v,w))\},$$

where $1 \leq n \leq N, 1 \leq m \leq M, 1 \leq l \leq L, 1 \leq u \leq U, 1 \leq v \leq V, 1 \leq w \leq W$ *for some integers* N, M, L, U, V *and W.*

Any member $conv((e_1,0,0),(0,e_2,0),(0,0,e_3),(e_4,e_5,e_6))$ *of* F *is a pyramid in* $\mathbb{R}^3$ *which is integrally indecomposable, by [2, Theorem 4.2] or [4, Example 3.17], if and only if*

$$gcd(e_1, e_2, e_3, e_4, e_5, e_6) = 1.$$

By a similar argument as in the Example 2.4 and Example 2.5, we see that

$$\begin{aligned} P_S &= \Big[r+\textstyle\sum_{i=1}^{N}\Big(|S_{1-M}(i)|+|S_{1-L}(i)|+|S_{1-U}(i)|+|S_{1-V}(i)|+|S_{1-W}(i)|\Big) \\ &\quad +\textstyle\sum_{i=1}^{M}\Big(|S_{1-L}(i)|+|S_{1-U}(i)|+|S_{1-V}(i)|+|S_{1-W}(i)|\Big)+\sum_{i=1}^{L}\Big(|S_{1-U}(i)|+|S_{1-V}(i)|+|S_{1-W}(i)|\Big) \\ &\quad +\textstyle\sum_{i=1}^{U}\Big(|S_{1-V}(i)|+|S_{1-W}(i)|\Big)+\sum_{i=1}^{V}\Big(|S_{1-W}(i)|\Big)\Big]/\Big[NMLUVW\Big] \end{aligned}$$

where r is the number of relatively prime exponents having three, four, five or six components in the related intervals.

Example 2.7 *Consider the family of polynomials*

$$S=\{ax^L+by^M+cz^N+dx^uy^vz^w+ex^ry^sz^t : a,b,c,d,e\in F^*\}$$

having the set of bipyramid Newton polytopes

$$F=\{conv((L,0,0),(0,M,0),(0,0,N),(u,v,w),(r,s,t))\}$$

with $L,M,N\geq 2$ such that

$$L+1\leq u\leq A,\quad M+1\leq v\leq B,\quad N+1\leq w\leq C$$

$$D\leq r\leq L-1,\quad E\leq s\leq M-1,\quad F\leq t\leq N-1$$

for some given positive integers A,B,C,D,E,F,L,M and N. Any bipyramid

$$conv((L,0,0),(0,M,0),(0,0,N),(e_1,e_2,e_3),(e_4,e_5,e_6))$$

in the family F is integrally indecomposable, by [4, Lemma 3.2 or Example 3.5,(2)], if and only if

$$gcd(L,M,N,e_1,e_2,e_3,e_4,e_5,e_6)=1.$$

Hence, we have

$$\begin{aligned} P_S \;=\; & \Big[|S_{M-M}(L)| + |S_{N-N}(L)| + |S_{(L+1)-A}(L)| + |S_{(M+1)-B}(L)| + |S_{(N+1)-C}(L)| \\ & + |S_{D-(L-1)}(L)| + |S_{E-(M-1)}(L)| + |S_{F-(N-1)}(L)| + |S_{N-N}(M)| + |S_{(L+1)-A}(M)| \\ & + |S_{(M+1)-B}(M)| + |S_{(N+1)-C}(M)| + |S_{D-(L-1)}(M)| + |S_{E-(M-1)}(M)| \\ & + |S_{F-(N-1)}(M)| + |S_{(L+1)-A}(N)| + |S_{(M+1)-B}(N)| + |S_{(N+1)-C}(N)| \\ & + |S_{D-(L-1)}(N)| + |S_{E-(M-1)}(N)| + |S_{F-(N-1)}(N)| \\ & + \sum_{i=L+1}^{A} |S_{(M+1)-B}(i)| + \sum_{i=L+1}^{A} |S_{(N+1)-C}(i)| + \sum_{i=L+1}^{A} |S_{D-(L-1)}(i)| \\ & + \sum_{i=L+1}^{A} |S_{E-(M-1)}(i)| + \sum_{i=L+1}^{A} |S_{F-(N-1)}(i)| + \sum_{i=M+1}^{B} |S_{(N+1)-C}(i)| \\ & + \sum_{i=M+1}^{B} |S_{D-(L-1)}(i)| + \sum_{i=M+1}^{B} |S_{E-(M-1)}(i)| + \sum_{i=M+1}^{B} |S_{F-(N-1)}(i)| \\ & + \sum_{i=N+1}^{C} |S_{D-(L-1)}(i)| + \sum_{i=N+1}^{C} |S_{E-(M-1)}(i)| + \sum_{i=N+1}^{C} |S_{F-(N-1)}(i)| \\ & + \sum_{i=D}^{L-1} |S_{E-(M-1)}(i)| + \sum_{i=D}^{L-1} |S_{F-(N-1)}(i)| \\ & + \sum_{i=E}^{M-1} |S_{F-(N-1)}(i)| + r\Big] \Big/ \Big[(A-L)(B-M)(C-N)(L-D)(M-E)(N-F)\Big] \end{aligned}$$

where r is the number of relatively prime exponents having three, four, five, six, seven, eight or nine components in the indicated intervals.

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