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Module classes and the lifting property

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Dedicated to the memory of Cemal KOÇ

Abstract

Let R be a ring. A collection of R-modules containing the zero module and closed under isomorphisms will be denoted by $\mathcal{X}$. An R-module M is said to be $\mathcal{X}$*-lifting* if for every $\mathcal{X}$-submodule N of M there exists $A \leq N$ such that $M = A \oplus B$ and $N \cap B$ is small in B [11]. In the present paper, we consider the question:

Can we characterize $\mathcal{X}$-lifting modules via objects of the class $\mathcal{X}$?

Key Words: Lifting module, torsion theory.

1. Introduction

Throughout this work all rings will be associative with identity and modules will be unital right modules.

Let R be a ring and M be an R-module. A submodule N of M is said to be a *small* in M, denoted by $N \ll M$, whenever $L \leq M$ and $M = N + L$ then $M = L$, and M is said to be a *lifting module* (or D_1-module) if for any submodule N of M there exists $A \leq N$ such that $M = A \oplus B$ and $N \cap B \ll B$.

By a class $\mathcal{X}$ of R-modules we mean a collection of R-modules containing the zero module and closed under isomorphisms, i.e., any module isomorphic to some module in $\mathcal{X}$ also belongs to $\mathcal{X}$. By a $\mathcal{X}$*-module* we mean any member of $\mathcal{X}$, and a submodule N of a module M is called $\mathcal{X}$*-submodule* of M if N is an $\mathcal{X}$-module. Doĝruöz and Smith [5] introduced the notion of $\mathcal{X}$-extending modules (see also [6] and [7]). Dually, Koşan and Harmanci [11] introduced $\mathcal{X}$-lifting modules. M is said to be a $\mathcal{X}$*-lifting module* if for every $\mathcal{X}$-submodule N of M there exists $A \leq N$ such that $M = A \oplus B$ and $N \cap B \ll B$.

Example 1.1 **(i)** *Let $\mathcal{X}$ be the class of all torsion $\mathbb{Z}$-modules. Then the $\mathbb{Z}$-module $\mathbb{Z}$ is an $\mathcal{X}$-lifting module.*
(ii) *Let $\mathcal{X}$ be the class of all torsion free $\mathbb{Z}$-modules. The zero submodule is the only small submodule of $\mathbb{Z}$, and for any non-zero submodules N and K with $N + K = \mathbb{Z}$, $N \cap K$ is not a small submodule of $\mathbb{Z}$ and so the $\mathbb{Z}$-module $\mathbb{Z}$ is not an $\mathcal{X}$-lifting module.*
(iii) *Let $\mathcal{X}$ denote the class of all finitely generated $\mathbb{Z}$-modules. Clearly, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are $\mathcal{X}$-lifting modules.*

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(iv) *Let $\mathcal{X}$ be the class of all torsion free $\mathbb{Z}$-modules and p any prime integer and $M = (\mathbb{Z}/p\mathbb{Z}) \oplus \mathbb{Z}$. It is clear that from (ii) and [11, Lemma 2.3], the $\mathbb{Z}$-module M is not $\mathcal{X}$-lifting.*
(v) *Let R be a ring and $\mathcal{X}$ denote the class of all injective R-modules. Then every R-module M is $\mathcal{X}$-lifting.*
(vi) *Let p be any prime integer and $\mathcal{X}_1 = \mathcal{X}_2 = \{T \in Mod - \mathbb{Z} : pT = 0\}$ and $M = (\mathbb{Z}/p\mathbb{Z}) \oplus (\mathbb{Z}/p^3\mathbb{Z})$. Let $M_1 = (\overline{1}, \overline{0})\mathbb{Z}$, $N = (\overline{1}, \overline{p})\mathbb{Z}$, $N_1 = (\overline{0}, \overline{p^2})\mathbb{Z}$, $N = M_1 \oplus N_1$. Then M_1, N_1 and N_2 are all $\mathcal{X}_1$ and $\mathcal{X}_2$ submodules of M, M_1 is a direct summand and N_1 is small in M. By [11, Lemma 2.3], M is both $\mathcal{X}_1$ and $\mathcal{X}_1$-lifting module.*

Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of modules. We write $\mathcal{X} \leq \mathcal{Y}$ in case every object of $\mathcal{X}$ is in $\mathcal{Y}$.

Lemma 1.2 ([11, Lemma 2.5]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be classes of modules with $\mathcal{X} \leq \mathcal{Y}$. Then every $\mathcal{Y}$-lifting module is $\mathcal{X}$-lifting.*

Example 1.3 *Let $\mathcal{X} = \{X \in Mod - \mathbb{Z} : 2X = 0\}$ and $\mathcal{Y} = \{Y \in Mod - \mathbb{Z} : 4Y = 0\}$ and let M be the $\mathbb{Z}$-module $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$. It is easy to see that $\mathcal{X} \leq \mathcal{Y}$ and M is $\mathcal{X}$-lifting but is not an $\mathcal{Y}$-lifting module.*
Let n be a positive integer and let $\mathcal{X}_i (1 \leq i \leq n)$ be classes of R-modules. Classes of R-modules can be combined in different ways to give other classes and we examine how lifting property behave under these constructions. Then $\oplus_{i=1}^n \mathcal{X}_i$ is defined to be the class of R-modules M such that $M = \oplus_{i=1}^n M_i$ is direct sum of $\mathcal{X}_i$-submodules M_i $(1 \leq i \leq n)$.

Lemma 1.4 ([11, Theorem 2.7]) *With the above notation, an R-module M is $(\oplus_{i=1}^n \mathcal{X}_i)$-lifting if and only if M is $\mathcal{X}_i$-lifting for all $1 \leq i \leq n$.*

Example 1.5 *Let M denote the $\mathbb{Z}$-module $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/3\mathbb{Z})$. Let $\mathcal{X}_1 = \{X \in Mod - \mathbb{Z} : 2X = 0\}$, $\mathcal{X}_2 = \{X \in Mod - \mathbb{Z} : 3X = 0\}$. Then M is $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_1 \oplus \mathcal{X}_2$-lifting.*

In [11], a referee asked the following question: Can we characterize $\mathcal{X}$-lifting modules via objects of the class $\mathcal{X}$? In this article, we will give some answers to this question.

The terminologies and notations of Anderson and Fuller [3], and Mohamed and Müller [12] will be freely used.

2. The results

Recall that a projective module P is called *a projective cover* of a module M if there exists an epimorphism $f : P \longrightarrow M$ with $Ker(f) \ll M$. A right R-module is said to be a *perfect* if M possesses a projective cover. So a ring R is called *perfect* if every right R-module is perfect.

Let $\mathcal{P}$ be any class of perfect R-modules. Note that $\mathcal{P}$ is closed under extensions. It is also easy to see that a module M is lifting if and only if M is Mod-R-lifting.

Proposition 2.1 *Let $\mathcal{P}$ be any class of perfect R-modules. Then*
(1) *R is semisimple if and only if $\mathcal{P} = \{M : M$ is a semisimple module $\}$.*
(2) *If R is semisimple, then M is lifting if and only if M is $\mathcal{P}$-lifting.*

Proof. Clear. □

Let $T_{\mathcal{X}}(M)$ denote the sum of $\mathcal{X}$-submodules of M.

Lemma 2.2 *Let $\mathcal{X}$ be any class of R-modules and M be an R-module.*
(1) *If M does not contain any non-zero $\mathcal{X}$-submodule, i.e. $T_{\mathcal{X}}(M) = 0$, then M is $\mathcal{X}$-lifting.*
(2) *Assume that $\mathcal{X}$ is closed under taking homomorphic images and direct sums. If M is $\mathcal{X}$-lifting module then M is $T_{\mathcal{X}}(M)$-lifting.*

Proof. (1) Obvious.
(2) Note that if $\mathcal{X}$ is closed under direct sums and homomorphic images, then $T_{\mathcal{X}}(M)$ belongs to $\mathcal{X}$. Hence if M is $\mathcal{X}$-lifting then M is $T_{\mathcal{X}}(M)$-lifting by Lemma 1.2. □

Proposition 2.3 *Let $\mathcal{X}$ be any class of R-modules and M be an R-module.*
(1) $T_{\mathcal{X}}(M) = \Sigma\{T_{\mathcal{X}}(N) : N$ *is a $\mathcal{X}$-submodule of* $M\}$.
(2) *Assume that $\mathcal{X}$ is closed under taking homomorphic images and direct sums.*
 (a) *For a homomorphism $f : M \longrightarrow N$, $f(T_{\mathcal{X}}(M)) \leq T_{\mathcal{X}}(N)$.*
 (b) Let a module $M = \oplus_{i \in I} M_i$ be a direct sum of modules M_i for all $i \in I$. Then $T_{\mathcal{X}}(M) = \oplus_{i \in I} T_{\mathcal{X}}(M_i)$.

Proof. (1) See [11, Lemma 2.18].
(2)(a) See [11, Lemma 2.19].
(2)(b) See [11, Corollary 2.20]. □

Let $\mathcal{X}$ be a class of right R-modules and M a right R-module. According to [3], the class of all modules generated by $\mathcal{X}$ is denoted by $\mathrm{Gen}(\mathcal{X})$. We denote $\mathrm{Tr}_M(\mathcal{X})$ the trace of $\mathcal{X}$ in M is defined by $\mathrm{Tr}_M(\mathcal{X}) = \sum\{\mathrm{Im}\, h \mid h : K \to M \text{ for some } K \in \mathcal{X}\}$.

Proposition 2.4 *Let $\mathcal{X}$ be any class of R-modules and M an R-module.*
(1) *If $\mathcal{X}$ is closed under taking homomorphic images then $T_{\mathcal{X}}(M) = \mathrm{Tr}_M(\mathcal{X})$.*
(2) $\mathrm{Tr}_M(\mathcal{X}) = \mathrm{Tr}_M(\mathrm{Gen}(\mathcal{X}))$.

Proof. Clear. □

Let $\mathcal{X}$ be the class of all torsion $\mathbb{Z}$-modules and M be the $\mathbb{Z}$-module $\mathbb{Z}$. Since the zero submodule of $\mathbb{Z}$ is the only $\mathcal{X}$-submodule of M, i.e. $T_{\mathcal{X}}(M) = 0$. By Lemma 2.2, the module M is $\mathcal{X}$-lifting.

Theorem 2.5 *Assume that $\mathcal{X}$ is closed under taking homomorphic images and direct sums. If an R-module M is $\mathcal{X}$-lifting then M is $\mathrm{Tr}_M(\mathrm{Gen}(\mathcal{X}))$-lifting.*

Proof. By Lemma 1.2 and Propositions 2.3. and 2.4. □

If $\mathcal{X}$ is a class of modules such that $\mathrm{Hom}_R(X, M) = 0$ for all $X \in \mathcal{X}$ then we shall write $\mathrm{Hom}_R(\mathcal{X}, M) = 0$. The class of all R-modules M with $\mathrm{Ext}_R(\mathcal{X}, M) = 0$ will be denoted by $\mathcal{X}^{\perp}$. This is usually called the right *orthogonal complement* relative to the functor $\mathrm{Ext}_R(-,-)$ of the class $\mathcal{X}$.

Lemma 2.6 *Let M be an R-module. If $M \in \mathcal{X}^{\perp}$, then $T_{\mathcal{X}}(E(M)/M) = 0$.*

Proof. Assume that $T_{\mathcal{X}}(E(M)/M) \neq 0$. Then we have split exact sequence $0 \to M \to U \to U/M \to 0$, where $U \leq E(M)$, $M \leq U$ and $U/M \in \mathcal{X}$. This implies that M is essential in U, a contradiction. □

Proposition 2.7 *Let $\mathcal{X}$ be a class of R-modules and let M be a nonzero R-module. If $M \in \mathcal{X}^{\perp}$, then $E(M)/M$ is an $\mathcal{X}$-lifting module.*

Proof. By Lemmas 1.2 and 2.6. □

Note that if $\mathcal{X}$ is closed under taking homomorphic images, then the converse of Lemma 2.6 is true because $M \in \mathcal{X}^{\perp}$ if and only if every X in $\mathcal{X}$ is projective with respect to the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$. But we do not know the converse of Proposition 2.7 is true or not.

To find a positive answer, we may need an answer to the following question.

Question Let $\mathcal{X}$ be any class of R-modules and M be an R-module. Assume that M is $\mathcal{X}$-lifting. Is $T_{\mathcal{X}}(M) = 0$?

Proposition 2.8 *Let $\mathcal{X}$ be a class of right R-modules and M be an R-module. If every nonzero cyclic singular module has a nonzero submodule in $\mathcal{X}$, then $M \in \mathcal{X}^{\perp}$ if and only if M is injective.*

Proof. Assume that every nonzero cyclic singular module has a nonzero submodule in $\mathcal{X}$. Then, for any nonzero singular module X, $T_{\mathcal{X}}(X) \neq 0$. Let $M \in \mathcal{X}^{\perp}$. If M is not injective, then $E(M)/M$ is a nonzero singular module and $T_{\mathcal{X}}(E(M)/M) = 0$ by Lemma 2.6. This is a contradiction. So M is injective. The converse is clear. □

Let R be a ring and $\mathcal{I}$ denote the class of all injective R-modules.

Theorem 2.9 *Let $\mathcal{X}$ be a class of right R-modules and M be a right R-module. Assume that every nonzero cyclic singular module has a nonzero submodule in $\mathcal{X}$. If $M \in \mathcal{X}^{\perp}$, then the following cases hold.*
(1) *M is an $\mathcal{I}$-lifting module.*
(2) *$E(M)/M$ is an $\mathcal{X}$-lifting module.*

Proof. (1) By Proposition 2.8 and Example 1.1(v).
(2) By Propositions 2.7 and 2.8. □

When $\mathcal{F}$ is the class of all flat right R-modules, then the modules of the class $\mathcal{F}^{\perp}$ are called *cotorsion modules* ([15]).

Lemma 2.10 *Let R be a ring and $(\mathcal{X}, \mathcal{X}^{\perp})$ a cotorsion theory. Then the following statements are equivalent:*
(1) $\mathcal{X} = \text{Mod-}R$.
(2) *Every nonzero cyclic singular R-module has a nonzero cyclic submodule in $\mathcal{X}$.*
(3) *Every nonzero cyclic singular R-module has a nonzero submodule in $\mathcal{X}$.*
(4) *Every nonzero singular R-module has a nonzero submodule in $\mathcal{X}$.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) Clear.
(4) $\Longrightarrow$ (1) By Proposition 2.8. □

Now we have the following theorem as a result of Lemma 2.10.

Theorem 2.11 *Let R be a ring and $(\mathcal{X}, \mathcal{X}^{\perp})$ be a cotorsion theory. If one of the following conditions satisfies, then any R-module is lifting if and only if any R-module is $\mathcal{X}$-lifting:*
(1) *Every nonzero cyclic singular R-module has a nonzero cyclic submodule in $\mathcal{X}$.*
(2) *Every nonzero cyclic singular R-module has a nonzero submodule in $\mathcal{X}$.*
(3) *Every nonzero singular R-module has a nonzero submodule in $\mathcal{X}$.*

Proof. Clear. □

Lemma 2.12 *Assume that $\mathcal{X}$ is closed under taking homomorphic images and M is an R-module. If $\mathcal{I} \subset \mathcal{X}$, then $M \in \mathcal{X}^{\perp}$ if and only if M is an injective module.*

Proof. :$\Rightarrow$ Let $M \in \mathcal{X}^{\perp}$. By Lemma 2.6, we have $T_{\mathcal{X}}(E(M)/M) = 0$. Since $\mathcal{I} \subset \mathcal{X}$ and $\mathcal{X}$ is closed under homomorphic images, then $T_{\mathcal{X}}(E(M)/M) = E(M)/M$, i.e., $M = E(M)$ is injective.
:$\Leftarrow$: Clear. □

Now we have the following corollary as a result of Theorem 2.9 and Lemma 2.12.

Corollary 2.13 *Let $\mathcal{X}$ be a class of R-modules closed under taking homomorphic images, $\mathcal{I} \subset \mathcal{X}$ and M be an R-module. If $M \in \mathcal{X}^{\perp}$, then the following cases hold.*
(1) *M is an $\mathcal{I}$-lifting module.*
(2) *M is a $T_{\mathcal{I}}(M)$-lifting module.*
(3) *M is a $Tr_M(\mathcal{I})$-lifting module.*
(4) *M is a $Tr_M(Gen(\mathcal{I}))$-lifting module.*
(5) *$E(M)/M$ is an $\mathcal{X}$-lifting module.*
(6) *$E(M)/M$ is an $\mathcal{I}$-lifting module.*

Lemma 2.14 *Let R be a ring.*
(1) *Assume that $\mathcal{X}$ is a class of R-modules which is closed under taking homomorphic images. Then $\mathcal{X}^{\perp} = (\mathrm{Gen}(\mathcal{X}))^{\perp}$.*
(2) *Let $\mathcal{C}$ be the class of all cyclic R-modules. Then $\mathcal{C}^{\perp} = (\mathrm{Gen}(\mathcal{C}))^{\perp} = (\mathrm{Mod}-R)^{\perp}$.*

Proof. (1) Let M be an R-module. By Proposition 2.4 and Lemma 2.6, we can obtain that $T_{\mathcal{X}}(M) = \mathrm{Tr}_M(\mathcal{X}) = \mathrm{Tr}_M(\mathrm{Gen}(\mathcal{X})) = T_{\mathrm{Gen}(\mathcal{X})}(M)$. This implies that $M \in \mathcal{X}^{\perp}$ if and only if $T_{\mathcal{X}}(E(M)/M) = 0$ if and only if $T_{\mathrm{Gen}(\mathcal{X})}(E(M)/M) = 0$ if and only if $M \in (\mathrm{Gen}(\mathcal{X}))^{\perp}$ by Lemma 2.6.
(2) is clear from (1). □

Example 2.15 *Let R be a ring and $\mathcal{I}$ denote the class of all injective R-modules. Then every R-module M*

is $\mathcal{I}$-lifting by Example 1.1(v) . Let $\mathcal{C}$ be the class of all cyclic right R-modules. By Lemma 2.14, we have $\mathcal{C}^{\perp} = (\mathrm{Gen}(\mathcal{C}))^{\perp} = (\mathrm{Mod} - R)^{\perp}$ *, i.e., Baer Criterion. So every R-module M is $\mathcal{C}$-lifting by Lemma 2.12 and Corollary 2.13.*

3. τ-lifting modules

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Then τ is uniquely determined by its associated class $\mathcal{T}$ of τ-torsion modules $\mathcal{T} = \{M \in \mathrm{Mod} - R \mid \tau(M) = M\}$ where for an R-module M, $\tau(M) = \{\sum N \mid N \leq M, N \in \mathcal{T}\}$ and $\mathcal{F}$ is referred to as the τ-torsion free class and $\mathcal{F} = \{M \in \mathrm{Mod} - R \mid \tau(M) = 0\}$. A module in $\mathcal{T}$ (or $\mathcal{F}$) is called a *τ-torsion module* (or *τ-torsionfree*). Every torsion class $\mathcal{T}$ determines in every module M a unique maximal $\mathcal{T}$-submodule $\tau(M)$, the *τ-torsion submodule* of M, and $\tau(M/\tau(M)) = 0$, i.e., $M/\tau(M)$ is $\mathcal{F}$-module and τ-torsionfree.

In what follows τ will represent a hereditary torsion theory, that is, if $\tau = (\mathcal{T}, \mathcal{F})$ then the class $\mathcal{T}$ is closed under taking submodules, direct sums, images and extensions by short exact sequences, equivalently the class $\mathcal{F}$ is closed under taking submodules, direct products, injective hulls and isomorphic copies. Hence, the class $\mathcal{F}$ is not, in general, closed under taking homomorphic images, if this happens to be true for a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, it is called that τ is *cohereditary*.

Recall that module M is called *τ-lifting* if for any τ-torsion free submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$ ([9] and [10]).

Note that

(1) Every lifting module is τ-lifting,

(2) If M is a τ-lifting module such that every proper submodule of M is contained in $\mathcal{F}$, then then M is a lifting module,

(3) If M is τ-torsion, then M is τ-lifting.

(4) Let $\mathbb{Z}$ denote the ring of integers and consider the $\mathbb{Z}$-module $M = N \oplus (U/V)$, where $N = \mathbb{Z}/8\mathbb{Z}$ and the submodules $U = 2\mathbb{Z}/8\mathbb{Z}$ and $V = 4\mathbb{Z}/8\mathbb{Z}$ of N. Let $\bar{\bar{0}}$ and $\bar{\bar{2}}$ denote the element of U/V. Let $\tau := (\mathcal{T}, \mathcal{F})$ denoted the torsion theory on Mod-$\mathbb{Z}$ where $\mathcal{F} = \{K \in Mod - \mathbb{Z} | \forall 0 \neq Y \subseteq K, \exists y \in Y$ such that for all positive integer t we have $3^t y \neq 0\}$. If $N_1 = (\bar{1}, \bar{\bar{2}})\mathbb{Z}, N_2 = (\bar{2}, \bar{\bar{0}})\mathbb{Z}, N_3 = (\bar{2}, \bar{\bar{2}})\mathbb{Z}, N_4 = (\bar{1}, \bar{\bar{0}})\mathbb{Z}, N_5 = (\bar{4}, \bar{\bar{0}})\mathbb{Z}, N_6 = (\bar{4}, \bar{\bar{2}})\mathbb{Z}$. Then N_1, N_2, N_3 and N_4 are τ-torsion free submodules of M, where N_1, N_4 are direct summands of M, $N_2 \ll M, M = N_1 + N_3, N_5 = N_1 \cap N_3, N_5 \ll M$ and $M = N_1 \oplus N_6$. It is easily checked that N_3 is neither small in M nor has any nonzero submodule which is direct summand of M. Hence M is not τ-lifting.

Let $(\mathcal{L}, \leq, 0, 1)$ be a modular lattice, τ be a hereditary torsion theory and M an R-module. We write

$$Sat_{\tau}(M) = \{N \leq M : M/N \in \mathcal{F}\}$$

by [14]. If $a \in \mathcal{L}$, then $b \in \mathcal{L}$ is said to be a *complement* of a (in $\mathcal{L}$), if $a \vee b = 1$ and $a \wedge b = 0$. If for each $a \in \mathcal{L}$, there exists $b \in \mathcal{L}$ such that $b \leq a$, $b \vee b' = 1$ and $b \wedge b' = 0$ and $a \wedge b$ is small in M holds then $\mathcal{L}$ is said to be *lifting- lattice*. If $Sat_{\tau}(M)$ is lifting-lattice, we say M is a *τ-lifting module.*

Proposition 3.1 *$Sat_{\tau}(M)$ is a complete upper-continuous modular lattice and if N is a τ-dense submodule of M, then there is a canonical bijection between $Sat_{\tau}(M)$ and $Sat_{\tau}(N)$ given by $A \longrightarrow A \cap N$ where $A \in Sat_{\tau}(M)$*

and this bijection is a lattice isomorphism.

Proof. A submodule N of M is τ-dense in M if and only if M/N is τ-torsion. $(Sat_\tau(M), \leq, 0, 1)$ is endowed the operations:

$\leq$: the inclusion operation of submodules of M,

$A \wedge B = A \cap B$, where $A, B \in Sat_\tau(M)$,

$A \vee B = \widetilde{A+B}$, where $A, B \in Sat_\tau(M)$ and $\widetilde{A+B}$ denotes the largest submodule of M satisfying $\widetilde{A+B}/(A+B) \in \mathcal{T}$, equivalently $\widetilde{A+B}/(A+B) = \tau(M/(A+B))$.

$1 = M$ and $0 = \tau(M)$.

Hence the proof is clear from [14]. □

Proposition 3.2 *Let M be an R-module. If $\tau(M) = 0$ and $\tau(M/N) = M/N$ for every proper submodule N of M, then M is indecomposable.*

Proof. Clear. □

Corollary 3.3 *Let M be a non indecomposable R-module. Then $Sat_\tau(M)$ contains elements other than* 0 *and* 1*.*

Proof. Clear from Example 1.1. □

Lemma 3.4 *Let M be an R-module.*

(1) *M is τ-lifting if and only if every submodule M' of M can be written as $M' = X \oplus Y$ with X is a summand of M and $\tau(Y) = 0$.*

(2) *Every submodule of a τ-lifting module is τ-lifting.*

Proof. Trivial. □

Recall that M is called *τ-cotorsionfree* if every proper submodule of M contains no τ-dense submodule.

Theorem 3.5 *Let M be a τ-cotorsionfree R-module.*

(1) *Any τ-torsion submodule of M is small in M.*

(2) *If M is τ-lifting, then M is indecomposable if and only if M is holllow.*

(3) *If every proper submodule of M is τ-torsion, then M is indecomposable.*

Proof. (1) Let N be a submodule of M with $\tau(N) = N$. Let $M = N + K$ for some submodule $K \leq M$. Then $M/K \cong N/(N \cap K)$. Since N is a τ-torsion submodule of M, $N/N \cap K$ and so M/K is τ-torsion. But M is τ-cotorsionfree, therefore $M = K$. Hence N is small in M.

(2) Assume that M is a τ-lifting module. Suppose that M is indecomposable. For $N \leq M$, we have two cases:

Case (i) If $\tau(M/N) = 0$, then $M/N \in \mathcal{F}$. Then M has a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B \ll B$. Since M is indecomposable, we have $M = A$ or $M = B$. If $M = A$ then $M = N$; otherwise $M = B$ then $N \ll M$. Therefore M is hollow.

Case (ii) Let $\tau(M/N) = M_1/N \neq 0$. Then $\tau(M/M_1) = 0$ and $M/M_1 \in \mathcal{F}$. Since M is a τ-lifting module, M

has a decomposition $M = A \oplus B$ such that $A \leq M_1$ and $M_1 \cap B \ll B$. By assumption, $M = A$ or $M = B$. If $M = A$ then $M = M_1$ and $\tau(M/N) = M/N$. By [8, Proposition 7.6], we have $M = N$. If $M = B$ then $N \ll M$. That is M is hollow. The converse is clear.
(3) Clear. □

Recall that M is called *τ-semisimple* if $N \in Sat_\tau(M)$ is a direct summand of M [14]. Clearly, if M is τ-semisimple, then M is τ-lifting.

In [13] (or [8]) , M is called *τ-complemented* (or τ-direct) if for every submodule N of M there exists a direct summand K of M such that K/N is τ-torsion.

Theorem 3.6 is clear from [13] and definitions.

Theorem 3.6 *Let M be an R-module. Then the following are equivalent:*
1. M is τ-semisimple.
2. $M = \tau(M) \oplus P$ for some τ-torsion free submodule P.
3. M is τ-complemented.

Proposition 3.7 *Let M be a τ-semisimple R-module. Then*
(1) *$M = \tau(M) \oplus K$ for some submodule K of M.*
(2) *If τ is a cohereditary torsion theory, then $Rad(M) \leq \tau(M)$.*
(3) *For every τ-dense submodule N of M, i.e $M/N \in \mathcal{T}$, $M = \tau(M) + N$.*
(4) *If M is τ-cotorsion free, then $Rad(M) \leq \tau(M)$.*

Proof. (1) Clear.
(2) Let L be a small submodule of M. By assumption, $M/(L + \tau(M)) = 0$. By hypothesis, let $M = (L + \tau(M)) \oplus X$ for some submodule X of M. Thus $L \leq \tau(M)$.
(3) Let N be a τ-dense submodule of M. As in the proof of (2), we can find a decomposition $M = (N + \tau(M)) \oplus Y$ for some submodule Y of M. It is easy to see that that Y is isomorphic to a submodule of M/N. Since M/N is τ-torsion and Y is τ-torsion free, we have $Y = 0$.
(4) This is Theorem 3.5 (1). □

Let $\mathcal{X}$ be any class of modules. The class $d\mathcal{X}$ consists of all modules M such that, for every submodule N of M, there exists a direct summand K of M such that $N \leq K$ and K/N is an $\mathcal{X}$-module. Dually, $d^*\mathcal{X}$ is defined to be the class of R- modules M such that each submodule N of M contains a direct summand K of M such that N/K is an $\mathcal{X}$-module. Properties of these classes are given in [2].

Definition 3.8 Let $\tau = (\mathcal{T}, \mathcal{F})$ a torsion theory and M be an R-module. We call M a *d^*F-lifting module*, if every submodule A of M has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \in \mathcal{F}$ (see [4] for more general cases).

Examples 3.9 (i) *Every simple module with respect to every $\tau = (\mathcal{T}, \mathcal{F})$ torsion theory is a $d^*\mathcal{F}$-lifting module.*
(ii) *Let $\tau = (\mathcal{T}_\mathbb{Z}, \mathcal{F}_\mathbb{Z})$ be a torsion theory on* Mod-$\mathbb{Z}$ *and $M_\mathbb{Z} = \mathbb{Z}_\mathbb{Z}$. Let $N = 2\mathbb{Z} \leq M$. M has only two direct summands which are* (0) *and M. Also every nonzero submodule of M is τ-torsion but, for any $0 \neq N$, M/N is τ-torsionfree. If N has a decomposition $N = A \oplus B$, we have $N = A$ or $N = B$. It is a contradiction. Hence $M_\mathbb{Z}$ is not a $d^*\mathcal{F}$-lifting module.*

Let R be a ring. Let $\mathcal{S}$ denote the class of simple R-modules. Then $T_{\mathcal{S}}(M)$ is the usual socle of M and is denoted simply by $Soc(M)$.

Proposition 3.10 *If M is a $d^*\mathcal{F}$-lifting R-module, then $M/T_{\mathcal{F}}(M)$ is semisimple.*

Proof. Any submodule of $M/T_{\mathcal{F}}(M)$ has the form $N/T_{\mathcal{F}}(M)$ for some submodule N of M which contains $T_{\mathcal{F}}(M)$. Since M is a $d^*\mathcal{F}$ lifting module, the module N has a decomposition $N = A \oplus B$ such that $A \leq_d M$ and $B \in \mathcal{F}$. Let $M = A \oplus C$ for some submodule C of M. Then, $M/T_{\mathcal{F}}(M) = N/T_{\mathcal{F}}(M) \oplus (C + T_{\mathcal{F}}(M))/T_{\mathcal{F}}(M)$. By [3, Theorem 9.6], M is a semisimple module. □

Proposition 3.11 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory such that $\mathcal{S} \subseteq \mathcal{F}$. Let M be a $d^*\mathcal{F}$-lifting R-module. Then $T_{\mathcal{F}}(M)$ is an essential submodule of M.*

Proof. Let N be any submodule of M with $N \cap T_{\mathcal{F}}(M) = 0$. Then N embeds in $M/T_{\mathcal{F}}(M)$. By Proposition 3.7, we have $N \in \mathcal{S}$. By hypothesis, $N \leq T_{\mathcal{F}}(M)$. Hence $N = 0$. This is a contradiction. Thus $T_{\mathcal{F}}(M)$ is an essential submodule of M. □

Theorem 3.12 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Let M be a $d^*\mathcal{F}$-lifting R-module. Then $\tau(M)$ is a direct summand of M. In general, every τ-torsion submodule of M is a direct summand.*

Proof. Let N be any submodule of M with $\tau(N) = N$. Then N has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \in \mathcal{F}$. Since $\tau(N) = N$ and $B \in \mathcal{F}$, we have $B = 0$. Therefore $N = A$ is a direct summand of M. □

Corollary 3.13 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory such that $\mathcal{S} \subseteq \mathcal{F}$. Let M be a $d^*\mathcal{F}$-lifting R-module. Then $\tau(M)$ is a semisimple direct summand of M. In particular, $\tau(M) \leq Soc(M)$.*

Theorem 3.14 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory and M be an R-module such that $\tau(M) = 0$. If M is a τ-lifting module then M is a $d^*\mathcal{F}$-lifting module.*

Proof. Let $N \leq M$. Since M is a τ-lifting module, by Lemma 3.4, N has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $\tau(B) = 0$. Since $\mathcal{F}$ is closed under submodules, then $B \in \mathcal{F}$. Hence M is a $d^*\mathcal{F}$ lifting module. □

Theorem 3.15 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory and M be an R-module such that $\tau(M) = M$. Then M is a $d^*\mathcal{F}$ lifting module if and only if M is semisimple.*

Proof. Let M be a module with $\tau(M) = M$ and M be a $d^*\mathcal{F}$-lifting module. Let $N \leq M$.Then N has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \in \mathcal{F}$. Since $\tau(M) = M$ and $B \in \mathcal{F}$, we have $B = 0$. Hence $N = A$ is a direct summand of M. By [3, Theorem 9.6], M is semisimple. Converse is clear. □

Example 3.16 *Let F be a field and R be the subring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ of all 3 by 3 matrices over F. Let M denote right R-module R. Clearly, every module over R is lifting. With respect to the idempotent ideals:*

$$X = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$$

1. Let $\mathcal{T}_X = \{M \in Mod - R : MX = 0\}$. Then $\mathcal{T}_X(M) = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$. If M is a $d^*\mathcal{F}$-lifting module, by Corollary 3.13, then $\mathcal{T}_X(M)$ is a direct summand of M. But $\mathcal{T}_X(R_R)$ is not a direct summand of M, so M is not a $d^*\mathcal{F}$-lifting module.
2. Let $\mathcal{T}_Y = \{M \in Mod - R : MY = 0\}$. Then $\mathcal{T}_Y(M) = 0$. Since M is a lifting module, then M is a $d^*\mathcal{F}$-lifting module by Theorem 3.15.

Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. In definition 3.8, we defined $d^*\mathcal{F}$-lifting module with respect to the $d^*\mathcal{F}$ class. Similarly, we can define $d^*\mathcal{T}$-lifting module with respect to the $d^*\mathcal{T}$ class (see [4] for more generally cases).

Definition 3.17 Let $\tau = (\mathcal{T}, \mathcal{F})$ a torsion theory and M be an R-module. We call M a *d^*T-lifting module*, if every submodule A of M has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \in \mathcal{T}$.

Examples 3.18 **(i)** *Every semisimple module with respect to a $\tau = (\mathcal{T}, \mathcal{F})$ torsion theory is a $d^*\mathcal{T}$-lifting module.*
(ii) *Let $\tau = (\mathcal{T}_{\mathbb{Z}}, \mathcal{F}_{\mathbb{Z}})$ be a torsion theory on* Mod-$\mathbb{Z}$ *and $M_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}}$. Clearly, $N \in \mathcal{T}_{\mathbb{Z}}$ if and only if for all $0 \neq n \in N$ there exists a $0 \neq t \in \mathbb{Z}$ such that $nt = 0$. Hence, for any submodule A of M, M is a $d^*\mathcal{T}$ lifting module since $A = A \oplus (0)$.*

Theorem 3.19 *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory and M be an R-module such that $\tau(M) = 0$. Then M is a $d^*\mathcal{T}$ lifting module if and only if M is semisimple.*

Proof. Let M be a $d^*\mathcal{T}$lifting module and $\tau(M) = 0$. Let $N \leq M$. Then N has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \in \mathcal{T}$. Since $B = \tau(B) \leq \tau(M) = 0$, we have $N = A$ is a direct summand of M. The converse is clear. □

Theorem 3.20 *If M is a $d^*\mathcal{T}$ lifting R-module, then $M/\tau(M)$ is semisimple.*

Proof. Let $\tau(M) \leq N \leq M$. Since M is a $d^*\mathcal{T}$-lifting module, N has a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \in \mathcal{T}$. Let $M = A \oplus C$ for some submodule C of M. Then $M/\tau(M) = (A + \tau(M))/\tau(M) \oplus (C + \tau(M))/\tau(M)$ by [3, Theorem 9.6]. □

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