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On generalized (α, β)-derivations of semiprime rings

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Abstract

We investigate some properties of generalized (α, β)-derivations on semiprime rings. Among some other results, we show that if g is a generalized (α, β)-derivation, with associated (α, β)-derivation δ, on a semiprime ring R such that $[g(x), \alpha(x)] = 0$ for all $x \in R$, then $\delta(x)[y, z] = 0$ for all $x, y, z \in R$ and δ is central. We also show that if α, ν, τ are endomorphisms and β, μ are automorphisms of a semiprime ring R and if R has a generalized (α, β)-derivation g, with associated (α, β)-derivation δ, such that $g([\mu(x), w(y)]) = [\nu(x), w(y)]_{\alpha,\tau}$, where $w : R \to R$ is commutativity preserving, then $[y, z]\delta(w(p)) = 0$ for all $y, z, p \in R$.

Key Words: Semiprime ring, derivation, generalized derivation, generalized (α, β)-derivation.

1. Introduction

Throughout, R denotes a ring with centre $Z(R)$. We denote $[x, y]$ for $xy - yx$, $x, y \in R$. Let σ, τ be endomorphisms of R, then for $x, y \in R$ we write $[x, y]_{\sigma,\tau}$ for $x\sigma(y) - \tau(y)x$. Obviously $[xy, z] = x[y, z] + [x, z]y, [x, yz] = y[x, z] + [x, y]z$, $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}\ y$ and $[x, yz]_{\sigma,\tau} = \tau(y)[x, z]_{\sigma,\tau} + [x, y]_{\sigma,\tau}\sigma(z)$. We shall use these identities without further mention.

The ring R is prime if $aRb = \{0\}$ implies either $a = 0$ or $b = 0$; it is semiprime if $aRa = \{0\}$ implies $a = 0$. A prime ring is obviously semiprime. An additive mapping δ from R into itself is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in R$. We call a mapping $f : R \to R$ central if $f(x) \in Z(R)$ for all $x \in R$. A mapping $f : R \to R$ is called strong commutativity preserving (SCP) on a set $S \subseteq R$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. For more information on SCP, we refer to [5,14] and references therein. We shall denote identity mapping of R by 1.

A more general concept of (α, β)-derivations have been extensively studied in prime and semiprime rings. They have played an important role in the solution of functional equations (see [4] and references therein). Let α, β be mappings from R into itself. An additive mapping δ of R into itself is called an (α, β)-derivation if $\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$ for all $x, y \in R$. Of course, a $(1, 1)$-derivation is a derivation.

Zalar [16] introduced the concept of a centralizer in a ring. An additive mapping f from R into itself is called a left (right) centralizer if $f(xy) = f(x)y(f(xy) = xf(y))$ for all $x, y \in R$. f is called a centralizer if

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it is a left as well as a right centralizer. Recently, Daif. et al. [7] have given the notion of a left θ-centralizer. An additive mapping f from R into itself is called a left θ-centralizer if $f(xy) = f(x)\theta(y)$ for all $x, y \in R$, where θ is a mapping from R into itself. For more information on centralizers we refer to [1, 15] and references therein.

The notion of a generalized derivation of a ring was introduced by Brešar [3] and Hvala [12]. They have studied some properties of such derivations. An additive mapping g of R into itself is called a generalized derivation of R, with associated derivation δ, if there is a derivation δ of R such that $g(xy) = g(x)y + x\delta(y)$ for all $x, y \in R$. For more information on generalized derivations we refer to [8, 14] and references therein.

Chang [6] introduced the notion of a generalized (α,β)-derivation of a ring R and investigated some properties of such derivations. Let α, β be mappings of R into itself. An additive mapping g of R into itself is called a generalized (α,β)-derivation of R, with associated (α,β)-derivation δ, if there exists an (α,β)-derivation δ of R such that $g(xy) = g(x)\alpha(y) + \beta(x)\delta(y)$ for all $x, y \in R$. Obviously this notion covers the notion of a generalized derivation (in case $\alpha = \beta = 1$), notion of a derivation (in case $g = \delta, \alpha = \beta = 1$), notion of a left centralizer (in case $\delta = 0, \alpha = 1$), notion of (α,β)-derivation (in case $g = \delta$) and the notion of left α-centralizer (in case $\delta = 0$). Thus it is interesting to investigate properties of this general notion. For more properties of generalized (α,β)-derivations we refer to [2, 9, 10, 13] and references therein.

The purpose of this paper is to investigate some more properties of generalized (α,β)-derivations and to prove a generalization, in the setting of a semiprime ring, of the following result (Theorem A) of Jung and Park [13, Theorem 2.2 (page 103)].

Theorem A. *Let R be a prime ring and I a nonzero ideal of R. Let α, ν, and τ be endomorphisms of R and β, μ automorphisms of R. If R admits a generalized (α,β)-derivation g with associated nonzero (α,β)-derivation δ such that $g([\mu(x), y]) = [\nu(x), y]_{\alpha,\tau}$ for all $x, y \in I$, then R is commutative.*

Among some other results, we prove the following:

(i) Let R be a semiprime ring and α, β automorphisms of R. Let g be a generalized (α,β)-derivation, with associated (α,β)-derivation δ, of R such that $[g(x), \alpha(x)] = 0$ for all $x \in R$, then $\delta(x)[y, z] = 0$ for all $x, y, z \in R$ and δ is central.

(ii) Let R be a semiprime ring. Let α, ν, τ be endomorphisms and β, μ automorphisms of R. If R has a generalized (α,β)-derivation g, with associated derivation δ, such that $g([\mu(x), w(y)]) = [\nu(x), w(y)]_{\alpha,\tau}$, where $w : R \to R$ is commutativity preserving, then $\delta(w(p))[y, z] = 0$ for all $y, z, p \in R$ and $\delta(w(p)) \in Z(R)$ for all $p \in R$.

We also deduce Theorem A, when the ideal I is replaced by R, as a corollary of the result (ii).

2. Results

We now prove our results. First we state the following lemma which will be used in the sequel.

Lemma 2.1 *[11, Lemma 1.1.4 (page 6)]. Suppose R is a semiprime ring and that $a \in R$ is such that $a[a, x] = 0$ for all $x \in R$. Then $a \in Z(R)$.*

Theorem 2.2 *Let R be a semiprime ring and g a generalized (α,β)-derivation of R with associated (α,β)-*

derivation δ, where α and β are automorphisms of R. If $[g(x),\alpha(x)]=0$ for all $x\in R$, then $\delta(x)[y,z]=0$ for all $x,y,z\in R$ and $\delta(x)\in Z(R)$ for all $x\in R$.

Proof. By hypothesis

$$[g(x),\alpha(x)]=0 \quad \text{for all} \quad x\in R. \tag{1}$$

Linearizing (1), we get

$$[g(x),\alpha(y)]+[g(y),\alpha(x)]=0 \quad \text{for all} \quad x,y\in R. \tag{2}$$

Replacing y by yx in (2), we get $[g(x),\alpha(yx)]+[g(yx),\alpha(x)]=0$. That is, $[g(x),\alpha(y)\alpha(x)]+[g(y)\alpha(x)+\beta(y)\delta(x),\alpha(x)]=0$. The last equation together with (1) implies $[g(x),\alpha(y)]\alpha(x)+[g(y),\alpha(x)]\alpha(x)+\beta(y)[\delta(x),\alpha(x)]+[\beta(y),\alpha(x)]\delta(x)=0$, which along with (2) gives

$$\beta(y)[\delta(x),\alpha(x)]+[\beta(y),\alpha(x)]\delta(x)=0 \quad \text{for all} \quad x,y\in R. \tag{3}$$

Replacing y by zy in (3), we get $\beta(z)\beta(y)[\delta(x),\alpha(x)]+[\beta(z)\beta(y),\alpha(x)]\delta(x)=0$. That is, $\beta(z)\beta(y)[\delta(x),\alpha(x)]+\beta(z)[\beta(y),\alpha(x)]\delta(x)+[\beta(z),\alpha(x)]\beta(y)\delta(x)=0$, which along with (3) implies

$$[\beta(z),\alpha(x)]\beta(y)\delta(x)=0 \quad \text{for all} \quad x,y,z\in R. \tag{4}$$

Replacing z by $\beta^{-1}(z)$ and y by $\beta^{-1}(y)$ in (4), we get

$$[z,\alpha(x)]y\delta(x)=0 \quad \text{for all} \quad x,y,z\in R. \tag{5}$$

Since R is semiprime, equality (5) implies

$$\delta(x)[z,\alpha(x)]=0 \quad \text{for all} \quad x,z\in R. \tag{6}$$

Linearizing (6) in x and then using (6), we get $\delta(y)[z,\alpha(x)]+\delta(x)[z,\alpha(y)]=0$, which implies

$$\delta(y)[z,\alpha(x)]=-\delta(x)[z,\alpha(y)] \quad \text{for all} \quad x,y,z\in R. \tag{7}$$

Replacing z by uz in (6) and then using (6), we get

$$\delta(x)u[z,\alpha(x)]=0 \quad \text{for all} \quad x,u,z\in R. \tag{8}$$

Replacing u by $[z,\alpha(y)]u\delta(y)$ in (8), we get $\delta(x)[z,\alpha(y)]u\delta(y)[z,\alpha(x)]=0$, which along with (7) and semiprimeness of R implies that

$$\delta(x)[z,\alpha(y)]=0 \quad \text{for all} \quad x,y,z\in R. \tag{9}$$

Replacing y by $\alpha^{-1}(y)$ in (9), we get

$$\delta(x)[z,y]=0 \quad \text{for all} \quad x,y,z\in R. \tag{10}$$

From (10) and Lemma 2.1, we get $\delta(x)\in Z(R)$ for all $x\in R$. □

Corollary 2.3 *Let R be a semiprime ring and $g:R\to R$ a generalized (α,β)-derivation such that $[g(x),\alpha(x)]=0$ for all $x\in R$, where α and β are automorphisms of R, then $(g(xu)-g(x)\alpha(u))\in Z(R)$ for all $x,u\in R$. If $Z(R)=\{0\}$, then g is a left α-centralizer.*

Proof. From (10) we have $\beta(x)\delta(u)[z,y]=0$ for all $x,u,y,z\in R$. Since $g(xu)-g(x)\alpha(u)=g(x)\alpha(u)+\beta(x)\delta(u)-g(x)\alpha(u)=\beta(x)\delta(u)$, therefore, $(g(xu)-g(x)\alpha(u))[z,y]=0$ for all $x,u,y,z\in R$. By Lemma 2.1, $(g(xu)-g(x)\alpha(u))\in Z(R)$. If $Z(R)=\{0\}$, then $g(xu)-g(x)\alpha(u)=0$. That is, $g(xu)=g(x)\alpha(u)$ for all $x,u\in R$. Thus g is a left α-centralizer. □

Corollary 2.4 *Let R be a semiprime ring. If R has a generalized (α,β)-derivation g with associated (α,β)-derivation δ, where α and β are automorphisms of R, such that $[g(x),\alpha(x)]=0$ for all $x,y\in R$ and δ is strong commutativity preserving, then R is commutative.*

Proof. Replacing z by uz in (10) and then using(10), we get

$$\delta(x)u[z,y]=0 \quad \text{for all} \quad x,u,y,z\in R. \tag{11}$$

Replacing u by $\delta(y)u$ and z by x in (11), we get

$$\delta(x)\delta(y)u[x,y]=0 \quad \text{for all} \quad x,y\in R. \tag{12}$$

Multiplying (11) on the left by $\delta(y)$ after replacing z by x, we get

$$\delta(y)\delta(x)u[x,y]=0 \quad \text{for all} \quad x,u,y\in R. \tag{13}$$

Subtracting (13) from (12), we get $[\delta(x),\delta(y)]u[x,y]=0$, which along with strong commutativity preserving property of δ and semiprimeness of R implies $[x,y]=0$ for all $x,y\in R$. Thus R is commutative. □

Corollary 2.5 *Let R be a prime ring with generalized (α,β)-derivation g having associated (α,β)-derivation δ, where α and β are automorphisms of R. If $[g(x),\alpha(x)]=0$ and $\delta\neq 0$, then R is commutative.*

Proof. Proof follows from (11) and primeness of R. □

Remark 2.6 Taking $\alpha=\beta=1$ in above theorem and corollaries, we get the corresponding results for generalized derivations.

Theorem 2.7 *Let R be a semiprime ring. Let α,ν,τ be endomorphisms and β,μ automorphisms of R. If R has a generalized (α,β)-derivation g, with associated derivation δ, such that $g([\mu(x),w(y)])=[\nu(x),w(y)]_{\alpha,\tau}$, where w is a strong commutativity preserving endomorphism of R, then $\delta(w(p))[y,z]=0$ for all $y,z,p\in R$ and $\delta(w(p))\in Z(R)$ for all $p\in R$.*

Proof. By hypothesis

$$g([\mu(x),w(y)])=[\nu(x),w(y)]_{\alpha,\tau}. \tag{14}$$

Replacing y by zy, we get $g([\mu(x),w(zy)])=[\nu(x),w(zy)]_{\alpha,\tau}$, which implies $g([\mu(x),w(z)w(y)])=[\nu(x),w(z)w(y)]_{\alpha,\tau}$. That is,

$g(w(z)[\mu(x), w(y)]+[\mu(x), w(z)]w(y)) = [\nu(x), w(z)w(y)]_{\alpha,\tau}$. From the last relation we have $g(w(z))\alpha[\mu(x), w(y)]+\beta(w(z))\delta[\mu(x), w(y)]+g([\mu(x), w(z)])\alpha(w(y))+\beta[\mu(x), w(z)]\delta(w(y)) = \tau(w(z))[\nu(x), w(y)]_{\alpha,\tau}+[\nu(x), w(z)]_{\alpha,\tau}\alpha(w(y)$ which along with (14) implies
$g(w(z))\alpha[\mu(x), w(y)] + \beta(w(z))\delta[\mu(x), w(y)] + \beta[\mu(x), w(z)]\delta(w(y)) = \tau(w(z))g([\mu(x), w(y)])$. Replacing x by $\mu^{-1}(w(y))$ in the last equation, we get $\beta[w(y), w(z)]\delta(w(y)) = 0$, which implies

$$[w(y), w(z)]\beta^{-1}\delta(w(y)) = 0. \qquad (15)$$

Since w is a strong commutativity preserving endomorphism, so the last equation gives

$$[y, z]\beta^{-1}\delta(w(y)) = 0. \qquad (16)$$

Linearizing (16), we have $[y + p, z]\beta^{-1}\delta(w(y + p)) = 0$. That is,
$([y, z] + [p, z])(\beta^{-1}(\delta(w(y))) + \beta^{-1}(\delta(w(p)))) = 0$, which along with (16) implies

$$[y, z]\beta^{-1}(\delta(w(p))) + [p, z]\beta^{-1}(\delta(w(y))) = 0. \qquad (17)$$

Now replacing z by zr in (16), we get $[y, zr]\beta^{-1}(\delta(w(y))) = 0$. That is, $(z[y, r] + [y, z]r)\beta^{-1}(\delta(w(y))) = 0$, which along with (16) gives $[y, z]r\beta^{-1}(\delta(w(y))) = 0$. Replacing r by $\beta^{-1}(\delta(w(p)))r(-[p, z])$ in the last equation, we get
$[y, z]\beta^{-1}(\delta(w(p)))r(-[p, z])\beta^{-1}(\delta(w(y))) = 0$, which along with (17) gives
$[y, z]\beta^{-1}(\delta(w(p)))r[y, z]\beta^{-1}(\delta(w(p))) = 0$. Since R is semiprime, the last equation implies $[y, z]\beta^{-1}(\delta(w(p))) = 0$, which gives $[\beta(y), \beta(z)](\delta(w(p))) = 0$. Replacing y by $\beta^{-1}(y)$ and z by $\beta^{-1}(z)$ in the last equation, we get

$$[y, z](\delta(w(p))) = 0. \qquad (18)$$

Further, Lemma 2.1 implies $\delta(w(p)) \in Z(R)$. □

Now we deduce Theorem A of Jung and Park [13], when ideal I is replaced by R, as a corollary of our Theorem 2.7.

Corollary 2.8 *Let R be a prime ring. Let α, ν, τ be endomorphisms and β, μ automorphisms of R. If R admits a generalized (α, β)-derivation g with associated nonzero derivation δ such that $g([\mu(x), y]) = [\nu(x), y]_{\alpha,\tau}$, then R is commutative.*

Proof. Taking $w = 1$, all conditions of Theorem 2.7 are satisfied. Therefore from (18), we get

$$[y, z]\delta(p) = 0 \quad \text{for all} \quad x, u, y \in R. \qquad (19)$$

Replacing z by zr, $r \in R$, in (19) and using it, we get $[y, z]r\delta(p) = 0$ for all $y, z, r, p \in R$. Since R is prime and $\delta \neq 0$, from the last equation, we get $[y, z] = 0$ for all $y, z \in R$. Thus R is commutative. □

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