Turk J Math
35 (2011) , 405 – 436.

doi:10.3906/mat-1003-201

On generalized Witt algebras in one variable

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Abstract

We study a class of infinite dimensional Lie algebras called generalized Witt algebras (in one variable). These include the classical Witt algebra and the centerless Virasoro algebra as important examples.

We show that any such generalized Witt algebra is a semisimple, indecomposable Lie algebra which does not contain any abelian Lie subalgebras of dimension greater than one.

We develop an invariant of these generalized Witt algebras called the spectrum, and use it to show that there exist infinite families of nonisomorphic, simple, generalized Witt algebras and infinite families of nonisomorphic, nonsimple, generalized Witt algebras.

We develop a machinery that can be used to study the endomorphisms of a generalized Witt algebra in the case that the spectrum is "discrete." We use this to show that, among other things, every nonzero Lie algebra endomorphism of the classical Witt algebra is an automorphism and every endomorphism of the centerless Virasoro algebra fixes a canonical element up to scalar multiplication.

However, not every injective Lie algebra endomorphism of the centerless Virasoro algebra is an automorphism.

Key Words: Infinite dimensional Lie algebra, Virasoro algebra

1. Introduction

Throughout this paper, we will work over a field $\mathbf{k}$ of characteristic zero. Also note that there will be no finiteness constraints on the dimension of the Lie algebras in this paper — in fact, most of the Lie algebras that we will consider will be infinite dimensional.

We now sketch the basic results and ideas of this paper in this introductory section. Precise definitions of the concepts can be found within the paper.

Let $\mathbf{R}$ be the field of fractions of the power series algebra $\mathbf{k}[[x]]$.

Following [6], we define a stable algebra to be a subalgebra of $\mathbf{R}$ which is closed under formal differentiation ∂. Notice that we confine ourselves to the one variable case throughout this paper.

Important examples of stable algebras are the polynomial algebra $\mathbf{k}[x]$, the power series algebra $\mathbf{k}[[x]]$ and the Laurent polynomial algebra $\mathbf{k}[x, x^{-1}]$.

1991 *AMS Mathematics Subject Classification:* Primary: 17B65, 17C20; Secondary: 17B40.

Following [8] and [10], to every stable algebra A, we associate a Lie algebra $Witt(A)$. We refer to $Witt(A)$ as a generalized Witt algebra. (The reader is warned that there are different definitions of a generalized Witt algebra in the literature. Please look at Definition 3.1 for ours.)

$Witt(\mathbf{k}[x])$ is the classical Witt algebra, (See [2]) and $Witt(\mathbf{k}[x, x^{-1}])$ is called the centerless Virasoro algebra in the literature. (See [7].)

A Lie algebra is called self-centralizing if it contains no abelian Lie subalgebras of dimension greater than one. We prove:

Theorem 1.1 (Theorem 3.8 and Proposition 3.11) *Every generalized Witt algebra is self-centralizing.*

Furthermore, if it is infinite dimensional (which is the case for all but one trivial example where $A = \mathbf{k}$), then a generalized Witt algebra must be semisimple and indecomposable.

To contrast, over an algebraically closed field, it is shown that the only finite dimensional Lie algebra which is self-centralizing, semisimple and indecomposable is $\mathfrak{sl}_2$, the Lie algebra of 2×2 matrices of trace zero.

However a generalized Witt algebra need not be simple; some are and some are not.

If a generalized Witt algebra has a nonzero ad-diagonal element, i.e., nonzero α such that $ad(\alpha)$ is diagonal in some basis, we show that the set of eigenvalues of $ad(\alpha)$ possesses the algebraic structure of a pseudomonoid.

We call this pseudomonoid the spectrum of α. We then show in Proposition 7.10 that any other nonzero ad-diagonal element of this Lie algebra has to have an equivalent spectrum. This allows us to define the spectrum of $\mathfrak{L}$ to be the spectrum of any nonzero ad-diagonal element. It is then shown that this is indeed an invariant for these kinds of Lie algebras, i.e., isomorphic Lie algebras have equivalent spectra.

The constraint that the Lie algebra possesses a nonzero ad-diagonal element, is not so bad as all the classical examples possess this property.

In these pseudomonoids, one can define the notion of an ideal subset. We show the following proposition.

Proposition 1.2 (Proposition 6.5) *Let $\mathfrak{L}$ be a generalized Witt algebra with nonzero ad-diagonal element and let G be its spectrum. Then there is a one-to-one correspondence between the ideal subsets of G and the ideals of $\mathfrak{L}$.*

If G is actually an abelian group then it is simple as a psuedomonoid and hence $\mathfrak{L}$ is simple.

Since the classical Witt algebra and centerless Virasoro algebra have nonzero ad-diagonal elements, and their spectra are simple pseudomonoids, we recover the well-known fact that they are simple, as a corollary.

Using this spectrum invariant, we can distinguish between nonisomorphic generalized Witt algebras and show that there is a rich variety of such algebras (with nonzero ad-diagonal element) by way of the following proposition.

Proposition 1.3 (Examples 5.9, 7.15 and 7.16) *There exist infinite families of nonisomorphic, simple, generalized Witt algebras and there exist infinite familes of nonisomorphic, nonsimple, generalized Witt algebras.*

In fact for every submonoid of $(\mathbf{k}, +)$, there is a generalized Witt algebra with that monoid as its spectrum.

Thus, in particular since every torsion-free abelian group embeds into the additive group of some rational vector space, we may get any torsion-free abelian group as the spectrum of a generalized Witt algebra in one variable by suitable choice of the base field $\mathbf{k}$*.*

A machinery is obtained to find the set of eigenvalues of any element in a generalized Witt algebra. It uses formal calculus and in particular, the logarithmic derivative. It is stated in Theorem 5.11.

Finally, motivated by [12], we discuss injective Lie algebra endomorphisms of generalized Witt algebras.

In the case where the generalized Witt algebra possesses a "discrete" spectrum, one can show that such an endomorphism must essentially fix a nonzero ad-diagonal element. (See Theorem 8.7.)

As corollaries of this fact, we can easily obtain information about endomorphisms of these Lie algebras and prove things such as this theorem:

Theorem 1.4 (Corollaries 8.8 and 8.9) *Any nonzero Lie algebra endomorphism* f *of the classical Witt algebra is actually an automorphism and furthermore,*

$$f(x\partial) = (x+b)\partial$$

for some $b \in \mathbf{k}$*.*

If f *is a nonzero Lie algebra endomorphism of the centerless Virasoro algebra, then* f *is injective and*

$$f(x\partial) = \frac{1}{a}x\partial$$

for some nonzero integer a*. However* f *need not be onto.*

More precisely, the centerless Virasoro algebra possesses injective Lie algebra endomorphisms which are not automorphisms.

One should compare this to the Jacobian conjecture for the classical Weyl algebra which states that any nonzero algebra endomorphism is an automorphism. This conjecture is still open. The classical Witt algebra is the Lie algebra of derivations of the classical Weyl algebra. (See [2].)

We remark that the automorphisms of the centerless Virasoro algebra were known and studied for example in [3].

This completes the introductory overview.

2. Generalized Weyl algebras

Let $\mathbf{k}[[x]]$ be the power series algebra over $\mathbf{k}$, and let $\mathbf{R}$ be its field of fractions. Note, since $\mathbf{k}[[x]]$ is a local ring with maximal ideal (x), $\mathbf{R}$ is obtained from $\mathbf{k}[[x]]$ by inverting x. Thus every element $g \in \mathbf{R}$ can be written in the form

$$g = \sum_{i=N}^{\infty} \alpha_i x^i,$$

for suitable $\alpha_i \in \mathbf{k}$ and $N \in \mathbb{Z}$.

Notice that $\mathbf{R}$ acts on itself by left multiplication and this gives us a monomorphism of $\mathbf{k}$ vector spaces:

$$\tau : \mathbf{R} \to \mathrm{End}_{\mathbf{k}}(\mathbf{R}).$$

Furthermore, there also exists $\partial \in \mathrm{End}_{\mathbf{k}}(\mathbf{R})$ which corresponds to formal differentiation with respect to x, i.e.

$$\partial(\sum_{i=N}^{\infty} \alpha_i x^i) = \sum_{i=N}^{\infty} i\alpha_i x^{i-1}.$$

It is easy to verify that $\partial(g) = 0$ if and only if g is a constant.

Definition 2.1 *A stable algebra A is a subalgebra of $\mathbf{R}$ with the property that $\partial(A) \subseteq A$.*

Remark 2.2 *Three important examples of stable algebras are the polynomial algebra $\mathbf{k}[x]$, the power series algebra $\mathbf{k}[[x]]$, and the Laurent polynomial algebra $\mathbf{k}[x, x^{-1}]$. (Recall a Laurent polynomial is an element of the form $\sum_{i=N}^{M} \alpha_i x^i$ for suitable $N, M \in \mathbb{Z}$ and $\alpha_i \in \mathbf{k}$.)*

Definition 2.3 *Given a stable algebra A, we define $Weyl(A)$ to be the subalgebra of $\mathrm{End}_{\mathbf{k}}(\mathbf{R})$ generated by $\tau(A)$ and ∂. Thus, $Weyl(A)$ is an associative algebra with identity element equal to the identity endomorphism of $\mathbf{R}$. We will identify A with its image $\tau(A) \subseteq \mathrm{End}_{\mathbf{k}}(\mathbf{R})$ from now on.*

Lemma 2.4 *Let A be a stable algebra. For any $f \in A$, one has $\partial f - f\partial = f'$ in $Weyl(A)$. Thus for any $\alpha \in Weyl(A)$, one has $\alpha = \sum_{i=0}^{N} \alpha_i \partial^i$ for suitable $N \in \mathbb{N}$ and $\alpha_i \in A$.*

Furthermore, if $\{e_i | i \in I\}$ is a $\mathbf{k}$-basis for A, then $\{e_i \partial^j | i \in I, j \in \mathbb{N}\}$ is a $\mathbf{k}$-basis for $Weyl(A)$.

Proof. The proof is standard and is left to the reader. □

Remark 2.5 *$Weyl(\mathbf{k}[x])$ is the classical Weyl algebra. It is a simple algebra which has no zero divisors (see [2]). In general, one can define an order on $Weyl(\mathbf{R})$ such that the order of a nonzero element is equal to the highest exponent of ∂ in its canonical expression and is defined to be $-\infty$ for the zero element.*

Then one shows that $\mathrm{ord}(\alpha\beta) = \mathrm{ord}(\alpha) + \mathrm{ord}(\beta)$ for any $\alpha, \beta \in Weyl(\mathbf{R})$ (see [2]) and it easily follows that $Weyl(\mathbf{R})$ has no zero divisors. Hence, $Weyl(A)$, which is a subalgebra of $Weyl(\mathbf{R})$, has no zero divisors in general. Note however, that in general, $Weyl(A)$ need not be simple.

3. Generalized Witt algebras

Definition 3.1 *Let $Witt(A)$ be the subspace of $Weyl(A)$ consisting of the order 1 elements together with zero. Thus $\alpha \in Witt(A)$ if α can be written as $f\partial$ for some $f \in A$.*

It is easy to check that $Witt(A)$ is a Lie subalgebra of $Weyl(A)$. (Note, it is not a subalgebra of $Weyl(A)$.)

If $\{e_i\}_{i\in I}$ is a $\mathbf{k}$-basis for A then $\{e_i\partial\}_{i\in I}$ is a $\mathbf{k}$-basis for $Witt(A)$.

Proposition 5.12 shows how our definition is related to the one found in [3].

Remark 3.2 *$Witt(\mathbf{k}[x])$ is the classical Witt algebra. It is the Lie algebra of derivations of the classical Weyl algebra (see [2]), and is a simple Lie algebra. However, in general, $Witt(A)$ is not neccessarily simple. $Witt(\mathbf{k}[x,x^{-1}])$ is called the centerless Virasoro algebra in the literature. (See [7].)*

In general, we cannot claim that $Witt(A)$ is simple, but these generalized Witt algebras do share one important common property: they are self-centralizing.

Definition 3.3 *Given a Lie algebra $\mathfrak{L}$ and an element $l \in \mathfrak{L}$, we define the centralizer of l, $C(l) = \{x \in \mathfrak{L} | [l,x] = 0\}$. Notice, by the Jacobi identity, $C(l)$ is always a Lie subalgebra of $\mathfrak{L}$ containing l.*

Proposition 3.4 *Given a Lie algebra $\mathfrak{L}$, the following conditions are equivalent.*
(a) For any nonzero $l \in \mathfrak{L}$, $[l,x] = 0$ implies $x = \beta l$ for some $\beta \in \mathbf{k}$.
(b) $C(l)$ is one dimensional for all nonzero $l \in \mathfrak{L}$.
(c) $\mathfrak{L}$ does not contain any abelian Lie algebras of dimension greater than one.
(d) If $\alpha, \beta \in \mathfrak{L}$ are linearly independent, then $[\alpha, \beta] \neq 0$.

Proof. The proof is easy and left to the reader. □

Definition 3.5 *A Lie algebra $\mathfrak{L}$ is said to be self-centralizing if it satisfies any of the equivalent conditions of Proposition 3.4.*

Remark 3.6 *Thus a self-centralizing Lie algebra is one where the centralizers have as small a dimension as possible. Notice that a self-centralizing Lie algebra of dimension strictly greater than one must have trivial center. Furthermore, a Lie algebra isomorphic to a self-centralizing one is itself self-centralizing.*

Remark 3.7 *It is easy to check that the nonabelian Lie algebra of dimension two is self-centralizing but is not simple. Similarly $\mathfrak{sl}_n$, the Lie algebra of $n \times n$, trace zero matrices is simple but contains an abelian Lie subalgebra of dimension greater than one for $n \geq 3$ and hence is not self-centralizing.*

We now make a useful observation

Theorem 3.8 *For any stable algebra A, $Witt(A)$ is a self-centralizing Lie algebra.*

Proof. Let $f\partial$ be a nonzero element of $Witt(A)$. Suppose $[f\partial, g\partial] = 0$. Then as $[f\partial, g\partial] = (fg' - gf')\partial$, we conclude that $fg' - gf' = 0$ in $A \subseteq \mathbf{R}$.

Then we can rewrite $fg' - gf' = 0$ as $(g/f)'f^2 = 0$ in $\mathbf{R}$ which is possible since f is not the zero element. Since the only elements in $\mathbf{R}$ which have zero derivative, are the constants, we conclude that g/f is a constant or that g is a multiple of f. Thus we conclude $C(f\partial)$ is one dimensional. This concludes the proof. □

Remark 3.9 *It follows immediately from Theorem 3.8, that the classical Witt algebra and the centerless Virasoro algebra are self-centralizing.*

Definition 3.10 *Recall that a Lie algebra is called semisimple if it does not possess any nontrivial solvable ideals. It is a standard fact that a Lie algebra is semisimple if it does not possess any nontrivial abelian ideals. (See [5].)*

Let us record some consequences of the self-centralizing property in the following proposition.

Proposition 3.11 *Let $\mathfrak{L}$ be a self-centralizing Lie algebra. Then:*
(a) Any Lie subalgebra is also self-centralizing.
(b) If $\mathfrak{L}$ possesses a finite dimensional ideal I of dimension $n > 1$, then $\dim(\mathfrak{L}) \leq n^2$. If $\mathfrak{L}$ possesses an ideal of dimension 1, then $\dim(\mathfrak{L}) \leq 2$.
(c) If $\mathfrak{L}$ is infinite dimensional, then $\mathfrak{L}$ does not possess any finite dimensional, nontrivial ideals.
(d) If α, β are two linearly independent elements of $\mathfrak{L}$ and x is a common eigenvector of $ad(\alpha)$ and $ad(\beta)$, then x is a multiple of $[\alpha, \beta]$.
(e) If α, β are two linearly independent elements of $\mathfrak{L}$, then there is no basis for $\mathfrak{L}$, in which both α and β are ad-diagonal.
(f) $\mathfrak{L}$ is indecomposable i.e., $\mathfrak{L}$ cannot be written as a direct sum of two nonzero Lie algebras.
(g) If $\dim(\mathfrak{L}) > 2$ then $\mathfrak{L}$ is semisimple.
(h) If $\mathfrak{L}$ is finite dimensional and $\mathbf{k}$ is algebraically closed, then $\mathfrak{L}$ is either isomorphic to the nonabelian Lie algebra of dimension two, $\mathfrak{sl}_2$, or a Lie algebra of dimension less than or equal to one.

Proof. (a) follows at once from the definition of a self-centralizing Lie algebra. To prove (b), suppose I is a nontrivial, finite dimensional ideal of dimension n. Then define $\theta : \mathfrak{L} \to \mathrm{End}_{\mathbf{k}}(I)$ by

$$\theta(x) = ad(x)|_I.$$

Note that $\mathrm{End}_{\mathbf{k}}(I)$ is finite dimensional of dimension n^2. If $n > 1$, then θ is injective by the self-centralizing property of $\mathfrak{L}$. This is because if z were a nonzero element in $\mathrm{Ker}(\theta)$, then $I \subseteq C(z)$. However, $C(z)$ has dimension 1 as $\mathfrak{L}$ is self-centralizing, while I is assumed to have dimension bigger than 1 giving a contradiction. It follows easily from the injectivity of θ that

$$\dim(\mathfrak{L}) \leq \dim(\mathrm{End}_{\mathbf{k}}(I)) = n^2.$$

If $n = 1$ and x is a generator of I, then $\mathrm{Ker}(\theta)$ is codimension at most one in $\mathfrak{L}$. However, $\mathrm{Ker}(\theta) = C(x) = I$ since $\mathfrak{L}$ is self-centralizing. Thus $\dim(\mathfrak{L}) \leq 2$.

(c) follows quickly from (b). (d) and (e) follow from quick calculations and the self-centralizing property. (f) is a trivial verification.

For (g), note that if $\dim(\mathfrak{L}) > 2$, then by (b), $\mathfrak{L}$ does not possess any nontrivial ideals of dimension one. On the other hand, because $\mathfrak{L}$ is self-centralizing, it cannot possess any abelian ideals of dimension greater than one and so we conclude that $\mathfrak{L}$ does not possess any nontrivial abelian ideals and hence is semisimple.

For (h), note that if $\dim(\mathfrak{L}) \leq 2$, the result is easy. So we can assume $2 < \dim(\mathfrak{L}) < \infty$, and so by (g), $\mathfrak{L}$ is semisimple. From standard results (see [5] or [4]), since we are over a field of characteristic zero, $\mathfrak{L}$ is the direct sum of simple Lie algebras. However by (f), we see that in fact $\mathfrak{L}$ must be simple.

If we assume $\mathbf{k}$ to be algebraically closed, then the Cartan subalgebra of $\mathfrak{L}$ is abelian, and since $\mathfrak{L}$ is self-centralizing, it must have rank one. From the classification of simple finite dimensional Lie algebras over an algebraically closed field, we see that $\mathfrak{L}$ is isomorphic to $\mathfrak{sl}_2$. □

Remark 3.12 *By Proposition 3.11, we see that there aren't very many finite dimensional self-centralizing Lie algebras. Thus it is somewhat striking that all of the generalized Witt algebras are self-centralizing.*

We will see later that we can find infinitely many nonisomorphic generalized Witt algebras so that the class of self-centralizing Lie algebras is pretty rich. In the class of infinite dimensional Lie algebras, Proposition 3.11 shows that being self-centralizing is a stronger condition than being semisimple and yet is usually easier to verify than simplicity.

Since stable algebras A are infinite dimensional in all but some trivial cases, $Witt(A)$ is usually infinite dimensional and since it is self-centralizing by Theorem 3.8, it follows by Proposition 3.11, that $Witt(A)$ is both semisimple and indecomposable. However there are examples where $Witt(A)$ is simple and there are examples where it is not. We will discuss this more later on.

4. Eigenvalues and eigenspaces

We have seen that all generalized Witt algebras are self-centralizing. Given a Lie algebra $\mathfrak{L}$, and $\alpha \in \mathfrak{L}$, let $E_a(\alpha) \subseteq \mathfrak{L}$ be the eigenspace of $ad(\alpha)$ corresponding to the eigenvalue $a \in \mathbf{k}$.

In this language, a self-centralizing Lie algebra $\mathfrak{L}$ is one such that

$$\dim(E_0(\alpha)) = 1$$

for all nonzero $\alpha \in \mathfrak{L}$. We have seen that a generalized Witt algebra is self-centralizing and hence satisfies this condition on the eigenspaces. We will now extend this result by studying further constraints on these eigenspaces in a generalized Witt algebra.

Before we can do this, we need to recall the concept of the logarithmic derivative on $\mathbf{R}$, and some of its basic properties.

Definition 4.1 *Let $\mathbf{R}^\sharp$ denote the group of nonzero elements in the field $\mathbf{R}$ under multiplication. (Recall $\mathbf{R}$ is the field of fractions of $\mathbf{k}[[x]]$.) The logarithmic derivative $LD : \mathbf{R}^\sharp \to \mathbf{R}$ is defined by*

$$LD(f) = \frac{f'}{f},$$

where f' is the formal derivative of f. It is easy to check that LD is a group homomorphism from $(\mathbf{R}^\sharp, \times)$ to $(\mathbf{R}, +)$.

It is also routine to see that $\mathrm{Ker}(LD)$ is exactly the constant functions. Thus if $u, v \in \mathbf{R}^\sharp$ have $LD(u) = LD(v)$, then u is a scalar multiple of v.

Now we are ready to prove an important lemma which generalizes Theorem 3.8.

Lemma 4.2 *If $f\partial \in Witt(\mathbf{R})$ is a nonzero element, then $\dim(E_a(f\partial)) \leq 1$ for all $a \in \mathbf{k}$. Furthermore, if $g\partial$ is a nonzero element in $E_a(f\partial)$, then $g = fu$ where $LD(u) = a/f$.*

Proof. Suppose $g\partial$ is a nonzero element in $E_a(f\partial)$. Then

$$[f\partial, g\partial] = ag\partial$$

$$(fg' - gf')\partial = ag\partial$$

$$(g/f)'f^2 = ag.$$

Thus we conclude $(g/f)'f = a(g/f)$. If we let $u = g/f$, this becomes $u'f = au$ or $LD(u) = a/f$. Thus we conclude $g = fu$ where $LD(u) = a/f$. If $h\partial$ is another nonzero element in $E_a(f\partial)$, then similarly we would conclude $h = fv$ where $LD(v) = a/f$. However, $LD(u) = LD(v) = a/f$ so v is a scalar multiple of u and hence h is a scalar multiple of g. Thus we see $\dim(E_a(f\partial)) \leq 1$ as we sought to show. □

Lemma 4.2 shows that for any nonzero $f\partial \in Witt(\mathbf{R})$, and $a \in \mathbf{k}$, the eigenspace of $ad(f\partial)$ corresponding to a is at most one dimensional. It remains to decide when this eigenspace is one dimensional and when it is zero dimensional. To do this, it turns out we need to find the image of $LD : \mathbf{R}^\sharp \to \mathbf{R}$. We will now introduce a few more concepts in formal calculus that will let us do this.

Definition 4.3 *Given a nonzero $f \in \mathbf{R}$, we can write*

$$f = \sum_{i=N}^{\infty} \alpha_i x^i,$$

where $\alpha_i \in \mathbf{k}$ for all $i \geq N$ and $\alpha_N \neq 0$. N is called the Weierstrass degree (see [9]) of f and will be denoted by $W(f)$. α_{-1} is called the residue of f and will be denoted $\operatorname{res}(f)$. We also define $W(0) = \infty$ and $\operatorname{res}(0) = 0$.

Definition 4.4 *Let $U = \{f \in \mathbf{R} | W(f) = 0\}$. Then $f \in U$ if and only if $f \in \mathbf{k}[[x]]$ and $f(0) \neq 0$ and this happens if and only if f is a unit of $\mathbf{k}[[x]]$. Thus U is the group of units of $\mathbf{k}[[x]]$ under multiplication.*

We now collect some elementary properties of the Weierstrass degree in the next lemma. The proof is simple and will be left to the reader.

Lemma 4.5 *Given nonzero $f \in \mathbf{R}$, we can write*

$$f = x^{W(f)}u$$

with $u \in U$. Furthermore such an expression for f is unique.
Given $f, g \in \mathbf{R}$,

$$W(fg) = W(f) + W(g).$$

We now define formal integration.

Definition 4.6 *Recall (x) is the unique maximal ideal of $\mathbf{k}[[x]]$. We define formal integration $\int : \mathbf{k}[[x]] \to (x)$ by*

$$\int (\sum_{i=0}^{\infty} \alpha_i x^i) = \sum_{i=0}^{\infty} \alpha_i \frac{x^{i+1}}{i+1}$$
$$= \sum_{i=1}^{\infty} \alpha_{i-1} \frac{x^i}{i}.$$

It follows easily that $\int \in \mathrm{End}_{\mathbf{k}}(\mathbf{k}[[x]])$ and that, if $f \in \mathbf{k}[[x]]$ is nonzero,

$$W(\int f) = W(f) + 1.$$

Furthermore, we have of course

$$\partial(\int f) = f$$

for all $f \in \mathbf{k}[[x]]$.

We will also need to compose two power series. Recall that given $g \in \mathbf{k}[[x]]$ and $f \in (x)$, we have a well-defined composition power series $g \circ f \in \mathbf{k}[[x]]$ given in the following manner: If $g = \sum_{i=0}^{\infty} \alpha_i x^i$ then $g \circ f \in \mathbf{k}[[x]]$ is given formally by $\sum_{i=0}^{\infty} \alpha_i f^i$.

We collect well-known results on this composition in the following proposition.

Proposition 4.7 *If $g \in \mathbf{k}[[x]]$ and $f \in (x)$. Then there exists a series $g \circ f \in \mathbf{k}[[x]]$ such that*

$$(g \circ f)' = (g' \circ f) f'.$$

Furthermore, $(g \circ f)(0) = g(0)$ and $g \circ x = g$.

We are now ready to study the image of the logarithmic derivative $LD : \mathbf{R}^\sharp \to \mathbf{R}$.

Lemma 4.8 *Let $u \in \mathbf{R}^\sharp$,*
(a) If $W(u) \neq 0$ then $W(LD(u)) = -1$ and $\mathrm{res}(LD(u))$ is equal to $W(u)$ which is of course an integer.
(b) If $W(u) = 0$ then $W(LD(u)) \geq 0$.
(c) If $W(g) < -1$ or if $W(g) = -1$ and $\mathrm{res}(g)$ is not an integer, then g is not in the image of $LD : \mathbf{R}^\sharp \to \mathbf{R}$.

Proof. The proof will be left to the reader. It follows from writing u as a Laurent series and explicitly calculating $LD(u)$. □

We have seen in Lemma 4.8, conditions that ensure an element $g \in \mathbf{R}$ is not in the image of $LD : \mathbf{R}^\sharp \to \mathbf{R}$. We now show that, in the remaining situations, the element g is in the image.

First recall $e^x \in \mathbf{k}[[x]]$ is the power series given by

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}.$$

It is easy to verify that $\partial e^x = e^x$ and that e^x evaluated at $x = 0$ is 1.

Given $g \in \mathbf{k}[[x]]$, $\int g$ lies in (x), the maximal ideal of $\mathbf{k}[[x]]$. Thus by Proposition 4.7 we can form the power series $e^x \circ (\int g)$ which we will denote by $e^{\int g}$. It follows from the same proposition that

$$\partial e^{\int g} = e^{\int g} \partial (\int g) = g e^{\int g}.$$

Furthermore, since $e^{\int g}(0) = e^x(0) = 1$, we see that $e^{\int g} \in U$ for all $g \in \mathbf{k}[[x]]$.

We will use these facts in the next theorem.

Theorem 4.9 *Let $g \in \mathbf{R}$. Then either:*
(a) $W(g) \geq 0$ and $g = LD(e^{\int g})$.
(b) $W(g) = -1$ and $\text{res}(g)$ is an integer then $g = \frac{\text{res}(g)}{x} + u$ for some unique $u \in \mathbf{k}[[x]]$ and we have $g = LD(x^{\text{res}(g)} e^{\int u})$.
(c) $W(g) < -1$ or $W(g) = -1$ and $\text{res}(g)$ is not an integer, in which case g is not in the image of $LD : \mathbf{R}^\sharp \to \mathbf{R}$.

Proof. (c) follows from Lemma 4.8. For (a), assume g has $W(g) \geq 0$ so that $e^{\int g} \in U$. Then we calculate

$$LD(e^{\int g}) = \frac{\partial e^{\int g}}{e^{\int g}} = \frac{g e^{\int g}}{e^{\int g}} = g$$

and so (a) is proven.

Assume g as in the statement of (b). Then it is obvious that we may write $g = \frac{\text{res}(g)}{x} + u$ with $u \in \mathbf{k}[[x]]$ determined uniquely. Since $\text{res}(g)$ is an integer $x^{\text{res}(g)} e^{\int u}$ certainly defines an element in $\mathbf{R}^\sharp$. We compute

$$\begin{aligned} LD(x^{\text{res}(g)} e^{\int u}) &= \text{res}(g) LD(x) + LD(e^{\int u}), \text{ since } LD \text{ is a homomorphism} \\ &= \text{res}(g)\frac{1}{x} + u, \text{ using the calculation in (a)} \\ &= g. \end{aligned}$$

Thus we are done.

□

We are now ready to complete the analysis of the eigenspaces of elements in $ad(Witt(\mathbf{R}))$ which was started in Lemma 4.2.

Theorem 4.10 (Spectral theorem for R) *Let $f\partial$ be a nonzero element in $Witt(\mathbf{R})$. Then:*
(a) If $W(f) > 1$, then $\dim(E_a(f\partial)) = 0$ for all nonzero $a \in \mathbf{k}$ and

$$\dim(E_0(f\partial)) = 1.$$

(b) If $W(f) \leq 0$*, then* $\dim(E_a(f\partial)) = 1$ *for all* $a \in \mathbf{k}$*. Furthermore,*

$$fe^{\int \frac{a}{f}}\partial \in E_a(f\partial).$$

(c) If $W(f) = 1$ *then* $\dim(E_a(f\partial)) = 0$ *if* $a \neq Nf'(0)$ *for some integer* N*.* $\dim(E_{Nf'(0)}(f\partial)) = 1$ *for all* $N \in \mathbb{Z}$*. Furthermore*

$$fx^N e^{\int (\frac{N(f'(0)x-f)}{fx})}\partial \in E_{Nf'(0)}(f\partial)$$

for all $N \in \mathbb{Z}$*.*

Proof. Let $f\partial \in Witt(\mathbf{R})$ be nonzero and let $a \in \mathbf{k}$. Then by Lemma 4.2, we see that $\dim(E_a(f\partial))$ is either zero or one and it is one if and only if $\frac{a}{f} = LD(u)$ for some $u \in \mathbf{R}^\sharp$. Furthermore, in this case, $fu\partial$ is a nonzero element of $E_a(f\partial)$. Since we know $\dim(E_0(f\partial)) = 1$ we can assume $a \neq 0$ for the rest of the proof. It follows that $W(\frac{a}{f}) = -W(f)$.

If $W(f) > 1$ then $W(\frac{a}{f}) < -1$ and so by Theorem 4.9, $\frac{a}{f}$ is not in the image of the logarithmic derivative and hence we have proven (a).

If $W(f) \leq 0$ then $W(\frac{a}{f}) \geq 0$ and so $\frac{a}{f} = LD(e^{\int \frac{a}{f}})$ by Theorem 4.9 giving us (b).

If $W(f) = 1$ then we can write $f = xf'(0)v$ where $v \in U$ has $v(0) = 1$. Then $W(\frac{a}{f}) = -1$ and $\mathrm{res}(\frac{a}{f}) = \frac{a}{f'(0)}$. Again by Theorem 4.9, $\frac{a}{f}$ is in the image of the logarithmic derivative if and only if this residue is an integer which happens if and only if a is an integral multiple of $f'(0)$. If this is the case, then $a = Nf'(0)$ and we can write

$$\frac{Nf'(0)}{f} = \frac{N}{x} + w$$

where $w \in \mathbf{k}[[x]]$. Theorem 4.9 then shows that $\frac{Nf'(0)}{f} = LD(x^N e^{\int w})$. Now it remains only to note that

$$w = \frac{Nf'(0)}{f} - \frac{N}{x} = \frac{N(f'(0)x - f)}{fx},$$

and we are done. □

5. Spectra

We now discuss the concept of a spectrum which we will find to be very useful in the remainder of this paper.

Definition 5.1 *Given a Lie algebra* $\mathfrak{L}$*, and* $\alpha \in \mathfrak{L}$*, we define the* $\mathfrak{L}$*-spectrum of* α *to be*

$$\mathrm{spec}_{\mathfrak{L}}(\alpha) = \{a \in \mathbf{k} | \dim(E_a(\alpha)) \neq 0\}.$$

We write $\mathrm{spec}(\alpha)$ *for* $\mathrm{spec}_{\mathfrak{L}}(\alpha)$ *when there is no danger of confusion. Thus the spectrum of* α *is the set of eigenvalues of* $ad(\alpha) \in \mathrm{End}_{\mathbf{k}}(\mathfrak{L})$*.*

Notice that in a nonzero Lie algebra $\mathfrak{L}$, $\text{spec}(0) = \{0\}$ and $0 \in \text{spec}(\alpha)$ for all $\alpha \in \mathfrak{L}$. In general the spectrum possesses no significant algebraic structure. However, we will soon see that if $\mathfrak{L}$ is self-centralizing, $\text{spec}(\alpha)$ possesses the structure of a pseudomonoid (which we will define shortly) for all $\alpha \in \mathfrak{L}$.

Definition 5.2 *A subset P of $\mathbf{k}$ is a pseudomonoid if it satisfies the following conditions:*
(a) $0 \in P$.
(b) If $a, b \in P$ and $a \neq b$ then $a + b \in P$ where $+$ is addition in $\mathbf{k}$.

Remark 5.3 *Notice that a pseudomonoid differs from a monoid because in a monoid we may also add an element to itself, i.e., if $a \in P$ and P is a monoid under $+$ then $a + a \in P$. This need not hold for a pseudomonoid as can be seen by the following example:*
Let $A = \{-1, 0, 1, \dots\}$ be the set of integers greater than or equal to negative one. This set is a pseudomonoid under addition but is not a monoid as

$$(-1) + (-1) = -2 \notin A.$$

The concept of a pseudomonoid turns out to be important for us because of the following lemma.

Lemma 5.4 *Let $\mathfrak{L}$ be a Lie algebra, and $\alpha \in \mathfrak{L}$ be a nonzero element. Then for any $a, b \in \mathbf{k}$, we have $[E_a(\alpha), E_b(\alpha)] \subseteq E_{a+b}(\alpha)$.*

Thus if $\mathfrak{L}$ is self-centralizing, then $\text{spec}(\alpha)$ *is a pseudomonoid for all $\alpha \in \mathfrak{L}$.*

Proof. For a proof of the first statement, take $\alpha \in \mathfrak{L}$ and $a, b \in \mathbf{k}$. Then for $e_a \in E_a(\alpha)$ and $e_b \in E_b(\alpha)$ we have by the Jacobi identity:

$$\begin{aligned} [\alpha, [e_a, e_b]] &= [[\alpha, e_a], e_b] + [e_a, [\alpha, e_b]] \\ &= [ae_a, e_b] + [e_a, be_b] \\ &= (a + b)[e_a, e_b]. \end{aligned}$$

Thus we see that $[e_a, e_b] \in E_{a+b}(\alpha)$ which proves the first statement.

Now suppose that $\mathfrak{L}$ is self-centralizing. $\text{spec}(0) = \{0\}$ is a pseudomonoid so assume $\alpha \neq 0$. Let $a, b \in \text{spec}(\alpha)$ with $a \neq b$. Then if we take nonzero $e_a \in E_a(\alpha)$ and $e_b \in E_b(\alpha)$, since $a \neq b$ it follows that e_a, e_b are linearly independent. Thus since $\mathfrak{L}$ is self-centralizing, it follows that $[e_a, e_b] \neq 0$, which shows that $E_{a+b}(\alpha) \neq 0$. Thus $a + b \in \text{spec}(\alpha)$ and so $\text{spec}(\alpha)$ is a pseudomonoid. □

Definition 5.5 *Let $\alpha \in \mathfrak{L}$. Then we define*

$$M_{\mathfrak{L}}(\alpha) = \oplus_{a \in \mathbf{k}} E_a(\alpha).$$

Thus $M_{\mathfrak{L}}(\alpha)$ is the subspace of $\mathfrak{L}$ spanned by the eigenspaces of α. It is the maximal subspace on which $ad(\alpha)$ is diagonal (with respect to some basis).

It is easy to argue that we can also write

$$M_{\mathfrak{L}}(\alpha) = \oplus_{a \in \mathrm{spec}(\alpha)} E_a(\alpha).$$

It follows from Lemma 5.4 that $M_{\mathfrak{L}}(\alpha)$ is a Lie subalgebra of $\mathfrak{L}$. We will write $M(\alpha)$ for $M_{\mathfrak{L}}(\alpha)$ when there is no danger of confusion.

We will now look at a few examples before proceeding any further. To do this, it is useful to introduce the concept of a differential spanning set.

Definition 5.6 *$S \subseteq \mathbf{R}$ is called a differential spanning set if it satisfies the following conditions:*
(a) $1 \in S$.
(b) If $f, g \in S$, then $fg \in S$.
(c) If $f \in S$ then ∂f is a linear combination of elements in S.

Given a differential spanning set S, the vector space A spanned by S in $\mathbf{R}$ is easily seen to be a stable algebra.

Example 5.7 *Let $S = \{x^n | n \in \mathbb{N}\}$, then it is easy to check that S is a differential spanning set which spans the polynomial stable algebra $\mathbf{k}[x]$ and is in fact a basis for this algebra. In $Witt(\mathbf{k}[x])$, one calculates the relation*

$$[x^n\partial, x^m\partial] = (x^n(x^m)' - x^m(x^n)')\partial$$
$$= (m-n)x^{m+n-1}\partial.$$

Thus we see easily that $M(x\partial) = Witt(\mathbf{k}[x])$ and that $\mathrm{spec}(x\partial) = \{-1, 0, 1, \ldots\}$.

Example 5.8 *Let $S = \{x^n | n \in \mathbb{Z}\}$, then S is a differential spanning set which forms a basis for the Laurent polynomial stable algebra $\mathbf{k}[x, x^{-1}]$. Exactly as in Example 5.7, one can show that $M(x\partial) = Witt(\mathbf{k}[x, x^{-1}])$ and that* $\mathrm{spec}(x\partial) = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \mathbb{Z}$.

Note that the spectrum of $x\partial$ depends on which Lie algebra we are in and so we stress that the reader should keep in mind the surpressed subscript $\mathfrak{L}$ in the notation for spec.

Example 5.9 *Let G be a submonoid of $(\mathbf{k}, +)$, then $S = \{e^{ax} | a \in G\}$, is a differential spanning set (since $e^{(a+b)x} = e^{ax}e^{bx}$ as the reader can verify). Let $A(G)$ be the stable algebra that this spanning set spans. In $Witt(A(G))$, we calculate*

$$[e^{ax}\partial, e^{bx}\partial] = (e^{ax}be^{bx} - e^{bx}ae^{ax})\partial$$
$$= (b-a)e^{(a+b)x}.$$

From this, it follows that $M(1\partial) = Witt(A(G))$ and that $\mathrm{spec}(1\partial) = G$. *Since $e^{bx}\partial \in E_b(1\partial)$ for all $b \in G$, it also follows that S is a basis for $A(G)$.*

Remark 5.10 *It is a standard fact that every torsion-free abelian group embeds into a torsion-free divisible group and that a torsion-free divisible group is isomorphic to the additive group of a rational vector space (see [11]).*

Any rational vector space is isomorphic to a subgroup of $(\mathbf{k}, +)$ *for suitable choice of* $\mathbf{k}$. *(We need the dimension of* $\mathbf{k}$ *over its characteristic subfield* $\mathbb{Q}$ *to be big enough.)*

Thus by Example 5.9, we conclude that every torsion-free abelian group is the spectrum of some ad-diagonal element in some generalized Witt algebra in one variable.

Finally we state a general spectral theorem for generalized Witt algebras. It is based on Theorem 4.10.

Theorem 5.11 (Spectral theorem) *Let* $Witt(A)$ *be a generalized Witt algebra and* $f\partial$ *be a nonzero element in* $Witt(A)$. *Then for all* $a \in \mathbf{k}$,

$$\dim(E_a(f\partial)) \leq 1,$$

and
(a) If $W(f) > 1$, *then* $\mathrm{spec}(f\partial) = \{0\}$.
(b) If $W(f) \leq 0$ *then for all* $a \in \mathbf{k}$,

$$a \in \mathrm{spec}(f\partial) \iff fe^{\int \frac{a}{f}} \in A.$$

(c) If $W(f) = 1$ *then* $\mathrm{spec}(f\partial) \subseteq \mathbb{Z}f'(0)$, *where* $\mathbb{Z}f'(0)$ *stands for the set of integral multiples of* $f'(0) \in \mathbf{k}$. *Furthermore, for all* $N \in \mathbb{Z}$,

$$Nf'(0) \in \mathrm{spec}(f\partial) \iff fx^N e^{\int \frac{N(f'(0)x - f)}{fx}} \in A.$$

Proof. First note that $A \subseteq \mathbf{R}$ so $Witt(A)$ is a Lie subalgebra of $Witt(\mathbf{R})$. Then if $f\partial \in Witt(A)$, and $a \in \mathbf{k}$, the a-eigenspace of $ad(f\partial)$ for $Witt(A)$ lies inside the one for $Witt(\mathbf{R})$. Thus $\mathrm{spec}_{Witt(A)}(f\partial) \subseteq \mathrm{spec}_{Witt(\mathbf{R})}(f\partial)$ and $a \in \mathrm{spec}_{Witt(\mathbf{R})}(f\partial)$ lies in $\mathrm{spec}_{Witt(A)}(f\partial)$ if and only if one of the eigenvectors in $\mathbf{R}$ corresponding to a actually lies in A. With these comments, the rest now follows from Theorem 4.10. □

Theorem 5.11 will show that our definition of generalized Witt algebras is related to the definition in papers such as [3]. We do this in the next proposition.

Proposition 5.12 *Let* $Witt(A)$ *be a generalized Witt algebra and let* $f\partial$ *be a nonzero element of* $Witt(A)$. *Then there exists a basis* $\{e_a\}_{a \in \mathrm{spec}(f\partial)}$ *of* $M(f\partial)$ *such that*

$$[e_a, e_b] = (b - a)e_{a+b}$$

for all $a, b \in \mathrm{spec}(f\partial)$. *(Here,* $(b-a)e_{a+b}$ *is considered to be zero for* $a = b$ *even though* $a + b$ *might not be in* $\mathrm{spec}(f\partial)$.*) Furthermore, we can take* $e_a \in E_a(f\partial)$ *for all* $a \in \mathrm{spec}(f\partial)$ *and* $e_0 = f\partial$.

Proof. By Theorem 5.11, if $W(f) > 1$, then $\mathrm{spec}(f\partial) = \{0\}$ and the result is obvious.

If $W(f) \leq 0$, then set

$$e_a = fe^{\int \frac{a}{f}}\partial$$

for all $a \in \mathrm{spec}(f\partial)$.

Then by Theorem 5.11, $\{e_a\}_{a \in \operatorname{spec} f\partial}$ is a basis for $M(f\partial)$.
One computes using $[g\partial, h\partial] = (gh' - hg')\partial$, that indeed

$$[e_a, e_b] = (b-a)e_{a+b}.$$

Similarly, in the remaining case where $W(f) = 1$, we set

$$e_{Nf'(0)} = fx^N e^{\int \frac{N(f'(0)x - f)}{fx}} \partial$$

for all $Nf'(0) \in \operatorname{spec}(f\partial)$, and again compute that

$$[e_{Nf'(0)}, e_{Mf'(0)}] = (M-N)f'(0)e_{(N+M)f'(0)}$$

for all $Mf'(0), Nf'(0) \in \operatorname{spec}(f\partial)$. □

We are now ready to study the issue of simplicity of a generalized Witt algebra. We will do this in the next section.

6. Simplicity

Definition 6.1 *A Lie algebra $\mathfrak{L}$ is said to be strongly graded if there exists a pseudomonoid G and a vector space decomposition:*

$$\mathfrak{L} = \oplus_{a \in G} E_a$$

with the properties
(a) $\dim(E_a) = 1$ *for all* $a \in G$.
(b) There is a basis $\{e_a\}_{a \in G}$ *of* $\mathfrak{L}$ *such that* $e_a \in E_a$ *for all* $a \in G$ *and*

$$[e_a, e_b] = (b-a)e_{a+b}.$$

Note that this means that $\operatorname{spec}(e_0) = G$.

Remark 6.2 *Of course, by Theorem 5.12 and Proposition 5.12, if $Witt(A)$ is a generalized Witt algebra, and $\alpha \in Witt(A)$ is nonzero, then $M(\alpha)$ is a strongly graded Lie algebra, graded by the pseudomonoid* $\operatorname{spec}(\alpha)$ *where α plays the role of e_0.*

Remark 6.3 *It is obvious that two strongly graded Lie algebras, graded by the same pseudomonoid $G \subseteq \mathbf{k}$, are isomorphic as Lie algebras.*

Now we set out to get a complete correspondence between the ideals of a strongly graded Lie algebra and the ideals of the pseudomonoid which grades it.

Definition 6.4 *Let G be a pseudomonoid.*

Then $S \subseteq G$ is called a closed subset if for all distinct $a, b \in S$, we have $a + b \in S$. Note that a closed subset S need not be a subpseudomonoid of G since we do not require that $0 \in S$. In fact the empty set $\emptyset$ is always a closed subset.

$I \subseteq G$ is called an ideal subset if for all $a \in I$ and all $b \in G$ such that $b \neq a$, we have $a + b \in I$. Again the empty set is always an ideal subset and G is always an ideal subset of G. These are called the trivial ideal subsets.

A pseudomonoid G which has no nontrivial ideal subsets is called a simple psuedomonoid.

A nonzero element $x \in G$ is called invertible if $-x \in G$. (Recall, all pseudomonoids are by definition in $\mathbf{k}$ and hence $-x$ exists in $\mathbf{k}$ and is distinct from x.)

Fix a strongly graded Lie algebra $\mathfrak{L}$, graded by a pseudomonoid G, then $\mathfrak{L} = \oplus_{g \in G} E_g$ such that there is $e_0 \in E_0$, with $\text{spec}(e_0) = G$ and E_g equal to the eigenspace of $ad(e_0)$ corresponding to g.

Then for any $S \subset G$, we define:

$$\Theta(S) = \oplus_{a \in S} E_a.$$

(We use the convention that $\Theta(\emptyset) = 0$.)

Thus Θ is a map from the subsets of G to the subspaces of $\mathfrak{L}$, which is obviously injective.

Notice that if S is a closed subset of G, then $\Theta(S)$ is a Lie subalgebra of $\mathfrak{L}$ (because $[E_a, E_a] = 0$ for all $a \in G$). Furthermore, if $0 \in S$, then $e_0 \in \Theta(S)$.

Similarly, if I is an ideal subset of G, then $\Theta(I)$ is an ideal of $\mathfrak{L}$.

Proposition 6.5 *Let $\mathfrak{L}$ be a strongly graded Lie algebra, graded by the pseudomonoid G.*

The map Θ defined above takes closed subsets of G to Lie subalgebras of $\mathfrak{L}$ and this correspondence is injective.

The map Θ takes closed subsets of G containing 0, to Lie subalgebras of $\mathfrak{L}$ containing e_0 and this correspondence is bijective.

The map Θ takes ideal subsets of G to ideals of $\mathfrak{L}$ and this correspondence is bijective.

Proof. All but the surjectivity of the last two correspondences has been proven.

So assume J is an ideal of $\mathfrak{L}$ (or a Lie subalgebra containing e_0). First, let us show that there is a subset I of G such that $\Theta(I) = J$. We can of course assume $J \neq 0$ as $\Theta(\emptyset) = 0$.

Define $I \subset G$ as follows. Recall that by the grading, for any $x \in \mathfrak{L}$, we can write x uniquely as

$$x = \sum_{a \in G} x_a,$$

with $x_a \in E_a$ and only finitely many x_a nonzero. We call x_a the a-th component of x. Then set

$$I = \{a \in G \text{ such that there exists } y \in J \text{ whose } a\text{-th component is nonzero}\}.$$

It is clear that $J \subseteq \Theta(I)$. So it remains only to show $\Theta(I) \subseteq J$. We do this by showing that $E_a \subseteq J$ for any $a \in I$. This follows immediately from the following fact:

Fact: If $y \in J$, then all of the components of y are also in J.

We will prove this fact by induction on n, the number of nonzero components of y. If $n = 0, 1$, it follows trivially. So assume $n > 1$ and we have proven the fact for all smaller n. So let $y \in J$ and assume we can write

$$y = \sum_{i=1}^{n} y_{a_i},$$

with $y_{a_i} \in E_{a_i}$ nonzero and $\{a_i\}_{i=1}^{n}$ a set of distinct elements in I. Also, without loss of generality, $a_1 \neq 0$. Then

$$[e_0, y] = \sum_{i=1}^{n} a_i y_{a_i}$$

is in J and so

$$y - \frac{1}{a_1}[e_0, y] = \sum_{i=2}^{n}(1 - \frac{a_i}{a_1})y_{a_i}$$

is in J. However by induction, it follows that the components of $y - \frac{1}{a_1}[e_0, y]$ lie in J and hence that y_{a_i} lie in J for all $2 \leq i \leq n$. However $y = y_{a_1} + \sum_{i=2}^{n} y_{a_i}$, so it also follows that y_{a_1} is in J. Thus by induction, we have proven the fact and hence that $J = \Theta(I)$.

All that remains is to show that I is an ideal subset if J is an ideal or that I is a closed subset containing zero if J is a Lie subalgebra containing e_0. We prove only the former, the proof of the latter being similar.

If $a \in I$ then by definition, there is $y \in J$ such that $y = \sum_{g \in G} y_g$ with $y_g \in E_g$ and $y_a \neq 0$. If $b \in G$ and $b \neq a$, take nonzero $z_b \in E_b$. Then $[z_b, y] = \sum_{g \in G}[z_b, y_g] \in J$ as J is an ideal. Notice that since our pseudomonoids are defined to be subpseudomonoids of $(\mathbf{k}, +)$, the only term in the sum that can lie in E_{a+b} is $[z_b, y_a]$ which is nonzero as z_b, y_a are nonzero and since we are in a strongly graded Lie algebra. All the other terms live in other eigenspaces and so we conclude $[z_b, y_a]$ has nonzero $(a + b)$-component and hence $a + b \in I$ showing that I, is an ideal subset of G.

□

Corollary 6.6 *If $\mathfrak{L}$ is a strongly graded Lie algebra, graded by a pseudomonoid G. Then $\mathfrak{L}$ is simple if and only if G is simple.*

Let $Witt(A)$ be a generalized Witt algebra and $\alpha \in Witt(A)$ be nonzero, then $M(\alpha)$ is a simple Lie algebra if and only if $\mathrm{spec}(\alpha)$ *is a simple pseudomonoid. (Note it is easy to see that* $\mathrm{spec}_{M(\alpha)}(\alpha) = \mathrm{spec}_{Witt(A)}(\alpha)$*.)*

Proof. Follows immediately from previous remarks and Proposition 6.5. □

So we see that it would be useful to have some conditions that ensure the simplicity of a pseudomonoid. This is the purpose of the next lemma.

Lemma 6.7 *Let G be a pseudomonoid. Then:*
(a) If I is an ideal subset, and $0 \in I$ then $I = G$.

(b) If I is an ideal subset, and there is an invertible element $x \in I$ then $I = G$.
(c) A pseudomonoid which is a group is a simple pseudomonoid.

Proof. Let I be an ideal subset with $0 \in I$. Then for any nonzero $a \in G$, we have $0 + a = a \in I$ since I is an ideal subset. Thus $I = G$. This proves (a).

Suppose I contained an invertible element x. Then as $x \neq -x$, and $-x \in G$, we have $x + (-x) = 0 \in I$ as I is an ideal subset. Thus $I = G$ by (a). So this proves (b).

If G is an (abelian) group and I a nonempty ideal subset. Then take $a \in I$. If $a = 0$ then $I = G$ by (a) and if a is nonzero then a is invertible as G is a group, and so $I = G$ by (b). Thus we conclude G is a simple pseudomonoid. □

Corollary 6.8 *If $\mathfrak{L}$ is a strongly graded Lie algebra, graded by an abelian group $A \subseteq \mathbf{k}$, then $\mathfrak{L}$ is simple.*

In Example 5.7 we saw that the classical Witt algebra, $Witt(\mathbf{k}[x])$ is strongly graded, graded by the pseudomonoid $G = \{-1, 0, 1, \dots\}$. If I is a nonempty ideal subset of this pseudomonoid, by adding -1 repeatedly to an element in I if necessary, we see $-1 \in I$. Since -1 is invertible in G, we conclude by Lemma 6.7, that $I = G$. So G is a simple pseudomonoid and so the classical Witt algebra is a simple Lie algebra.

In Example 5.8 we saw that the centerless Virasoro algebra, $Witt(\mathbf{k}[x, x^{-1}])$ is strongly graded, graded by the pseudomonoid $\mathbb{Z}$. Since this is a group, it is simple as a pseudomonoid and we have proven the following corollary.

Corollary 6.9 *The classical Witt algebra and the centerless Virasoro algebra are simple.*

Example 6.10 *The natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$ is a monoid which is not simple as a pseudomonoid. In fact if we define $I_k = \{k, k+1, \dots\}$ for all $k \in \mathbb{N}$, then the reader can easily verify that I_k is an ideal subset of $\mathbb{N}$. (There is exactly one more nonempty ideal subset not covered by these which we leave the reader to find if they wish.) So from Example 5.9, $Witt(A(\mathbb{N}))$ gives us an example of a generalized Witt algebra which is not simple.*

Definition 6.11 *Two subsets S_1, S_2 of $\mathbf{k}$ are said to be equivalent if there exists nonzero $k \in \mathbf{k}$ such that*

$$S_1 = kS_2 \equiv \{kx | x \in S_2\}.$$

It is easy to see that this defines an equivalence relation on the subsets of $\mathbf{k}$. We write $[[S]]$ for the equivalence class of the set S under this equivalence relation.

For any Lie algebra $\mathfrak{L}$, nonzero $\alpha \in \mathfrak{L}$, and nonzero $k \in \mathbf{k}$, it is easy to see that $M(\alpha) = M(k\alpha)$ and $\mathrm{spec}(k\alpha) = k\,\mathrm{spec}(\alpha)$. Thus we have

$$[[\mathrm{spec}(k\alpha)]] = [[\mathrm{spec}(\alpha)]].$$

It is also easy to see that two strongly graded Lie algebras, graded by equivalent pseudomonoids, are isomorphic as Lie algebras.

Given a strongly graded Lie algebra $\mathfrak{L}$, graded by the pseudomonoid G, we have $\mathfrak{L} = M(e_0)$ with $\mathrm{spec}(e_0) = G \subseteq \mathbf{k}$ where e_0 is obtained from the definition of a strongly graded Lie algebra.

We would like to define $[[\mathrm{spec}(e_0)]]$ as an invariant of $\mathfrak{L}$. However, it turns out that this is not apriori, intrinsic enough to be useful, i.e., it is not obvious that we might not find another nonzero element f such that $\mathfrak{L} = M(f)$ and $[[\mathrm{spec}(f)]] \neq [[\mathrm{spec}(e_0)]]$.

In the next section, we show that this in fact cannot occur, and hence define an invariant which helps us find infinite families of nonisomorphic generalized Witt algebras!

7. Invariance of the spectrum

Before we proceed any further, we need to develop a somewhat technical tool. We need to weakly order any field (of characteristic zero). We define this notion now.

Definition 7.1 *A weak order on* $\mathbf{k}$ *is a linear order* $\preceq$ *on* $\mathbf{k}$ *such that if* $x \preceq y$ *then* $x + z \preceq y + z$ *for all* $z \in \mathbf{k}$. *(Recall a linear order is a partial order with the property that for any two elements* e, f *either* $e \preceq f$ *or* $f \preceq e$ *(or both).)*

Note the field of real numbers $\mathbb{R}$ has a weak order (the usual one) and so any subfield of $\mathbb{R}$ has a weak order.

A weak order on an abelian group is defined in exactly the same way.

As is common, we will write $x \prec y$ if $x \preceq y$ and $x \neq y$.

There is also a stronger notion of ordered field in the literature (see [9]). However for example $\mathbb{C}$, the field of complex numbers, cannot be made into an ordered field. However, we show in the next proposition, that any field (of characteristic zero) has a weak order.

Proposition 7.2 *Any field* $\mathbf{k}$ *(of characteristic zero) possesses a weak order.*

Proof. We identify the characteristic subfield of $\mathbf{k}$ with the rational numbers $\mathbb{Q}$ as is usual. Then of course, k is a vector space over $\mathbb{Q}$. Define the set S as

$$S = \{(A, \preceq) | A \text{ is a } \mathbb{Q}\text{-subspace of } \mathbf{k} \text{ and } \preceq \text{ is a weak order on } A.\}.$$

We make S into a partially ordered set $(S, \leq)$ as follows:

$$(A_1, \preceq_1) \leq (A_2, \preceq_2) \iff A_1 \subseteq A_2 \text{ and } \preceq_2 |_{A_1} = \preceq_1 .$$

The characteristic subfield $\mathbb{Q}$ of $\mathbf{k}$ can be viewed as the characteristic subfield of the real numbers and so we can put the standard order on it. Thus S is not empty.

It is easy to verify that any chain $\{(A_i, \preceq_i)_{i \in I}\}$ in $(S, \leq)$ has an upper bound $(\cup_{i \in I} A_i, \preceq)$ and thus Zorn's lemma gives us a maximal element $(M, \preceq)$ of $(S, \leq)$.

Suppose $M \neq \mathbf{k}$, then we can find $a \in \mathbf{k} \setminus M$ and thus $M' = M \oplus \mathbb{Q}a$ is a $\mathbb{Q}$-subspace of $\mathbf{k}$. We define $\preceq'$ on M' as follows:

$$m_1 + q_1 a \prec' m_2 + q_2 a \iff m_1 \prec m_2 \text{ or } m_1 = m_2 \text{ and } q_1 < q_2.$$

It is easy to verify that $\preceq'$ is a weak order on M' which restricts to $\preceq$ on M.

Thus $(M, \preceq) < (M', \preceq')$ which is a contradiction as $(M, \preceq)$ is maximal. Thus we conclude $M = \mathbf{k}$ and hence that we can weakly order $\mathbf{k}$. □

We now use Proposition 7.2 to weakly order any pseudomonoid.

Definition 7.3 *Let $G \subseteq \mathbf{k}$ be a psuedomonoid. A weak order on G is the restriction of some weak order on $\mathbf{k}$.*

Proposition 7.2 shows all pseudomonoids possess a weak order (since we require our pseudomonoids to be in $\mathbf{k}$, by definition).

Definition 7.4 *Let G be a pseudomonoid with weak order $\preceq$.*

We say that $x \in (G, \preceq)$ is positive if $0 \prec x$ and we say x is negative if $x \prec 0$.

If we set P to be the set of positive elements in $(G, \preceq)$ and N to be the set of negative elements in $(G, \preceq)$, then it is easy to see that $\{P, N, \{0\}\}$ is a partition of G.

A maximum element M of $(G, \preceq)$ is an element such that $x \preceq M$ for all $x \in G$. Similarly, a minimum element m of $(G, \preceq)$ is an element such that $m \preceq x$ for all $x \in G$. Notice that if there is a maximum element, it is unique as $\preceq$ is a linear order and similarly for a minimum element.

An extreme element of $(G, \preceq)$ is either a maximum or a minimum element.

We collect in the next lemma some basic but useful facts about ordered psuedomonoids.

Lemma 7.5 *Let $(G, \preceq)$ be a pseudomonoid with a weak order. Then:*
(a) If G possesses a minimum element m then either $m = 0$ or m is the unique negative element in $(G, \preceq)$.
(b) If G possesses a maximum element M then either $M = 0$ or M is the unique positive element in $(G, \preceq)$.
(c) If G possesses a minimum and a maximum element then the order of G is less than or equal to 3.
(d) If the order of G is infinite, then G possesses at most one extreme element.
(e) If G is a finite pseudomonoid, then the order of G is either one, two or three. Furthermore, for each of these orders, there is a unique pseudomonoid up to equivalence.

Proof. For (a), let m be a minimum element and assume m is not zero. Then we must have $m \prec 0$ as m is a minimum element.

Suppose there were $x \prec 0$ with $x \neq m$, then $x+m \prec 0+m = m$ with $x+m \in G$ as G is a pseudomonoid. This contradicts the minimality of m and thus we conclude there is no such x, i.e., m is the unique negative element.

The proof of (b) is similar to (a) and is left to the reader. For (c), note that if G has a minimum element m and a maximum element M then it follows that the set of nonpositive elements is $\{0, m\}$ by (a) and the set of nonnegative elements is $\{0, M\}$ by (b). Thus $G = \{0, m, M\}$ and hence G has order less than or equal to 3. (Exact order depends on whether or not the elements $\{0, m, M\}$ are distinct or not.)

(d) follows immediately from (c). The first part of (e) also follows immediately from (c) since any weak order on a finite pseudomonoid has a maximum and a minimum element.

Note that if the order of G is three and $G = \{0, m, M\}$, then we must have $M = -m$ since $M + m \in G$. Thus it is easy to see that $[[G]] = [[\{-1, 0, 1\}]]$. If G has order two, obviously $[[G]] = [[\{0, 1\}]]$ and if G has order one, then $G = \{0\}$. So we are done. □

Remark 7.6 *From Proposition 3.11, we have a complete list of finite dimensional, self-centralizing Lie algebras (in the case that* $\mathbf{k}$ *is algebraically closed). The reader can easily verify that each of these is strongly graded, graded by a finite pseudomonoid of size one, two or three.*

Definition 7.7 *Let* $\mathfrak{L} = \oplus_{g \in G} E_g$ *be a strongly graded Lie algebra and suppose we have a weak order* $\preceq$ *on the pseudomonoid* G*. Then if* $\alpha \in \mathfrak{L}$ *is nonzero we can uniquely write*

$$\alpha = \sum_{i=1}^{n} e_{g_i}$$

where $g_1 \prec g_2 \prec \cdots \prec g_n \in G$ *and* $e_{g_i} \in E_{g_i}$ *is nonzero for all* $1 \leq i \leq n$*.*

We call $g_1 \in (G, \preceq)$ *the initial index of* α *and write* $g_1 = \mathrm{Init}(\alpha)$*.*

We call $g_n \in (G, \preceq)$ *the terminal index of* α *and write* $g_n = \mathrm{Term}(\alpha)$*.*

Lemma 7.8 *Let* $\mathfrak{L}$ *be a strongly graded Lie algebra, graded by a weakly ordered pseudomonoid* $(G, \preceq)$*. Then if* x, y *are nonzero elements of* $\mathfrak{L}$*, we have:*
(a) If $\mathrm{Term}(x) \neq \mathrm{Term}(y)$ *then* $[x, y] \neq 0$ *and*

$$\mathrm{Term}([x, y]) = \mathrm{Term}(x) + \mathrm{Term}(y).$$

(b) If $\mathrm{Init}(x) \neq \mathrm{Init}(y)$ *then* $[x, y] \neq 0$ *and* $\mathrm{Init}([x, y]) = \mathrm{Init}(x) + \mathrm{Init}(y)$*.*

Proof. $\mathfrak{L} = \oplus_{g \in G} E_g$ so we can take a basis $\{e_g\}_{g \in G}$ of $\mathfrak{L}$ with $e_g \in E_g$ for all $g \in G$. First we expand x in the basis $\{e_g\}_{g \in G}$. Thus

$$x = \sum_{i=1}^{n} x_{g_i} e_{g_i},$$

with $g_1 \prec g_2 \prec \cdots \prec g_n$ and $x_{g_i} \neq 0$ for all $1 \leq i \leq n$. Thus $\mathrm{Init}(x) = g_1$ and $\mathrm{Term}(x) = g_n$.

We can expand y in a similar manner.

$$y = \sum_{j=1}^{m} y_{h_j} e_{h_j}$$

with $h_1 \prec \cdots \prec h_m$ and $y_{h_j} \neq 0$ all $1 \leq j \leq m$. Thus $\mathrm{Init}(y) = h_1$ and $\mathrm{Term}(y) = h_m$.

Then we calculate that

$$[x, y] = \sum_{i=1}^{n} \sum_{j=1}^{m} x_{g_i} y_{h_j} [e_{g_i}, e_{h_j}].$$

Hence, if $g_n \neq h_m$ then $0 \neq [e_{g_n}, e_{h_m}] \in E_{g_n+h_m}$ and $g_n + h_m$ is easily seen to be the terminal index of $[x, y]$, and similarly, if $g_1 \neq h_1$ then $g_1 + h_1$ is the initial index of $[x, y]$. □

Corollary 7.9 *Let $\mathfrak{L}$ be as in Lemma 7.8. Suppose $\alpha \in \mathfrak{L}$ is nonzero. Then:*
(a) If $\mathrm{Init}(\alpha) \neq 0$ then every eigenvector x of $ad(\alpha)$ has $\mathrm{Init}(x) = \mathrm{Init}(\alpha)$.
(b) If $\mathrm{Term}(\alpha) \neq 0$ then every eigenvector x of $ad(\alpha)$ has $\mathrm{Term}(x) = \mathrm{Term}(\alpha)$.
(c) $\dim(E_a(\alpha)) \leq 1$ for all $a \in \mathbf{k}$.

Proof. For (a), let α have $\mathrm{Init}(\alpha) \neq 0$ and assume x is an eigenvector of $ad(\alpha)$ with $\mathrm{Init}(x) \neq \mathrm{Init}(\alpha)$. Then by Lemma 7.8 we have $[\alpha, x]$ is nonzero and

$$\mathrm{Init}([\alpha, x]) = \mathrm{Init}(x) + \mathrm{Init}(\alpha).$$

However, as x is an eigenvector, we also have $[\alpha, x] = \mu x$ for some $\mu \in \mathbf{k}$. Since $[\alpha, x] \neq 0$ we conclude $\mu \neq 0$ and hence that

$$\mathrm{Init}(\alpha) + \mathrm{Init}(x) = \mathrm{Init}([\alpha, x]) = \mathrm{Init}(\mu x) = \mathrm{Init}(x).$$

Thus $\mathrm{Init}(\alpha) = 0$ which contradicts our hypothesis. Thus we conclude every eigenvector of α must have the same initial index as α. The proof of (b) is similar and is left to the reader.

For (c), note that if both $\mathrm{Init}(\alpha)$ and $\mathrm{Term}(\alpha)$ are zero, then α is a nonzero scalar multiple of e_0 and the result is clear. So we can assume one of $\mathrm{Init}(\alpha)$ or $\mathrm{Term}(\alpha)$ is nonzero. For concreteness, let us assume $\mathrm{Init}(\alpha) \neq 0$, the proof for the case where $\mathrm{Term}(\alpha) \neq 0$ being similar and left to the reader.

Then if $\dim(E_a(\alpha)) \geq 2$ for some $a \in \mathbf{k}$. We can find linearly independent $x, y \in E_a(\alpha)$. By (a), we have $\mathrm{Init}(x) = \mathrm{Init}(y) = \mathrm{Init}(\alpha)$. Then it is clear we can form a nonzero linear combination of x and y whose $\mathrm{Init}(\alpha)$-component is zero. Call this element z then this means that $\mathrm{Init}(z)$ is not $\mathrm{Init}(\alpha)$. This is a contradiction as z is nonzero and in $E_a(\alpha)$ and so, by (a) again, must have $\mathrm{Init}(z) = \mathrm{Init}(\alpha)$. □

We are now ready to prove an important proposition. This proposition will enable us to define the spectrum of a strongly graded Lie algebra and use it as a tool to distinguish between two such Lie algebras.

Proposition 7.10 *Let $\mathfrak{L}$ be an infinite dimensional, strongly graded Lie algebra, graded by a pseudomonoid G. Choose a weak order $\preceq$ on G and let $\{e_g\}_{g \in G}$ be the usual basis of $\mathfrak{L}$.*

Suppose we have nonzero $\alpha \in \mathfrak{L}$ such that $M(\alpha) = \mathfrak{L}$, then:
(a) If $(G, \preceq)$ has no nonzero extreme elements, $\alpha = ke_0$ for some nonzero $k \in \mathbf{k}$. Thus $\mathrm{spec}(\alpha) = k\,\mathrm{spec}(e_0)$ and

$$[[\mathrm{spec}(\alpha)]] = [[\mathrm{spec}(e_0)]] = [[G]].$$

(b) If $(G, \preceq)$ has a nonzero extreme element m, then m is unique and

$$\alpha = ke_0 + k'e_m$$

for some $k, k' \in \mathbf{k}$ with $k \neq 0$. Furthermore we still have

$$[[\mathrm{spec}(\alpha)]] = [[\mathrm{spec}(e_0)]] = [[G]].$$

Proof. Assume the setup as in the statement of the proposition.

First note that if $\text{Init}(\alpha) \neq 0$ then Corollary 7.9 shows that all the eigenvectors of α have initial index equal to $\text{Init}(\alpha)$. However these eigenvectors span $\mathfrak{L}$ as $M(\alpha) = \mathfrak{L}$ and so it follows easily that $\text{Init}(\alpha)$ is a nonzero minimal element of $(G, \preceq)$.

Similarly if $\text{Term}(\alpha) \neq 0$ then $\text{Term}(\alpha)$ is a nonzero maximal element of $(G, \preceq)$.

For (a), note that our previous arguments show that if $(G, \preceq)$ has no nonzero extreme elements, that $\text{Init}(\alpha) = 0 = \text{Term}(\alpha)$ and hence that $\alpha = ke_0$ for some nonzero $k \in \mathbf{k}$ from which the rest of the conclusion in (a), is obvious.

For (b), note that we can assume that at least one of $\text{Term}(\alpha), \text{Init}(\alpha)$ is a nonzero extreme element of $(G, \preceq)$ or else the conclusion would follow from our argument for (a).

Since $\mathfrak{L}$ is infinite dimensional, G is infinite and hence $(G, \preceq)$ can possess at most one extreme element by Lemma 7.5, part (d). Thus for (b), we can assume $(G, \preceq)$ has exactly one extreme element m and that it is a minimum. (If it was a maximum, reorder G by setting $x \prec' y \iff y \prec x$. This reordering switches $\text{Init}(\alpha)$ and $\text{Term}(\alpha)$ but does not change the conclusions of this proposition.)

Thus we have that without loss of generality, $\text{Init}(\alpha) = m \prec 0$ is the minimum of $(G, \preceq)$ and that $\text{Term}(\alpha) = 0$ (Recall if $\text{Term}(\alpha) \neq 0$, we showed before that it would be a nonzero maximum which is a contradiction to our assumption). Thus we have

$$\alpha = k'e_m + T + ke_0$$

where $k, k' \in \mathbf{k}$ are nonzero and T consists of terms which have components corresponding to elements in $g \in G$ which have $m \prec g \prec 0$. By Lemma 7.5, part (a), there are no such elements g, and so we conclude that $\alpha = k'e_m + ke_0$.

It remains to show that $[[\text{spec}(\alpha)]] = [[G]]$. Since $[[\text{spec}(\alpha)]]$ does not change if we scale α, we will assume from now on that $k = 1$. So $\alpha = k'e_m + e_0$.

Suppose x is an eigenvector of $ad(\alpha)$ corresponding to eigenvalue $\mu \in \mathbf{k}$ with $0 \prec \text{Term}(x)$. Then $x = ae_{\text{Term}(x)} + D$ where D has nonzero components only in indices $g \in G$ with $g \prec \text{Term}(x)$, and $a \in \mathbf{k}$ is nonzero. Then

$$[\alpha, x] = [k'e_m + e_0, ae_{\text{Term}(x)} + D] = a\,\text{Term}(x)e_{\text{Term}(x)} + D'.$$

where D' has nonzero components only in indices $g \in G$ with $g \prec \text{Term}(x)$.

However, $[\alpha, x] = \mu x$ and so we have

$$a\,\text{Term}(x)e_{\text{Term}(x)} + D' = \mu a e_{\text{Term}(x)} + \mu D$$

from which it follows that $\mu = \text{Term}(x) \in G$.

Now if x is an eigenvector of $ad(\alpha)$ with $\text{Term}(x) \preceq 0$ then $x = ce_m + de_0$ and it is easy to check that x must be a scalar multiple of e_m or of α corresponding to the eigenvalues m and 0 respectively. In any case, we have $[\alpha, x] = \text{Term}(x)x$.

Thus we see that if x is any eigenvector of $ad(\alpha)$, then x corresponds to the eigenvalue $\text{Term}(x) \in G$. So $\text{spec}(\alpha) \subseteq \text{spec}(e_0) = G$. Furthermore, $m, 0 \in \text{spec}(\alpha)$, with m a minimum element of $\text{spec}(\alpha)$ under the ordering inherited from G.

However, we also see that if x, y are eigenvectors of $ad(\alpha)$ corresponding to different eigenvalues, then $\text{Term}(x) \neq \text{Term}(y)$ and we must have $[x, y] \neq 0$ by Lemma 7.8. Since $M(\alpha) = \mathfrak{L}$ and $\dim(E_a(\alpha)) \leq 1$ for all $a \in \mathbf{k}$ by Corollary 7.9, we conclude that $\mathfrak{L}$ is strongly graded with respect to the eigenspaces of α.

Thus reversing the roles of α and e_0 in the part of the proof where we showed $\text{spec}(\alpha) \subseteq \text{spec}(e_0)$, and noting that $e_0 = \alpha - k' e_m$, we conclude that $\text{spec}(e_0) \subseteq \text{spec}(\alpha)$ and hence that $\text{spec}(e_0) = \text{spec}(\alpha)$ and thus we are done. □

Definition 7.11 *Let $\mathfrak{L}$ be a strongly graded Lie algebra. We define*

$$\text{spec}(\mathfrak{L}) = [[\text{spec}(\alpha)]]$$

where α is a nonzero element in $\mathfrak{L}$ with $M(\alpha) = \mathfrak{L}$.

Note that $\text{spec}(\mathfrak{L})$ is well-defined if $\mathfrak{L}$ is infinite dimensional, by Proposition 7.10.

If $\mathfrak{L}$ is finite dimensional, then Corollary 7.9, part (c), shows that

$$\dim(E_a(\alpha)) \leq 1$$

for all $a \in \mathbf{k}$ and so we must have the order of $\text{spec}(\alpha)$ is equal to the dimension of $\mathfrak{L}$ for any nonzero α with $M(\alpha) = \mathfrak{L}$. Since there is exactly one pseudomonoid of order $\text{spec}(\alpha)$ up to equivalence by Lemma 7.5, $\text{spec}(\mathfrak{L})$ is well-defined in this case also.

We now show that $\text{spec}(\mathfrak{L})$ is truly an invariant of $\mathfrak{L}$.

Proposition 7.12 *Let $\mathfrak{L}, \mathfrak{L}'$ be two Lie algebras and $f : \mathfrak{L} \to \mathfrak{L}'$ be a Lie algebra homomorphism. Then:*
(a) For every $\alpha \in \mathfrak{L}$ and $a \in \mathbf{k}$, we have

$$f(E_a(\alpha)) \subseteq E_a(f(\alpha)).$$

Hence $f(M(\alpha)) \subseteq M(f(\alpha))$.
(b) If f is injective, then $\text{spec}(\alpha) \subseteq \text{spec}(f(\alpha))$.
(c) If f is bijective, then $\text{spec}(\alpha) = \text{spec}(f(\alpha))$ and furthermore

$$f(M(\alpha)) = M(f(\alpha)).$$

(d) If $\mathfrak{L}, \mathfrak{L}'$ are two strongly graded Lie algebras, and f is an isomorphism, then $\text{spec}(\mathfrak{L}) = \text{spec}(\mathfrak{L}')$.

Proof. For (a), notice that if $x \in E_a(\alpha)$, then $[\alpha, x] = ax$ and hence

$$f([\alpha, x]) = af(x).$$

Since f is a Lie algebra homomorphism, we have $f([\alpha, x]) = [f(\alpha), f(x)]$ and so we conclude $[f(\alpha), f(x)] = af(x)$ and thus $f(x) \in E_a(f(\alpha))$. Also $M(\alpha) = \oplus_{a \in \mathbf{k}} E_a(\alpha)$ and so

$$f(M(\alpha)) = \oplus_{a \in \mathbf{k}} f(E_a(\alpha)) \subseteq \oplus_{a \in \mathbf{k}} E_a(f(\alpha)) = M(f(\alpha)).$$

This gives us (a).

For (b), notice that if f is injective, and we had nonzero $x \in E_a(\alpha)$, then $f(x)$ would be nonzero, and by (a), it would lie in $E_a(f(\alpha))$. This proves (b).

For (c), notice that since f is bijective, f^{-1} exists and is in fact a Lie algebra homomorphism. Thus from (a) and (b) applied to (f, α) and $(f^{-1}, f(\alpha))$ we get

$$f(M(\alpha)) \subseteq M(f(\alpha)) \text{ and } f^{-1}(M(f(\alpha))) \subseteq M(f^{-1}(f(\alpha)))$$

giving us $f(M(\alpha)) = M(f(\alpha))$. We also get

$$\text{spec}(\alpha) \subseteq \text{spec}(f(\alpha)) \text{ and } \text{spec}(f(\alpha)) \subseteq \text{spec}(f^{-1}(f(\alpha))),$$

giving us $\text{spec}(\alpha) = \text{spec}(f(\alpha))$.

For (d), note that $\text{spec}(\mathfrak{L}) = [[\text{spec}(\alpha)]]$ for some nonzero $\alpha \in \mathfrak{L}$ with $M(\alpha) = \mathfrak{L}$. Since f is an isomorphism, we have $f(\alpha)$ is nonzero with

$$M(f(\alpha)) = f(M(\alpha)) = f(\mathfrak{L}) = \mathfrak{L}'.$$

Hence by Proposition 7.10, we have

$$\text{spec}(\mathfrak{L}') = [[\text{spec}(f(\alpha))]] = [[\text{spec}(\alpha)]] = \text{spec}(\mathfrak{L}).$$

Thus we are done. □

Definition 7.13 *Two pseudomonoids G and G' are isomorphic if there is a bijection $f : G \to G'$ such that*
(a) $f(0) = 0$ and
(b) $f(x + y) = f(x) + f(y)$ for all distinct $x, y \in G$.

It is easy to see that if $[[G]] = [[G']]$, then G is isomorphic to G'.

Example 7.14 *The field $\mathbf{k}$ is a vector space over its characteristic subfield $\mathbb{Q}$. If $\dim_{\mathbb{Q}}(\mathbf{k}) = \infty$ then we can find $\mathbb{Q}$-vector subspaces V_n of $\mathbf{k}$ of dimension n for every $n \in \mathbb{N}$. Certainly the $\{V_n\}_{n\in\mathbb{N}}$ are a family of nonisomorphic pseudomonoids which are simple pseudomonoids by Lemma 6.7 as they are abelian groups.*

Thus the construction of Example 5.9 gives us a family $Witt(A(V_n))$ of simple, strongly graded Lie algebras by Corollary 6.6.

Furthermore since $\text{spec}(Witt(A(V_n))) = [[V_n]]$ we see that

$$\{Witt(A(V_n))\}_{n\in\mathbb{N}}$$

is an infinite family of nonisomorphic, simple, generalized Witt algebras.

Example 7.15 *Let $\mathbb{N}$ be the monoid of natural numbers. For every pair of relatively prime integers $n, m > 1$, we define $M_{n,m}$ to be the submonoid of $\mathbb{N}$ generated by n and m. It is easy to see that $M_{n,m}$ is never* *simple as a pseudomonoid as one can find nontrivial restrictions of ideal subsets from* $\mathbb{N}$. *(See Example 6.10.) Furthermore* $M_{n,m}$ *is isomorphic to* $M_{n',m'}$ *if and only if* $\{n,m\} = \{n',m'\}$.

Thus again using the construction of Example 5.9, we get an infinite family

$$Witt(A(M_{n,m}))_{1<n<m, gcd(n,m)=1}$$

of nonisomorphic, nonsimple, generalized Witt algebras. By Proposition 3.11, all of these Lie algebras are semisimple and indecomposable and have no abelian Lie subalgebras of dimension greater than one.

In contrast, over an algebraically closed field, the only finite dimensional Lie algebra which is indecomposable, semisimple and has no abelian Lie subalgebras of dimension greater than one is $\mathfrak{sl}_2$.

Example 7.16

$$\text{spec}(Witt(\mathbf{k}[x])) = [[\{-1,0,1,\dots\}]]$$

and

$$\text{spec}(Witt(\mathbf{k}[x,x^{-1}])) = \mathbb{Z}$$

by examples 5.7 and 5.8. These spectra are easily seen not to be isomorphic to those discussed in examples 7.14 and 7.15, and not isomorphic to each other of course.

Thus the following is a list of nonisomorphic generalized Witt algebras: the classical Witt algebra, the centerless Virasoro algebra, $Witt(A(M_{m,n}))$ for relatively prime $m,n > 1$ and $Witt(A(V_n))$ for $\mathbb{Q}$-vector subspaces V_n of $\mathbf{k}$, where $\dim_{\mathbb{Q}}(V_n) = n$ for all $n \in \mathbb{N}$.

Thus, we hope we have conveyed the rich variety of generalized Witt algebras available!

In the final section, we verify the Jacobian conjecture for a class of generalized Witt algebras. That is, we show that under suitable hypothesis, any nonzero Lie algebra endomorphism of a generalized Witt algebra is actually an automorphism.

8. The Jacobian conjecture

A polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ is a map with the property that each of its components is a complex polynomial in n-variables. Such a map is called invertible if it is bijective, and if its inverse is a polynomial map also. It is easily seen that an invertible polynomial map has the property that the determinant of its Jacobian matrix is a nonzero constant as a function on $\mathbb{C}^n$. (See [2]). The classical Jacobian conjecture is that the converse is true and remains open for all $n \geq 2$.

One can ask the following question about the classical Weyl algebra in n-variables. (Defined similarly as we did in the beginning of the paper, but using n-variables instead of one.) Is every nonzero algebra endomorphism of a classical Weyl algebra actually an automorphism? The answer to this question is unknown for all $n \geq 1$. If the statement is true for some n, then it implies the classical Jacobian conjecture in dimension n. (See [2].)

One can generalize to the following definition.

Definition 8.1 *Given a Lie algebra* $\mathfrak{L}$, *one says that the Jacobian conjecture holds for* $\mathfrak{L}$, *if every nonzero Lie algebra endomorphism is actually an automorphism.*

Certainly the Jacobian conjecture does not hold for all Lie algebras but does hold for finite dimensional, simple Lie algebras.

We will show, among other things that the Jacobian conjecture holds for the classical Witt algebra which is the Lie algebra of derivations of the classical Weyl algebra where the corresponding conjecture remains open.

One can see immediately, the spectral theory machinery developed earlier has a lot to say about this. For example one has:

Corollary 8.2 *If $Witt(A)$ is a generalized Lie algebra and $f\partial$ is a nonzero element such that* $\mathrm{spec}(f\partial) \neq \{0\}$*. Then for every injective Lie algebra endomorphism F of $Witt(A)$, one has $F(f\partial) = g\partial$ with $W(g) \leq 1$.*

Proof. This follows immediately from Theorem 5.11 and Proposition 7.12. □

Corollary 8.2 shows that the image of an element under an injective endomorphism, is reasonably constrained by its spectrum. Of course, Corollary 8.2 is a rough application of these ideas and we will have to refine them a bit to get our desired result. To this end, we define:

Definition 8.3 *A pseudomonoid $G \subseteq \mathbf{k}$ is called self-containing if there is nonzero $a \in \mathbf{k}$ such that $aG \subset G$ and $aG \neq G$.*

Notice in this case that aG is a subpseudomonoid of G which is equivalent to G so we could also define a pseudomonoid to be self-containing if it possesses a proper subpseudomonoid equivalent to itself.

The integers $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$ is an example of a self-containing pseudomonoid since $n\mathbb{Z}$ is a proper subpseudomonoid equivalent to $\mathbb{Z}$ for all natural numbers $n \geq 2$. The reader can verify that this is in fact a complete list of all such proper subpseudomonoids.

We next give examples of pseudomonoids which are not self-containing.

Lemma 8.4 *Any subfield E of $\mathbf{k}$ is not a self-containing pseudomonoid.*
$\{-1, 0, 1, \dots\} \subseteq \mathbf{k}$ is not a self-containing psuedomonoid.

Proof. Suppose $aE \subseteq E$ for some nonzero $a \in \mathbf{k}$. Since $1 \in E$, it follows that $a \in E$. Then given $x \in E$, $xa^{-1} \in E$ and $x = a(xa^{-1})$. Thus $aE = E$. So E is not self-containing.

Give $G = \{-1, 0, 1, \dots\}$ the weak order inherited by viewing it as the usual subset of the real numbers. If $aG \subseteq G$ for some nonzero $a \in \mathbf{k}$, it again follows as $1 \in G$, that $a \in G$.

Clearly $a \neq -1$ so $a > 0$. Then we must have $a(-1) = -a \in G$ and hence $-a = -1$ and $a = 1$. Thus $aG = G$ and so G is not a self-containing pseudomonoid either. □

Definition 8.5 *Let $\mathfrak{L}$ be a strongly graded Lie algebra, graded by G. Then we can write $\mathfrak{L} = \oplus_{g \in G} E_g$ as usual. For nonzero $x \in \mathfrak{L}$, we let x_g be the g-component of x.*

We define the support of x as

$$\mathrm{Supp}(x) = \{g \in G | x_g \neq 0\}.$$

We also define $\mathrm{Supp}(0) = \emptyset$.

Definition 8.6 *A weak order $\preceq$ on a pseudomonoid G is called discrete if for every $a, b \in G$, the order of the set $\{g \in G | a \preceq g \preceq b\}$ is finite.*

A pseudomonoid which possesses a discrete order is called discrete.

Every subpseudomonoid of the integers is discrete by restricting the standard weak order.

We are now ready to prove:

Theorem 8.7 *Let $\mathfrak{L}$ be an infinite dimensional, strongly graded Lie algebra, graded by a pseudomonoid G. Suppose G possesses a discrete order $\preceq$.*

Write $\mathfrak{L} = \oplus_{g \in G} E_g$ as usual and let $\{e_g\}_{g \in G}$ be a basis of $\mathfrak{L}$ with the usual properties. Let Θ be the correspondence map of Proposition 6.5.

Then for every injective Lie algebra endomorphism f of $\mathfrak{L}$, we have one of the following two possibilities:
(a)

$$f(e_0) = \frac{1}{a} e_0$$

for some nonzero $a \in \mathbf{k}$ such that $aG \subseteq G$. In this case $f(\mathfrak{L}) = \Theta(aG)$. Hence if G is not self-containing, then f is onto.
(b)

$$f(e_0) = \frac{1}{a} e_0 + D$$

for some nonzero $a \in \mathbf{k}$ such that $aG \subseteq G$ and $\operatorname{Supp}(D)$ consists of elements $\prec' 0$. (Here $\preceq'$ is either equal to $\preceq$, or is $\preceq$ reversed.) Furthermore, there is $\preceq'$-minimal $I \in \operatorname{Supp}(D)$ such that $I \preceq' ag$ for all $g \in G$.

In the situation of (b), if G is not self-containing, then I is actually a minimum element of $(G, \preceq')$, and

$$f(e_0) = \frac{1}{a} e_0 + k' e_I.$$

Furthermore f is onto.

Proof. Let $f : \mathfrak{L} \to \mathfrak{L}$ be an injective endomorphism of Lie algebras. Then $f(\mathfrak{L})$ is an infinite dimensional Lie subalgebra of $\mathfrak{L}$.

Write $\mathfrak{L} = \oplus_{g \in G} E_g$ as in the statement of the theorem and let $\preceq$ be a discrete order on G.

Now

$$f(\mathfrak{L}) = f(M(e_0)) \subseteq M(f(e_0))$$

and

$$\operatorname{spec}(e_0) \subseteq \operatorname{spec}(f(e_0))$$

by Proposition 7.12. Thus $f(e_0) \in f(\mathfrak{L})$ is ad-diagonalizable on $f(\mathfrak{L})$. (In other words, there is a basis for $f(\mathfrak{L})$ consisting of eigenvectors of $ad(f(e_0))$.)

Using the chosen order on G, we can speak of $I = \operatorname{Init}(f(e_0))$ and $T = \operatorname{Term}(f(e_0))$ which both lie in G.

By Corollary 7.9, we conclude that if $I \neq 0$ then every eigenvector x of $ad(f(e_0))$ has $\operatorname{Init}(x) = I$. Similarly, if $T \neq 0$, then every eigenvector x of $ad(f(e_0))$ has $\operatorname{Term}(x) = T$.

Let us assume both I and T are nonzero to derive a contradiction. Let $S = \{g \in G | I \preceq g \preceq T\}$. Since $(G, \preceq)$ is discrete, S is finite. Since I, T are nonzero, we have seen that every eigenvector of $ad(f(e_0))$ will lie in $\Theta(S)$, and hence $f(\mathfrak{L}) \subseteq \Theta(S)$ which is a contradiction as $f(\mathfrak{L})$ is infinite dimensional.

So at least one of I, T is zero. By reordering G if necessary, we can assume $T = 0$. (Notice, if you reverse a discrete order by setting $x \prec' y \iff y \prec x$, you get a discrete order where T and I interchange. Also notice that this reordering will not affect the conclusion of the theorem.)

Now if $I = 0$ also then $f(e_0) = ke_0$ for nonzero $k \in \mathbf{k}$. Now by Proposition 7.12,

$$G = \mathrm{spec}(e_0) \subseteq \mathrm{spec}(f(e_0)) = \mathrm{spec}(ke_0) = kG.$$

Thus $\frac{1}{k}G \subseteq G$. Then notice that $f(E_b(e_0)) \subseteq E_b(ke_0) = E_{\frac{b}{k}}(e_0)$ for all $b \in G$ by Proposition 7.12. Since $E_{\frac{b}{k}}$ is one dimensional, we conclude that $f(E_b) = E_{\frac{b}{k}}$ for all $b \in G$ and hence that

$$f(\mathfrak{L}) = f(\oplus_{g \in G} E_g) = \oplus_{g \in G} E_{\frac{g}{k}} = \Theta(\frac{1}{k}G).$$

So in this case, we get the situation described in (a) of the theorem if we set $a = \frac{1}{k}$.

So we may now assume $I \neq 0$, and hence that $I \prec 0$.

Thus $f(e_0) = ke_0 + D$ where every element of $\mathrm{Supp}(D)$ is negative with minimum element I.

Now if x is an eigenvector of $ad(f(e_0))$ corresponding to $\mu \in \mathrm{spec}(f(e_0))$, we may write

$$x = \sum_{i=1}^{n} x_{g_i},$$

where $g_1 \prec \cdots \prec g_n \in G$ and $x_{g_i} \in E_{g_i}$ is nonzero for all $1 \leq i \leq n$.

Then a simple calculation shows that

$$[f(e_0), x] = kg_n x_{g_n} + D',$$

where $\mathrm{Supp}(D') \subseteq \{g \in G | g \prec g_n\}$. Since this must equal μx, we conclude that $kg_n = \mu$ or, in other words, $k\,\mathrm{Term}(x) = \mu$. Thus we conclude that $\mathrm{spec}(f(e_0)) \subseteq k\,\mathrm{spec}(e_0)$. However, by Proposition 7.12, it follows that $\mathrm{spec}(e_0) \subseteq \mathrm{spec}(f(e_0))$. Thus $G = \mathrm{spec}(e_0) \subseteq \mathrm{spec}(f(e_0)) \subseteq k\,\mathrm{spec}(e_0)$. Hence $\frac{1}{k}G \subseteq G$ in this case also.

Now since $I \neq 0$, every eigenvector x corresponding to μ of $f(e_0)$ has $\mathrm{Init}(x) = I$. Thus $I = \mathrm{Init}(x) \preceq \mathrm{Term}(x) = \mu/k$ and we conclude that $I \preceq \frac{g}{k}$ for all $g \in G$ since $G \subseteq \mathrm{spec}(f(e_0))$.

Now if G is not self-containing, we must have $\frac{1}{k}G = G$ and hence I is a mimimum element of G. Since $I \prec 0$, it is the unique such element. Thus since we had $f(e_0) = ke_0 + D$ where $\mathrm{Supp}(D) \subseteq \{g \in G | g \prec 0\}$, we conclude that $f(e_0) = ke_0 + k'e_I$.

Now $kI \in G$ as $\frac{1}{k}G = G$. Then by Proposition 7.12, we have $0 \neq f(e_{kI}) \in E_{kI}(f(e_0))$.

By our previous analysis, $k\,\mathrm{Term}(f(e_{kI})) = kI$ and so $\mathrm{Term}(f(e_{kI})) = I$. Since I is a minimum of $(G, \preceq)$, we conclude $f(e_{kI})$ is a nonzero multiple of e_I. Hence $e_I \in f(\mathfrak{L})$.

Since $f(e_0) = ke_0 + k'e_I$ in $f(\mathfrak{L})$, we conclude that $f(\mathfrak{L})$ contains e_0. Now by Proposition 6.5, it follows that $f(\mathfrak{L}) = \Theta(S)$ where S consists of the union of the supports of the elements in $f(\mathfrak{L})$.

However for every $g \in G$, $kg \in G$ and $\mathrm{Term}(f(e_{kg})) = g$ by an analysis similar to the one done previously. Hence $S = G$ and f is onto. Thus we are done.

□

Corollary 8.8 *Let $\mathfrak{L}$ be a strongly graded Lie algebra, graded by a discrete pseudomonoid which is not self-containing. Then every injective Lie algebra endomorphism of $\mathfrak{L}$ is an automorphism.*

If f is any nonzero Lie algebra endomorphism of the classical Witt algebra, then f is an automorphism, and furthermore

$$f(x\partial) = (x + b)\partial$$

for some $b \in \mathbf{k}$. Thus the Jacobian conjecture holds for the classical Witt algebra.

Proof. The first part follows immediately from Theorem 8.7.

By Example 5.7, the classical Witt algebra is a strongly graded Lie algebra graded by the pseudomonoid $G = \{-1, 0, 1, \dots\}$, which is obviously discrete and is not self-containing by Lemma 8.4. We have already seen that this Lie algebra is simple, hence any nonzero Lie algebra endomorphism f is injective and hence an automorphism by Theorem 8.7.

Furthermore, in the strong grading of the classical Witt algebra, we can take $x\partial = e_0$ and $x^n\partial \in E_{n-1}$ for all $n \in \mathbb{N}$.

Notice further that if $aG \subseteq G$, in fact $a = 1$ as we saw in the proof of Lemma 8.4. Thus applying Theorem 8.7 again and noting that we must have $I = -1$ if we are in situation (b), we conclude furthermore that

$$f(x\partial) = (x + b)\partial$$

for some $b \in \mathbf{k}$. □

Corollary 8.9 *If f is a nonzero Lie algebra endomorphism of the centerless Virasoro algebra then f is injective and*

$$f(x\partial) = \frac{1}{a}x\partial$$

for some nonzero integer a.

However, the Jacobian conjecture is false for this Lie algebra. Thus there exist injective Lie algebra endomorphisms of the centerless Virasoro algebra which are not automorphisms.

Proof. By Example 5.8, the centerless Virasoro algebra is strongly graded by the pseudomonoid $G = \mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$, with basis $e_n = x^{n+1}\partial \in E_n$ for all $n \in \mathbb{Z}$. G is obviously discrete.

Let f be a nonzero Lie algebra endomorphism. Since the centerless Virasoro algebra is simple, f is injective. It is easy to see that $a\mathbb{Z} \subseteq \mathbb{Z}$ if and only if a is an integer. Also if we use the standard order of $\mathbb{Z}$, then there is no I as in situation (b) of Theorem 8.7, and so we immediately conclude from the same theorem that:

$$f(x\partial) = \frac{1}{a}x\partial$$

for some nonzero integer a and Image$(f) = \Theta(a\mathbb{Z})$.

We will now construct such a Lie algebra endomorphism for every nonzero intger a. Thus for $a \neq \pm 1$, we obtain injective Lie algebra endomorphisms which are not onto.

Define $f_a(e_n) = a^{-(n+1)} e_{an}$ for all $n \in \mathbb{Z}$. Certainly this defines a vector space endomorphism which is not onto if $a \neq \pm 1$.

We calculate

$$\begin{aligned}
[f_a(e_n), f_a(e_m)] &= a^{-(n+m+2)}[e_{an}, e_{am}] \\
&= (am - an)a^{-(n+m+2)} e_{a(n+m)} \\
&= (m-n)a^{-(n+m+1)} e_{a(n+m)} \\
&= f_a((m-n)e_{n+m}) \\
&= f_a([e_n, e_m]).
\end{aligned}$$

Hence f is a homomorphism of Lie algebras and we are done. □

This concludes our initial study of generalized Witt algebras. One sees that for this family of self-centralizing Lie algebras, spectral analysis provides a powerful tool to answer basic questions locally. (On $M(\alpha)$ for nonzero $\alpha \in \mathfrak{L}$.)

We found this extremely useful in the case where $\mathfrak{L} = M(\alpha)$ for some nonzero α, but it should be possible to push these results to the more general case by patching together the local spectra to get some sort of global scheme.

References

[1] Amayo, R., Stewart, I.: *Infinite-dimensional Lie Algebras,* 115-120, Noordhoff Int. Publishing (1974).

[2] Coutinho, S.: *A Primer of Algebraic D-modules,* London Math. Soc. Student Texts **33**, Cambridge Univ. Press, 1995.

[3] Doković, D., Zhao, K.: *Derivations, isomorphisms and second cohomology of a generalized Witt algebra,* Trans. A.M.S. **350**, 2-7 (1998).

[4] Humphreys, J.: *Introduction to Lie Algebras and Representation Theory,* Springer-Verlag, **G.T.M. 9**, 1-21, (1987).

[5] Jacobson, N.: *Lie Algebras,* Dover Publications, 1979.

[6] Kac, V.: *Description of Filtered Lie Algebra with which Graded Lie algebras of Cartan type are Associated,* Izv. Akad. Nauk SSSR, Ser. Mat. Tom **38**, 832-834 (1974).

[7] Kaplansky, I.: *The Virasoro algebra,* Comm. in Mathematical Physics, **86**, 49-52 (1982).

[8] Kawamoto, N.: *Generalizations of Witt algebras over a field of characteristic zero,* Hiroshima Math. J.,**16**, 417-426 (1986).

[9] Lang, S.: *Algebra,* 3rd ed., Addison-Wesley Pub. Co., 1993.

[10] Nam, K.: *Generalized W and H type Lie algebras,* Algebra Colloquium, Springer Verlag, 6:3, 329-340 (1999).

[11] Robinson, D.: *A Course in the Theory of Groups,* Springer-Verlag, **G.T.M. 80**, 95-98 (1996).

[12] Rudakov, A.: *Groups of Automorphisms of Infinite-Dimensional Simple Lie Algebras,* Math. USSR-Izvestija, **3**, 836-837 (1969).

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Received: 16.03.2010

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