Turk J Math
35 (2011) , 443 – 456.
© TÜBİTAK
doi:10.3906/mat-1004-233

Approximation by complex potentials generated by the Gamma function

Sorin G. Gal

Abstract

In this paper we find the exact orders of approximation of analytic functions by the complex versions of several potentials (including the Flett potential) generated by the Gamma function and by some singular integrals.

Key words and phrases: Complex potentials, singular integrals, order of approximation

1. Introduction

In the real case, the approximation properties of the potentials such as those of Riesz, Bessel, generalized Riesz, generalized Bessel and Flett have been studied by many authors, see e.g. Kurokawa [5], Gadjiev-Aral-Aliev [3], Uyhan-Gadjiev-Aliev [7], Sezer [6], Aliev-Gadjiev-Aral [1] and their references.

Let us recall that in the real case, the classical Bessel type parabolic potential is defined for any $f \in L^p(\mathbb{R}^2)$, $1 \le p < \infty$, by

$$B^\alpha(f)(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \left[\int_{-\infty}^\infty \tau^{(\alpha/2)-1} e^{-\tau} W(y,\tau) f(x-y, t-\tau) dy \right] d\tau,$$

where $\alpha > 0$, $\Gamma(\alpha)$ is the Gamma function and $W(y,\tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-y^2/(4\tau)}$ is the Gauss-Weierstrass kernel.

It is known that formally we can write

$$B^\alpha(f)(x,t) = \left(I - \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t} \right)^{-\alpha/2} f(x,t),$$

and the following convergence properties hold (see Uyhan-Gadjiev-Aliev [7]):

(i) if $f \in L^p(\mathbb{R}^2)$, $1 \le p < \infty$, *is continuous at* $(x,t) \in \mathbb{R}^2$ *then* $\lim_{\alpha \to 0^+} B^\alpha(f)(x,t) = f(x,t)$;

(ii) if $f \in L^p(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$, *where* $C_0(\mathbb{R}^2)$ *denotes the space of all continuous functions on* $\mathbb{R}^2$ *vanishing at infinity, then* $\lim_{\alpha \to 0^+} B^\alpha(f) = f$ *uniformly on* $\mathbb{R}^2$;

(iii) if $f \in L^p(\mathbb{R}^2) \cap C(\mathbb{R}^2)$, *where* $C(\mathbb{R}^2)$ *denotes the space of all continuous functions on* $\mathbb{R}^2$, *then* $\lim_{\alpha \to 0^+} B^\alpha(f) = f$ *uniformly on every compact* $K \subset \mathbb{R}^2$;

2000 *AMS Mathematics Subject Classification:* 30E10, 41A35, 41A25.

(iv) in addition, for f in some suitable Lipschitz-type classes, quantitative upper estimates of order $O(\alpha)$ are obtained.

Also, let us recall that the classical Flett potential is defined for any $f \in L^p(\mathbb{R})$ by (see Flett [2])

$$F^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} Q_t(f)(x) dt,$$

where $Q_t(f)(x) = \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{f(x-u)}{u^2+t^2} du$ is the classical Poisson-Cauchy singular integral.

It is known that the following convergence properties hold (see Sezer [6]):

(i) if $f \in L^p(\mathbb{R}) \cap C_0(\mathbb{R})$, then $\lim_{\alpha \to 0^+} F^{\alpha}(f) = f$ uniformly on $\mathbb{R}$;

(ii) for f in some suitable Lipschitz-type classes, quantitative upper estimates of order $O(\alpha)$ are obtained.

Remark. The form of the Flett's potential suggests to study the approximation properties as $\alpha \to 0^+$ of new potentials, as

$$F_U^{\alpha}(f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} U_t(f)(x) dt, x \in \mathbb{R}, \tag{1}$$

where $U_t(f)(x)$ can be any from $P_t(f)(x) = \frac{1}{2t} \int_{-\infty}^{+\infty} f(x-u) e^{-|u|/t} du$ (the Picard singular integral), $R_t(f)(x) = \frac{2t^3}{\pi} \int_{-\infty}^{+\infty} \frac{f(x-u)}{(u^2+t^2)^2} du$ (a Poisson-Cauchy-type singular integral) and $W_t^*(f)(x) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(x-u) e^{-u^2/t} du$ (the Gauss-Weierstrass singular integral).

In this paper, the exact order of approximation by the complex versions of the potentials $F_U^{\alpha}(f)(x)$ given by (1) (including the Flett potential) is obtained. The complex versions are obtained from their real versions by replacing in the formula of any $U_t(f(x)$, the translation $x - y$ by the rotation ze^{-iy}, where $z = re^{ix} \in \mathbb{C}$, that is obtaining the form

$$F_U^{\alpha}(f)(z) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} U_t(f)(z) dt, \tag{2}$$

where

$$U_t(f)(z) = Q_t(f)(z) = \frac{t}{\pi} \int_{-\infty}^{\infty} \frac{f(ze^{-iu})}{u^2+t^2} du,$$

or

$$U_t(f)(z) = P_t(f)(z) = \frac{1}{2t} \int_{-\infty}^{+\infty} f(ze^{-iu}) e^{-|u|/t} du,$$

or

$$U_t(f)(z) = R_t(f)(z) = \frac{2t^3}{\pi} \int_{-\infty}^{+\infty} \frac{f(ze^{-iu})}{(u^2+t^2)^2} du,$$

or

$$U_t(f)(z) = W_t^*(f)(z) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} f(ze^{-iu}) e^{-u^2/t} du.$$

2. Main results

Note that in order to exist $F_U^\alpha(f)(z)$ in (2) for all $|z|<R$, it is enough to suppose that the function $f(z)$ is analytic in $|z|<R$, with $R>1$.

For $R>0$ let us denote $\mathbb{D}_R=\{z\in\mathbb{C};|z|<R\}$.

The main result is the following.

Theorem 2.1. *Let us suppose that $\alpha>0$ and that $f:\mathbb{D}_R\to\mathbb{C}$, with $R>1$, is analytic in $\mathbb{D}_R$, that is $f(z)=\sum_{k=0}^{\infty}a_k z^k$, for all $z\in\mathbb{D}_R$.*

(i) For $U_t(f)(z)=\frac{t}{\pi}\int_{-\infty}^{\infty}\frac{f(ze^{-iu})}{u^2+t^2}du$ we have that $F_U^\alpha(f)(z)$ given by (2) is analytic in $\mathbb{D}_R$ and we can write

$$F_U^\alpha(f)(z)=\sum_{k=0}^{\infty}a_k\cdot\frac{1}{(k+1)^\alpha}\cdot z^k, z\in\mathbb{D}_R.$$

Also, if f is not constant function for $q=0$, and not a polynomial of degree $\le q-1$ for $q\in\mathbb{N}$, then for all $1\le r<r_1<R$, $q\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,1]$, we have

$$\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r\sim\alpha,$$

where $\|f\|_r=\sup\{|f(z)|;|z|\le r\}$ and the constants in the equivalence depend only on f, q, r and r_1.

(ii) For $U_t(f)(z)=\frac{1}{2t}\int_{-\infty}^{+\infty}f(ze^{-iu})e^{-|u|/t}du$ we have that $F_U^\alpha(f)(z)$ given by (2) is analytic in $\mathbb{D}_R$ and we can write

$$F_U^\alpha(f)(z)=\sum_{k=0}^{\infty}a_k b_{k,\alpha}z^k, z\in\mathbb{D}_R,$$

where $b_{k,\alpha}=\frac{1}{\Gamma(\alpha)}\int_0^\infty\frac{t^{\alpha-1}e^{-t}}{1+t^2k^2}dt$.

Also, if f is not constant function for $q=0$, and not a polynomial of degree $\le q-1$ for $q\in\mathbb{N}$, then for all $1\le r<r_1<R$, $q\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,1]$ we have

$$\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r\sim\alpha,$$

where the constants in the equivalence depend only on f, q, r and r_1.

(iii) For $U_t(f)(z)=\frac{2t^3}{\pi}\int_{-\infty}^{+\infty}\frac{f(ze^{-iu})}{(u^2+t^2)^2}du$ we have that $F_U^\alpha(f)(z)$ given by (2) is analytic in $\mathbb{D}_R$ and we can write

$$F_U^\alpha(f)(z)=\sum_{k=0}^{\infty}a_k\frac{1}{(k+1)^{\alpha+1}}[k(\alpha+1)+1]\cdot z^k, z\in\mathbb{D}_R.$$

Also, there exists $\alpha_0\in(0,1]$ (absolute constant) such that if f is not constant function for $q=0$, and not a polynomial of degree $\le q-1$ for $q\in\mathbb{N}$, then for all $1\le r<r_1<R$, $q\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,\alpha_0]$ we have

$$\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r\sim\alpha,$$

where the constants in the equivalence depend only on f, q, r and r_1.

(iv) For $U_t(f)(z)=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{+\infty}f(ze^{-iu})e^{-u^2/t}\,du$ *we have that* $F_U^{\alpha}(f)(z)$ *given by* (2) *is analytic in* $\mathbb{D}_R$ *and we can write*

$$F_U^{\alpha}(f)(z)=\sum_{k=0}^{\infty}a_k\cdot\frac{1}{(1+k^2/4)^{\alpha+1}}\cdot z^k, z\in\mathbb{D}_R.$$

Also, if f *is not constant function for* $q=0$, *and not a polynomial of degree* $\le q-1$ *for* $q\in\mathbb{N}$, *then for all* $1\le r<r_1<R$, $q\in\mathbb{N}\cup\{0\}$, $\alpha\in(0,1]$ *we have*

$$\|[F_U^{\alpha}(f)]^{(q)}-f^{(q)}\|_r\sim\alpha,$$

where the constants in the equivalence depend only on f, q, r *and* r_1.

Proof. (i) By Gal [4], p. 213, Theorem 3.2.5, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z)=\sum_{k=0}^{\infty}a_ke^{-kt}z^k,\ \text{for all } |z|<R \text{ and } t\ge 0.$$

Since $|\sum_{k=0}^{\infty}a_ke^{-kt}z^k|\le\sum_{k=0}^{\infty}|a_k|\cdot|z|^k<\infty$, this implies that for fixed $|z|<R$, the series in t, $\sum_{k=0}^{\infty}a_ke^{-kt}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$F_U^{\alpha}(f)(z)=\sum_{k=0}^{\infty}a_kz^k\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-(k+1)t}dt,$$

where by making use of the change of variable $(k+1)t=s$, we easily get that $\int_0^{\infty}t^{\alpha-1}e^{-(k+1)t}dt=\frac{\Gamma(\alpha)}{(k+1)^{\alpha}}$.

In other order of ideas, we easily can write

$$F_U^{\alpha}(f)(z)-f(z)=\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty}t^{\alpha-1}e^{-t}[U_t(f)(z)-f(z)]dt, \tag{3}$$

which together with the estimate $|U_t(f)(z)-f(z)|\le C_r(f)t$ in Gal [4], p. 213, Theorem 3.2.5, (iii), implies

$$|F_U^{\alpha}(f)(z)-f(z)|\le\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty}t^{\alpha-1}e^{-t}|U_t(f)(z)-f(z)|dt$$

$$\le C_r(f)\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty}t^{\alpha}e^{-t}dt=C_r(f)\cdot\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)}=C_r(f)\alpha, \tag{4}$$

for all $|z|\le r$, where $C_r(f)>0$ is independent of z (and α) but depends on f and r.

Now, let $q\in\mathbb{N}\cup\{0\}$ and $1\le r<r_1<R$. Denoting by γ the circle of radius r_1 and center 0, since for any $|z|\le r$ and $v\in\gamma$ we have $|v-z|\ge r_1-r$, by using the Cauchy's formula, for all $|z|\le r$ and $\alpha>0$ we get

$$|[F_U^{\alpha}(f)]^{(q)}(z)-f^{(q)}(z)|=\frac{q!}{2\pi}\left|\int_{\gamma}\frac{F_U^{\alpha}(f)(z)-f(z)}{(v-z)^{q+1}}dv\right|$$

$$\le C_{r_1}(f)\alpha\cdot\frac{q}{2\pi}\cdot\frac{2\pi r_1}{(r_1-r)^{q+1}}, \tag{5}$$

which proves the upper estimate

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \leq C^* \alpha, \tag{6}$$

with C^* depending only on f, q, r and r_1.

It remains to prove the lower estimate. For this purpose, reasoning exactly as in the proof of Theorem 3.2.5, at pages 218-219 in the book Gal [4], for $z = re^{i\varphi}$ and $p \in \mathbb{N} \cup \{0\}$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p[1 - e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$ and then integrating with respect to t, it follows

$$I := \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}e^{-t}dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}[1 - e^{-(q+p)t}]dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p\left[1 - \frac{1}{(q+p+1)^{\alpha}}\right],$$

because taking into account that by making use of the change of variable $(q+p+1)t = s$, we easily get that

$$\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}[1 - e^{-(q+p)t}]dt = 1 - \frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-(q+p+1)t}dt$$

$$= 1 - \frac{1}{(q+p+1)^{\alpha}}.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p\left[1 - \frac{1}{(q+p+1)^{\alpha}}\right].$$

Since

$$\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt = f^{(q)}(z) - [F_U^{\alpha}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z) - (F_U^{\alpha}(f))^{(q)}(z)\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p\left[1 - \frac{1}{(q+p+1)^{\alpha}}\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p\left[1-\frac{1}{(q+p+1)^\alpha}\right]\le \|f^{(q)}-(F_U^\alpha(f))^{(q)}\|_r.$$

First take $q=0$. From the previous inequality we immediately obtain

$$|a_p|r^p\left(1-\frac{1}{(p+1)^\alpha}\right)\le \|f-F_U^\alpha(f)\|_r.$$

In what follows, denoting $V_\alpha=\inf_{p\ge 1}\left(1-\frac{1}{(p+1)^\alpha}\right)$, we clearly get $V_\alpha=1-\frac{1}{2^\alpha}$.

Denoting $g(x)=2^{-x}$, by the mean value theorem, there exists $\xi\in(0,\alpha)\subset(0,1]$ such that

$$V_\alpha=g(0)-g(\alpha)=-\alpha g'(\xi)=\alpha\cdot 2^{-\xi}ln(2)\ge \alpha 2^{-\alpha}ln(2)\ge \alpha 2^{-1}ln(2),$$

which immediately implies

$$\alpha\cdot\frac{ln(2)}{2}\cdot r^p\cdot|a_p|\le\|f-F_U^\alpha(f)\|_r,$$

that is

$$\frac{ln(2)}{2}\cdot r^p\cdot|a_p|\le\frac{\|f-F_U^\alpha(f)\|_r}{\alpha},$$

for all $p\ge 1$ and $\alpha\in(0,1]$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty}\alpha_k=0$ and such that $\lim_{k\to\infty}\frac{\|F_U^\alpha(f)-f\|_r}{\alpha_k}=0$, then $a_p=0$ for all $p\ge 1$, that is f is constant on $\overline{\mathbb{D}}_r$.

Therefore, if f is not a constant function, then $\inf_{\alpha\in(0,1]}\frac{\|F_U^\alpha(f)-f\|_r}{\alpha}>0$, which implies that there exists a constant $C_r(f)>0$ such that $\frac{\|F_U^\alpha(f)-f\|_r}{\alpha}\ge C_r(f)$, for all $\alpha\in(0,1]$, that is

$$\|F_U^\alpha(f)-f\|_r\ge C_r(f)\alpha,\text{ for all }\alpha\in(0,1].$$

Now, consider $q\ge 1$ and denote $V_{q,\alpha}=\inf_{p\ge 0}(1-\frac{1}{(q+p+1)^\alpha})$. Evidently that we have $V_{q,\alpha}\ge\inf_{p\ge 1}(1-\frac{1}{(p+1)^\alpha})\ge\alpha\cdot\frac{ln(2)}{2}$.

Reasoning as in the case of $q=0$, we obtain

$$\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha}\ge|a_{q+p}|\frac{(q+p)!}{p!}\cdot\frac{ln(2)}{2}\cdot r^p,$$

for all $p\ge 0$ and $\alpha\in(0,1]$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty}\alpha_k=0$ and such that $\lim_{k\to\infty}\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha_k}=0$ then $a_{q+p}=0$ for all $p\ge 0$, that is f is a polynomial of degree $\le q-1$ on $\overline{\mathbb{D}}_r$.

Therefore, because by hypothesis f is not a polynomial of degree $\le q-1$, we obtain $\inf_{\alpha\in(0,1]}\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha}$ >0, which implies that there exists a constant $C_{r,q}(f)>0$ such that $\frac{\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r}{\alpha}\ge C_{r,q}(f)$, for all $\alpha\in(0,1]$, that is

$$\|[F_U^\alpha(f)]^{(q)}-f^{(q)}\|_r\ge C_{r,q}(f)\alpha,\text{ for all }\alpha\in(0,1].$$

(ii) By Gal [4], p. 206, Theorem 3.2.1, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z)=\sum_{k=0}^{\infty}\frac{a_k}{1+t^2k^2}z^k,\ \text{ for all } |z|<R \text{ and } t\ge 0.$$

Since $|\sum_{k=0}^{\infty}\frac{a_k}{1+t^2k^2}z^k|\le\sum_{k=0}^{\infty}|a_k|\cdot|z|^k<\infty$, this implies that for fixed $|z|<R$, the series in t, $\sum_{k=0}^{\infty}\frac{a_k}{1+t^2k^2}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$F_U^{\alpha}(f)(z)=\sum_{k=0}^{\infty}a_kz^k\frac{1}{\Gamma(\alpha)}\int_0^{\infty}\frac{t^{\alpha-1}e^{-t}}{1+t^2k^2}dt.$$

In other order of ideas, (3) together with the estimate $|U_t(f)(z)-f(z)|\le C_r(f)t^2$ in Gal [4], p. 207, Theorem 3.2.1, (iv), implies

$$|F_U^{\alpha}(f)(z)-f(z)|\le\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty}t^{\alpha-1}e^{-t}|U_t(f)(z)-f(z)|dt$$

$$\le C_r(f)\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty}t^{\alpha+1}e^{-t}dt=C_r(f)\cdot\frac{\Gamma(\alpha+2)}{\Gamma(\alpha)}=C_r(f)\alpha(\alpha+1)\le 2C_r(f)\alpha, \tag{7}$$

for all $|z|\le r$, where $C_r(f)>0$ is independent of z (and α) but depends on f and r.

Now, reasoning formally exactly as in the proof at the above point (i), by (5) we get again the upper estimate in (6).

It remains to prove the lower estimate. For this purpose, starting exactly as in the proof of the above point (i), we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$=a_{q+p}(q+p)(q+p-1)...(p+1)r^p\cdot\frac{t^2(q+p)^2}{1+t^2(q+p)^2}.$$

Multiplying above with $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$ an then integrating with respect to t, it follows

$$I:=\frac{1}{\Gamma(\alpha)}\cdot\int_0^{\infty}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}e^{-t}dt$$

$$=a_{q+p}(q+p)(q+p-1)...(p+1)r^p\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt\right]d\varphi\right|$$

$$=|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right].$$

Since

$$\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z)-[U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt=f^{(q)}(z)-[F_U^{\alpha}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-ip\varphi}\left[f^{(q)}(z)-(F_U^{\alpha}(f))^{(q)}(z)\right]d\varphi\right|$$

$$=|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2}\right]dt\right]$$

$$\leq \|f^{(q)}-(F_U^{\alpha}(f))^{(q)}\|_r.$$

First take $q=0$. From the previous inequality we immediately obtain

$$|a_p|r^p\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2p^2}{1+t^2p^2}\right]dt\right)\leq \|f-F_U^{\alpha}(f)\|_r.$$

In what follows, denoting $V_\alpha=\inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2p^2}{1+t^2p^2}\right]dt\right)$, we clearly get

$$V_\alpha=\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha-1}e^{-t}\left[\frac{t^2}{1+t^2}\right]dt.$$

Taking into account that $1+t^2\leq 2e^t$ for all $t\geq 0$, we obtain

$$V_\alpha\geq\frac{1}{\Gamma(\alpha)}\int_0^{\infty}t^{\alpha+1}e^{-2t}dt=\frac{\Gamma(\alpha+2)}{2^{\alpha+2}\Gamma(\alpha)}=\frac{\alpha}{4}\cdot\frac{\alpha+1}{2^{\alpha}}\geq C\alpha,$$

since the function $f(x)=\frac{x+1}{2^x}$ is strictly positive and continuous in $[0,1]$.

This immediately implies

$$C\cdot r^p\cdot|a_p|\leq\frac{\|f-F_U^{\alpha}(f)\|_r}{\alpha},$$

for all $p\geq 1$ and $\alpha\in(0,1]$.

Now, if would exist a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty}\alpha_k=0$ and such that $\lim_{k\to\infty}\frac{\|F_U^{\alpha}(f)-f\|_r}{\alpha_k}=0$, then $a_p=0$ for all $p\geq 1$, that is f would be constant on $\overline{\mathbb{D}}_r$.

Therefore, if f is not a constant function, then $\inf_{\alpha\in(0,1]}\frac{\|F_U^{\alpha}(f)-f\|_r}{\alpha}>0$, which implies that there exists a constant $C_r(f)>0$ such that $\frac{\|F_U^{\alpha}(f)-f\|_r}{\alpha}\geq C_r(f)$, for all $\alpha\in(0,1]$, that is

$$\|F_U^{\alpha}(f)-f\|_r\geq C_r(f)\alpha,\ \text{for all }\alpha\in(0,1].$$

Now, consider $q \geq 1$ and denote

$$V_{q,\alpha} = \inf_{p \geq 0} \left(\frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} \left[\frac{t^2(q+p)^2}{1+t^2(q+p)^2} \right] dt \right).$$

Evidently that we have $V_{q,\alpha} \geq \inf_{p\geq 1} \left(\frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-t} \left[\frac{t^2p^2}{1+t^2p^2} \right] dt \right) \geq \alpha \cdot C$.

Reasoning as in the case of $q = 0$, we obtain

$$\frac{\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r}{\alpha} \geq |a_{q+p}| \frac{(q+p)!}{p!} \cdot C \cdot r^p,$$

for all $p \geq 0$ and $\alpha \in (0,1]$.

This implies that if there exists a subsequence $(\alpha_k)_k$ in $(0,1]$ with $\lim_{k\to\infty} \alpha_k = 0$ and such that $\lim_{k\to\infty} \frac{\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r}{\alpha_k} = 0$ then $a_{q+p} = 0$ for all $p \geq 0$, that is f is a polynomial of degree $\leq q-1$ on $\overline{\mathbb{D}}_r$.

Therefore, because by hypothesis f is not a polynomial of degree $\leq q-1$, we obtain $\inf_{\alpha\in(0,1]} \frac{\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r}{\alpha} > 0$, which implies that there exists a constant $C_{r,q}(f) > 0$ such that $\frac{\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r}{\alpha} \geq C_{r,q}(f)$, for all $\alpha \in (0,1]$, that is

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \geq C_{r,q}(f)\alpha, \text{ for all } \alpha \in (0,1].$$

(iii) By Gal [4], p. 213, Theorem 3.2.5, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^{\infty} a_k e^{-kt}(1+kt)z^k| \leq 2\sum_{k=0}^{\infty} |a_k| \cdot |z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^{\infty} a_k(1+kt)e^{-kt}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k z^k \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1}(1+kt)e^{-(k+1)t}dt,$$

where by making use of the change of variable $(k+1)t = s$, we easily get that $\int_0^{\infty} t^{\alpha-1}e^{-(k+1)t}dt = \frac{\Gamma(\alpha)}{(k+1)^{\alpha}}$ and therefore we immediately obtain

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \frac{1}{(k+1)^{\alpha+1}}[k(\alpha+1)+1]z^k.$$

In other order of ideas, (3) together with the estimate $|U_t(f)(z) - f(z)| \leq C_r(f)t^2$ in Gal [4], p. 213-214, Theorem 3.2.5, (iv), implies the upper estimate in (7).

Now, reasoning exactly as in the proof of the above point (i), by (5)and (7) we again get the upper estimate in (6).

To prove the lower estimate, we reason exactly as in the proof of (i). We get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p[1-(1+(q+p)t)e^{-(q+p)t}].$$

Multiplying above with $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$ and then integrating with respect to t, it follows

$$I := \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty} \left\{ \frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi \right\} t^{\alpha-1}e^{-t}dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right].$$

Since

$$\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt = f^{(q)}(z) - [F_U^{\alpha}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[f^{(q)}(z) - (F_U^{\alpha}(f))^{(q)}(z)\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot \left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]dt\right] \le \|f^{(q)} - (F_U^{\alpha}(f))^{(q)}\|_r.$$

First take $q = 0$. From the previous inequality we immediately obtain

$$|a_p|r^p\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+pt)e^{-pt}\right]dt\right) \le \|f - F_U^{\alpha}(f)\|_r.$$

In what follows, denoting

$$V_{\alpha} = \inf_{p\ge 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1-(1+pt)e^{-pt}\right]dt\right),$$

by simple calculation we get

$$V_\alpha = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-(1+t)e^{-t}\right]dt = 1-\frac{1}{2^\alpha}-\frac{\alpha}{2^{\alpha+1}}.$$

But there exists $\alpha_0 \in (0,1]$, such that if C is an absolute constant with $0 < C < ln(2) - \frac{1}{2}$, then we have

$$1-\frac{1}{2^\alpha}-\frac{\alpha}{2^{\alpha+1}} \geq C\alpha, \text{ for all } \alpha \in [0,\alpha_0].$$

Indeed, denoting $g(\alpha) = 1-\frac{1}{2^\alpha}-\frac{\alpha}{2^{\alpha+1}} - C\alpha$, we have $g(0)=0$ and $g'(\alpha) = 2^{-\alpha}ln(2) - \frac{1}{2^{\alpha+1}} + \frac{\alpha(\alpha+1)ln(2)}{2^{\alpha+1}} - C$, which implies $g'(0) = ln(2) - \frac{1}{2} - C > 0$. Since $g'(\alpha)$ obviously is continuous with respect to α, there exists $\alpha_0 > 0$ such that $g'(\alpha) > 0$ for all $\alpha \in [0,\alpha_0]$, that is $V_\alpha \geq C\alpha$, for all $\alpha \in [0,\alpha_0]$.

This immediately implies

$$C \cdot r^p \cdot |a_p| \leq \frac{\|f - F_U^\alpha(f)\|_r}{\alpha},$$

for all $p \geq 1$ and $\alpha \in (0,\alpha_0]$.

Now, if would exist a subsequence $(\alpha_k)_k$ in $(0,\alpha_0]$ with $\lim_{k\to\infty}\alpha_k = 0$ and such that $\lim_{k\to\infty}\frac{\|F_U^{\alpha}(f)-f\|_r}{\alpha_k} = 0$, then $a_p = 0$ for all $p \geq 1$, that is f would be constant on $\overline{\mathbb{D}}_r$.

Therefore, if f is not a constant function, then $\inf_{\alpha\in(0,\alpha_0]}\frac{\|F_U^\alpha(f)-f\|_r}{\alpha} > 0$, which implies that there exists a constant $C_r(f) > 0$ such that $\frac{\|F_U^\alpha(f)-f\|_r}{\alpha} \geq C_r(f)$, for all $\alpha \in (0,\alpha_0]$, that is

$$\|F_U^\alpha(f) - f\|_r \geq C_r(f)\alpha, \text{ for all } \alpha \in (0,\alpha_0].$$

Now, consider $q \geq 1$ and denote

$$V_{q,\alpha} = \inf_{p\geq 0}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-(1+(q+p)t)e^{-(q+p)t}\right]\right).$$

Evidently that we have $V_{q,\alpha} \geq \inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-(1+pt)e^{-pt}\right]dt\right) \geq \alpha\cdot C$, for $C \in (0, ln(2)-1/2)$ and $\alpha \in [0,\alpha_0]$.

Reasoning exactly as in the case of $q = 0$ and as in the previous case (ii), we easily obtain that because by hypothesis f is not a polynomial of degree $\leq q-1$, there exists a constant $C_{r,q}(f) > 0$ such that

$$\|[F_U^\alpha(f)]^{(q)} - f^{(q)}\|_r \geq C_{r,q}(f)\alpha, \text{ for all } \alpha \in (0,\alpha_0].$$

(iv) By Gal [4], p. 223, Theorem 3.2.8, (i), $U_t(f)(z)$ is analytic (as function of z) in $\mathbb{D}_R$ and we can write

$$U_t(f)(z) = \sum_{k=0}^\infty a_k e^{-k^2t/4}z^k, \text{ for all } |z| < R \text{ and } t \geq 0.$$

Since $|\sum_{k=0}^\infty a_k e^{-k^2t/4}z^k| \leq \sum_{k=0}^\infty |a_k|\cdot|z|^k < \infty$, this implies that for fixed $|z| < R$, the series in t, $\sum_{k=0}^\infty a_k e^{-k^2t/4}z^k$ is uniformly convergent on $[0,\infty)$, and therefore we immediately can write

$$F_U^\alpha(f)(z) = \sum_{k=0}^\infty a_k z^k \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-(1+k^2/4)t}dt,$$

where by making use of the change of variable $(1+k^2/4)t = s$, we easily get that $\int_0^{\infty} t^{\alpha-1}e^{-(1+k^2/4)t}dt = \frac{\Gamma(\alpha)}{(1+k^2/4)^{\alpha}}$ and therefore we immediately obtain

$$F_U^{\alpha}(f)(z) = \sum_{k=0}^{\infty} a_k \frac{1}{(1+k^2/4)^{\alpha+1}} z^k.$$

In other order of ideas, (3) together with the estimate $|U_t(f)(z) - f(z)| \le C_r(f)t$ in Gal [4], p. 224, Theorem 3.2.8, (iv), implies the upper estimate in (4).

Now, reasoning exactly as in the proof of (i), by (5) and (4) we get again the upper estimate in (6).

To prove the lower estimate, again we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p[1 - e^{-(q+p)^2t/4}].$$

Multiplying above with $\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$ an then integrating with respect to t, it follows

$$I := \frac{1}{\Gamma(\alpha)} \cdot \int_0^{\infty}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]e^{-ip\varphi}d\varphi\right\}t^{\alpha-1}e^{-t}dt$$

$$= a_{q+p}(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1 - e^{-(q+p)^2t/4}\right]dt.$$

Applying the Fubini's result to the double integral I and then passing to modulus, we easily obtain

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1 - e^{-(q+p)^2t/4}\right]dt\right].$$

Since

$$\frac{1}{\Gamma(\alpha)}\int_0^{\infty}[f^{(q)}(z) - [U_t(f)]^{(q)}(z)]t^{\alpha-1}e^{-t}dt = f^{(q)}(z) - [F_U^{\alpha}(f)]^{(q)}(z),$$

the previous equality immediately implies

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip\varphi}\left[f^{(q)}(z) - (F_U^{\alpha}(f))^{(q)}(z)\right]d\varphi\right|$$

$$= |a_{q+p}|(q+p)(q+p-1)...(p+1)r^p$$

$$\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^{\infty} t^{\alpha-1}e^{-t}\left[1 - e^{-(q+p)^2t/4}\right]dt\right]$$

and

$$|a_{q+p}|(q+p)(q+p-1)...(p+1)r^{p}$$

$$\cdot\left[\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-e^{-(q+p)^2t/4}\right]dt\right]\leq\|f^{(q)}-(F_U^\alpha(f))^{(q)}\|_r.$$

First take $q=0$. From the previous inequality we immediately obtain

$$|a_p|r^p\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-e^{-p^2t/4}\right]dt\right)\leq\|f-F_U^\alpha(f)\|_r.$$

In what follows, denoting

$$V_\alpha=\inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-e^{-p^2t/4}\right]dt\right),$$

by simple calculation we get

$$V_\alpha=\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-e^{-t/4}\right]dt=1-\left(\frac{4}{5}\right)^\alpha.$$

Denoting $g(x)=\left(\frac{4}{5}\right)^x$, by the mean value theorem, there exists $\xi\in(0,\alpha)\subset(0,1]$ such that

$$V_\alpha=g(0)-g(\alpha)=-\alpha g'(\xi)=\alpha\cdot\left(\frac{4}{5}\right)^\xi ln\left(\frac{4}{5}\right)\geq\alpha\left(\frac{4}{5}\right)^\alpha ln\left(\frac{4}{5}\right)$$

$$\geq\alpha\left(\frac{4}{5}\right)ln\left(\frac{4}{5}\right),$$

which immediately implies

$$\left(\frac{4}{5}\right)ln\left(\frac{4}{5}\right)\cdot r^p\cdot|a_p|\leq\frac{\|f-F_U^\alpha(f)\|_r}{\alpha},$$

for all $p\geq 1$ and $\alpha\in(0,1]$.

Reasoning now exactly as in the proof of the above point (i), we similarly get that if f is not a constant function, then there exists a constant $C_r(f)>0$ such that

$$\|F_U^\alpha(f)-f\|_r\geq C_r(f)\alpha,\text{ for all }\alpha\in(0,1].$$

Now, consider $q\geq 1$ and denote

$$V_{q,\alpha}=\inf_{p\geq 0}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-e^{-(q+p)^2t/4}\right]\right).$$

Evidently that we have

$$V_{q,\alpha}\geq\inf_{p\geq 1}\left(\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t}\left[1-e^{-p^2t/4}\right]dt\right)\geq\alpha\cdot C,$$

for all $\alpha\in[0,1]$.

Reasoning in continuation exactly as in the case of $q=0$ and as in the previous case (i), we easily obtain that because by hypothesis f is not a polynomial of degree $\le q-1$, there exists a constant $C_{r,q}(f)>0$ such that

$$\|[F_U^{\alpha}(f)]^{(q)} - f^{(q)}\|_r \ge C_{r,q}(f)\alpha, \text{ for all } \alpha \in (0,1].$$

The theorem is proved. □

References

[1] Aliev, I. A., Gadjiev, A.D. and Aral, A.: On approximation properties of a family of linear operators at critical value of parameter, J. Approx. Theory, Vol. 138, 242–253(2006).

[2] Flett, F. M.: Temperatures, Bessel potentials and Lipschitz space, Proc. London Math. Soc., Vol. 22, No. 3, 385–451(1971).

[3] Gadjiev, A. D., Aral, A. and Aliev, I. A.: On behaviour of Riesz and generalized Riesz potentials as order tends to zero, Math. Inequalities and Appl., Vol. 4, 875–888(2007).

[4] Gal, S. G.: Approximation by Complex Bernstein and Convolution Type Operators, World Scientific Publishing Company, New Jersey, London, Singapore, Beijing, Shanghai, Hong Kong, Taipei, Chennai, 2009.

[5] Kurokawa, T.: On the Riesz and Bessel kernels as approximations of the identity, Sci. Rep. Kagoshima Univ., Vol. 30, 31–45(1981).

[6] Sezer, S.: On approximation properties of the families of Flett and generalized Flett potentials, Int. Journal Math. Analysis, Vol. 3, No. 39, 1905–1915(2009).

[7] Uyhan, S. B., Gadjiev, A. D. and Aliev, I. A.: On approximation properties of the parabolic potentials, *Bull. Austral. Math. Soc.*, Vol. 74, 449–460(2006).

Sorin G. GAL
Department of Mathematics and Computer Science,
University of Oradea,
Str. Universitatii No. 1,
410087, Oradea-ROMANIA
e-mail: galso@uoradea.ro

Received: 06.04.2010