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Existence of mild solutions for abstract mixed type semilinear evolution equations*

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Abstract

This paper is concerned with the existence of global mild solutions and positive mild solutions to initial value problem for a class of mixed type semilinear evolution equations with noncompact semigroup in Banach spaces. The main method is based on a new fixed point theorem with respect to convex-power condensing operator.

Key Words: Semilinear evolution equation; convex-power condensing operator; fixed point theorem; C_0-semigroup; measure of noncompactness.

1. Introduction

In this paper, we are interested in the following initial value problem (IVP) of mixed type semilinear evolution equation in Banach space E:

$$\begin{cases} u'(t) + Au(t) = f\left(t, u(t), \int_0^t k(t,s)u(s)ds, \int_0^a h(t,s)u(s)ds\right), & t \in J, \\ u(0) = x_0, \end{cases} \tag{1.1}$$

where $A : D(A) \to E$ is a dense and closed linear operator, $-A$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geq 0)$ in E, and $J = [0, a]$, $x_0 \in E$. For convenience, we denote

$$(Ku)(t) = \int_0^t k(t,s)u(s)ds, \quad (Su)(t) = \int_0^a h(t,s)u(s)ds.$$

Then IVP (1.1) can be rewritten as

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), (Ku)(t), (Su)(t)), & t \in J, \\ u(0) = x_0. \end{cases}$$

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This kind of equation (1.1) and other special forms serve as models for various partial differential equations or partial integro-differential equations arising in heat flow in material with memory, viscoelasticity and reaction diffusion problems (see [16, 20]). In recent years, the existence, uniqueness and some other properties of solutions to semilinear evolution equations similar to (1.1) have been extensively studied. We can refer to [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 18, 20] and references cited therein.

In particular, we would like to mention the results due to Li [8], Sun and Zhang [18]. First, we point out that many authors applied the famous Sadovskii's fixed point theorem to investigate similar problems and used the following hypothesis with respect to the Kuratowski measure of noncompactness $\alpha(\cdot)$: there exists a constant $L>0$ such that for any bounded and equicontinuous set $D\subset C(J,E)$ and $t\in J$,

$$\alpha(f(t,D(t)))\le L\alpha(D(t)). \tag{1.2}$$

What's more, they required a stronger condition, i.e., the constant L satisfies a strong inequality (see Remark 3.5 below).

In [8], based on the Sadovskii's fixed point theorem for condensing operator, Li discussed the existence of mild solutions to the following initial value problem for semilinear evolution equations

$$\begin{cases} u'(t)+Au(t)=f(t,u(t)), & t\ge 0,\\ u(0)=x_0 \end{cases} \tag{1.3}$$

in Banach spaces, and required that f satisfies a suitable condition on the measure of noncompactness similar to (1.2). The author first proved the local existence of mild solutions for IVP (1.3) on interval $[t_0,t_0+h]$, $t_0\in[0,T)$, where

$$h=h(t_0,\|x_0\|)=\min\left\{1,\frac{\|x_0\|+1}{C(t_0)},\frac{1}{4M(t_0)L(t_0)+1}\right\},$$

$M(t_0)=\sup\{\|T(t)\|:t\in[0,t_0+1]\}$, $C(t_0)=\sup\{\|f(t,x)\|:\|x\|\le R(t_0),t\in[0,t_0+1]\}$, $R(t_0)=2M(t_0)(\|x_0\|+1)$ and $L(t_0)=L(t_0+1,R(t_0))$. The constant $L(t_0)$ satisfies $4M(t_0)L(t_0)h<1$, which can guarantee $\alpha(Q(B))\le 4M(t_0)L(t_0)h\alpha(B)<\alpha(B)$, herein the operator Q is defined by the formula

$$(Qu)(t)=T(t-t_0)x_0+\int_{t_0}^{t}T(t-s)f(s,u(s))ds,\quad t\in I.$$

However, if we apply the new fixed point theorem obtained in [18] stated below, we can prove the existence of mild solution to (1.3) on the interval $[t_0,t_0+h]$ without requiring the condition $4M(t_0)L(t_0)h<1$.

Sun and Zhang [18] defined a new kind of operator, i.e., convex-power condensing operator, which is a generalization of condensing operator. Furthermore, they established a new fixed point theorem for this kind of operator which generalizes the famous Schauder's fixed point theorem and Sadovskii's fixed point theorem. As an application, they proved the existence theorems of global mild solutions and positive mild solutions to the following IVP for a class of semilinear evolution equation with noncompact semigroup

$$\begin{cases} u'(t)+Au(t)=f(t,u(t)), & t\in J,\\ u(0)=x_0 \end{cases} \tag{1.4}$$

in Banach spaces, and also assumed that (1.2) holds.

Clearly, equation (1.1) is more general than the corresponding equations discussed in previous literature, see [6, 7, 8, 13, 16, 18]. In case that $f(s,u(s),(Ku)(s),(Su)(s)) = f(s,u(s))$, equation (1.1) reduces to equation (1.4), which has been considered in [6, 7, 18]. It is natural to ask if we can apply this new fixed point theorem to more general problem (1.1). Furthermore, we can replace the constant L in (1.2) used in [18] by nonnegative Lebesgue integrable functions $L_i \in L(J,R^+)$, $i = 1,2,3$ (see Theorem 3.6 in the sequel). Hence, our results generalize and partially improve the main results in [18].

Recently, Zhang et al. [21] have extended the fixed point theorems of Rothe and Altman types to convex-power condensing operator and considered the existence of solutions to a first-order differential equation with integral boundary conditions.

Motivated by the above works, the main aim of this paper is to study the existence of mild solutions and positive mild solutions to IVP (1.1) by using the fixed point theorem for convex-power condensing operator due to Sun and Zhang [18].

The rest of this paper is arranged as follows. In Section 2, we will introduce the definition of convex-power condensing operator and the corresponding fixed point theorem from [18]. In Section 3, we prove existence theorems of global mild solutions and positive mild solutions to IVP (1.1).

2. Preliminaries

In this section, we introduce some definitions and fixed point theorems.

From now on, without any special statement, we always assume that E is a real Banach space. For any bounded set $D \subset E$, we denote by $\alpha(D)$ the Kuratowski measure of noncompactness.

Definition 2.1 *An operator $F : D \to E$ is said to be a condensing operator, if F is continuous, bounded and for any nonprecompact bounded subset $S \subset D$, $\alpha(F(S)) < \alpha(S)$.*

For the condensing operator, we introduce the following well-known fixed point theorem.

Lemma 2.2 (Sadovskii's fixed point theorem [17]) *Let $D \subset E$ be a closed, bounded and convex set. Assume that $F : D \to D$ is a condensing operator. Then F has at least one fixed point in D.*

In order to introduce the definition of convex-power operator, we give some notations (see [18]). Let $D \subset E$ be closed and convex, $F : D \to D$, $x_0 \in D$. For any subset $S \subset D$, set

$$\begin{cases} F^{(1,x_0)}(S) \equiv F(S), \\ F^{(n,x_0)}(S) = F\left(\overline{\mathrm{co}}\left\{F^{(n-1,x_0)}(S), x_0\right\}\right), n = 2,3,\cdots. \end{cases} \tag{2.5}$$

Definition 2.3 ([18]) *Let $D \subset E$ be closed and convex. An operator $F : D \to D$ is said to be a convex-power condensing operator, if F is continuous, bounded and there exist $x_0 \in D$ and a positive integer n_0 such that for any bounded nonprecompact subset $S \subset D$,*

$$\alpha\left(F^{(n_0,x_0)}(S)\right) < \alpha(S). \tag{2.6}$$

Remark 2.4 By Definition 2.3, we can see that if $\alpha(F^{(n_0,x_0)}(S)) = \alpha(S)$, then S is a precompact set in E. Obviously, a condensing operator is convex-power condensing. Thus, the convex-power condensing operator is a generalization of the condensing operator. What's more, we can see that the convex-power condensing operator arises naturally from many problems in the sequel.

In [18], Sun and Zhang proved the following fixed point theorem with respect to the convex-power condensing operator, which is the main tool for the proof of our main results.

Lemma 2.5 ([18]) *Let $D \subset E$ be closed, bounded and convex, and $F : D \to D$ be a convex-power condensing operator. Then F has at least one fixed point in D.*

Lemma 2.6 ([18]) *Let $D \subset E$ be closed, bounded and convex, and $F : D \to D$ be continuous. If there exist $x_0 \in D$, $0 \leq k < 1$ and a positive integer n_0 such that for any bounded subset $S \subset D$,*

$$\alpha\left(F^{(n_0,x_0)}(S)\right) \leq k\alpha(S). \tag{2.7}$$

Then F has at least one fixed point in D.

Remark 2.7 Lemma 2.5 shows that the operator F is not required to be condensing and completely continuous, thus this fixed point theorem is a generalization of the well-known Sadvoskii's fixed point theorem, since when $n_0 = 1$, Lemma 2.5 is the latter.

3. Main results

In this section, we shall establish the existence theorems of global mild solutions and positive mild solutions for IVP (1.1). For convenience, we give some notations.

Let E be a real Banach space, $J = [0, a]$, $u_0 \in E$ and $C(J, E)$ be the space of all continuous functions from J into E with the supremum norm $\|u\|_C = \sup\{\|u(t)\| : t \in J\}, \quad u \in C(J, E)$. For $B \subset C(J, E)$, set $B(t) = \{u(t) : u \in B\}$,

$$(KB)(t) = \{(Ku)(t) : u \in B\}, \quad (SB)(t) = \{(Su)(t) : u \in B\}.$$

Let $f : J \times E \times E \times E \to E$, $D_0 = \{(t, s) \in R^2 : 0 \leq s \leq t \leq a\}$, $D = \{(t, s) \in R^2 : 0 \leq s, t \leq a\}$, $k \in C(D_0, R^+)$, $h \in C(D, R^+)$, $k_0 = \max\{k(t, s) : (t, s) \in D_0\}$, $h_0 = \max\{h(t, s) : (t, s) \in D\}$. For $R > 0$, denote $B_R = \{x \in E : \|x\| \leq R\}$. Let

$$M_R = \sup\{\|f(t, u, v, w)\| : (t, u, v, w) \in J \times B_R \times B_R \times B_R\}. \tag{3.8}$$

Lemma 3.1 ([5]) *Let $B \subset C(J, E)$ be bounded and equicontinuous. Then $m(t) = \alpha(B(t))$ is continuous on J and*

$$\alpha\left(\int_J B(s)ds\right) \leq \int_J \alpha(B(s))ds \quad \textit{and} \quad \alpha(B) = \max_{t\in J} \alpha(B(t)).$$

Lemma 3.2 ([5]) *Let* $B \subset C(J,E)$ *be equicontinuous,* $u_0 \in C(J,E)$. *Then* $\overline{\text{co}}\{B, u_0\}$ *is equicontinuous in* $C(J,E)$.

Lemma 3.3 *Assume that for all* $R > 0$, f *is bounded and uniformly continuous on* $J \times B_R \times B_R \times B_R$, $H \subset C(J,E)$ *is bounded and equicontinuous, and the* C_0*-semigroup* $T(t)(t \geq 0)$ *generated by* $-A$ *is an equicontinuous semigroup. Then for all* $t, s \in J$, $t \geq s$, $\{T(t-s)f(s,u(s),(Ku)(s),(Su)(s)) : u \in H\}$ *is equicontinuous in* $C(J,E)$.

Proof. Since $H \subset C(J,E)$ is bounded, equicontinuous and k, h are uniformly continuous, we have KH, SH are continuous and bounded in $C(J,E)$. Thus there exists a real number $R_0 > 0$ such that for all $s \in J$ and $u \in H$, $(s,u(s),(Ku)(s),(Su)(s)) \in J \times B_{R_0} \times B_{R_0} \times B_{R_0}$. By the uniform continuity of f on $J \times B_{R_0} \times B_{R_0} \times B_{R_0}$, we know that for any $\epsilon > 0$, there exists $\eta_1 > 0$ such that when $(s_i, u_i, v_i, w_i) \in J \times B_{R_0} \times B_{R_0} \times B_{R_0} (i = 1,2)$, $|s_1 - s_2| < \eta_1$, $\|u_1 - u_2\| < \eta_1$, $\|v_1 - v_2\| < \eta_1$, $\|w_1 - w_2\| < \eta_1$,

$$\|f(s_1,u_1,v_1,w_1) - f(s_2,u_2,v_2,w_2)\| < \frac{\epsilon}{2M}, \tag{3.9}$$

where $M = \sup\{\|T(t)\| : t \in J\}$.

By virtue of the fact that $T(t)$ is continuous in the sense of operator norm, there exists η_2 such that for any $t, s_i \in J, t \geq s_i (i = 1,2)$, $|s_1 - s_2| < \eta_2$,

$$\|T(t-s_1) - T(t-s_2)\| = \|T(s_2 - s_1 + s) - T(s)\| < \frac{\epsilon}{2M_0}, \tag{3.10}$$

where $M_0 = \sup\{\|f(t,u(t),(Ku)(t),(Su)(t))\| : t \in J, u \in H\}$. Set $\eta = \min\{\eta_1, \eta_2\}$. Since H, KH, SH are equicontinuous, then there exists $\delta \in (0,\eta)$, such that when $t, s_i \in J, t \geq s_i (i = 1,2)$, $|s_1 - s_2| < \delta$, for any $u \in H$, $\|u(s_1) - u(s_2)\| < \eta$, $\|(Ku)(s_1) - (Ku)(s_2)\| < \eta$, $\|(Su)(s_1) - (Su)(s_2)\| < \eta$.

It follows from (3.9) and (3.10) that

$$\begin{aligned}
&\|T(t-s_1)f(s_1,u(s_1),(Ku)(s_1),(Su)(s_1)) - T(t-s_2)f(s_2,u(s_2),(Ku)(s_2),(Su)(s_2))\| \\
&\leq \|T(t-s_1) - T(t-s_2)\| \cdot \|f(s_1,u(s_1),(Ku)(s_1),(Su)(s_1))\| \\
&+ \|T(t-s_2)\| \cdot \|f(s_1,u(s_1),(Ku)(s_1),(Su)(s_1)) - f(s_2,u(s_2),(Ku)(s_2),(Su)(s_2))\| \\
&\leq \frac{\epsilon}{2M_0}M_0 + \frac{\epsilon}{2M}M = \epsilon.
\end{aligned}$$

Hence, $\{T(t-s)f(s,u(s),(Ku)(s),(Su)(s)) : u \in H\}(\forall t, s \in J, t \geq s)$ is equicontinuous in $C(J,E)$. Thus we complete the proof. □

We introduce the definition of mild solutions to IVP (1.1) (see [16]). If $u \in C(J,E)$ satisfies the following integral equation,

$$u(t) = T(t)x_0 + \int_0^t T(t-s)f(s,u(s),(Ku)(s),(Su)(s))ds, \quad t \in J,$$

then u is called a mild solution to IVP (1.1) on J.

Now, we state and prove our main results.

Theorem 3.4 *Let E be a real Banach space, and C_0-semigroup $T(t)(t \geq 0)$ generated by $-A$ be an equicontinuous semigroup. Assume that*

(H1) *for any $R > 0$, f is bounded and uniformly continuous on $J \times B_R \times B_R \times B_R$, and*

$$\limsup_{R\to\infty} \frac{M_R}{R} < \frac{1}{aa_0 M}, \tag{3.11}$$

where $a_0 = \max\{1, ak_0, ah_0\}$, $M = \sup\{\|T(t)\| : t \in J\}$, M_R is defined by (3.8);

(H2) *there exist constants $L_i > 0$ $(i = 1, 2, 3)$ such that for any bounded and equicontinuous sets $D_i \subset C(J, E)$ $(i = 1, 2, 3)$ and $t \in J$,*

$$\alpha(f(t, D_1(t), D_2(t), D_3(t))) \leq L_1\alpha(D_1(t)) + L_2\alpha(D_2(t)) + L_3\alpha(D_3(t)).$$

Then IVP (1.1) has at least one mild solution in $C(J, E)$.

Proof. Define the operator $Q : C(J, E) \to C(J, E)$ by

$$(Qu)(t) = T(t)x_0 + \int_0^t T(t-s)f\,(s, u(s), (Ku)(s), (Su)(s))\,ds, \quad t \in J. \tag{3.12}$$

Then, $u \in C(J, E)$ is the mild solution to IVP (1.1) if and only if $u = Qu$.

Since f is uniformly continuous on $J \times B_R \times B_R \times B_R$, we can see that $Q : C(J, E) \to C(J, E)$ is continuous and bounded. It follows from (3.11) that there exist $0 < r < \frac{1}{aa_0M}$ and $R_0 > 0$ such that for any $R \geq a_0R_0$, we have $M_R < rR$.

Let $R^* = \max\{R_0, M\|x_0\|(1 - aa_0rM)^{-1}\}$, $B_{R^*} = \{u \in C(J, E) : \|u\|_C \leq R^*\}$. For any $u \in B_{R^*}$, we have $\|u\|_C \leq R^* \leq a_0R^*$, $\|Ku\|_C \leq ak_0\|u\|_C \leq ak_0R^* \leq a_0R^*$, and $\|Su\|_C \leq ah_0\|u\|_C \leq ah_0R^* \leq a_0R^*$. So, by the definition of Q, we have

$$\begin{aligned}\|(Qu)(t)\| &\leq \|T(t)x_0\| + \int_0^t \|T(t-s)\| \cdot \|f(s, u(s), (Ku)(s), (Su)(s))\|ds \\ &\leq M\|x_0\| + MaM_{a_0R^*} \leq M\|x_0\| + Mara_0R^* \leq R^*,\end{aligned}$$

which shows that the operator $Q : B_{R^*} \to B_{R^*}$ is continuous and bounded.

We shall prove $Q(B_{R^*}) \subset C(J, E)$ is equicontinuous. For any $u \in B_{R^*}$, $0 \leq t_1 \leq t_2 \leq a$, we have

$$\begin{aligned}&\|(Qu)(t_2) - (Qu)(t_1)\| \\ &\leq \|T(t_2)x_0 - T(t_1)x_0\| + \int_{t_1}^{t_2} \|T(t_2-s)\| \cdot \|f(s, u(s), (Ku)(s), (Su)(s))\|ds \\ &\quad + \int_0^{t_1} \|T(t_2-s) - T(t_1-s)\| \times \|f(s, u(s), (Ku)(s), (Su)(s))\|ds \\ &\leq \|T(t_2)x_0 - T(t_1)x_0\| + \int_0^{t_1} \|T(t_2-s) - T(t_1-s)\| \cdot M_{a_0R^*}ds \\ &\quad + MM_{a_0R^*}|t_2 - t_1|.\end{aligned}$$

Since $T(t)x_0$ is continuous on J, then $T(t)x_0$ is uniformly continuous on J. Note that $T(t)$ is continuous in the sense of operator norm. It follows from the Lebesgue dominated convergence theorem that

$$\int_0^{t_1} \|T(t_2-s)-T(t_1-s)\| \cdot M_{a_0R^*}ds \to 0, \quad t_2-t_1 \to 0.$$

Thus, $Q(B_{R^*})$ is equicontinuous.

Let $F=\overline{\text{co}}Q(B_{R^*})$. Then $Q: F \to F$. By Lemma 3.3, $F \subset C(J,E)$ is equicontinuous.

Now, we are in the position to prove that $Q: F \to F$ is a convex-power condensing operator. First, it is easy to see that Q is continuous. Let $u_0 \in F$. We shall prove there exists a positive integer n_0 such that for any nonprecompact set $B \subset F$,

$$\alpha\left(Q^{(n_0,u_0)}(B)\right) < \alpha(B).$$

For any $B \subset F$, by the definition of $Q^{(n,u_0)}(B)$ and Lemma 3.2, we get $Q^{(n,u_0)}(B) \subset B_{R^*}$ is equicontinuous. Hence , we know from Lemma 3.1 that

$$\alpha\left(Q^{(n,u_0)}(B)\right) = \max_{t\in J}\alpha\left(\left(Q^{(n,u_0)}(B)\right)(t)\right), \quad n=1,2,\cdots. \tag{3.13}$$

Since $B \subset F$ is bounded and equicontinuous, by Lemma 3.1, we get

$$\alpha(B) = \max_{t\in J}\alpha(B(t)).$$

It follows from (H1) and Lemma 3.3 that $\{T(t-s)f(s,u(s),(Ku)(s),(Su)(s)) : u \in B\}$ $(\forall t,s \in J, t \geq s)$ is equicontinuous in $C(J,E)$.

Hence from (H2) and Lemma 3.1, we have

$$\begin{aligned}
&\alpha\left(\left(Q^{(1,u_0)}(B)\right)(t)\right) = \alpha\left(\left(Q\left(B\right)\right)(t)\right) \\
&= \alpha\left(T(t)x_0 + \int_0^t T(t-s)f\left(s,B(s),(KB)(s),(SB)(s)\right)ds\right) \\
&= \alpha\left(\int_0^t T(t-s)f\left(s,B(s),(KB)(s),(SB)(s)\right)ds\right) \\
&\leq \int_0^t \alpha(T(t-s)f\left(s,B(s),(KB)(s),(SB)(s)\right))\,ds \\
&\leq \int_0^t \|T(t-s)\|\alpha\left(f\left(s,B(s),(KB)(s),(SB)(s)\right)\right)ds \\
&\leq \int_0^t M\left(L_1+ak_0L_2+ah_0L_3\right)\alpha(B(s))ds \\
&\leq Mt\left(L_1+ak_0L_2+ah_0L_3\right)\alpha(B).
\end{aligned} \tag{3.14}$$

By the equicontinuity of $Q^{(1,u_0)}(B)=Q(B)$ and the uniform continuity of f, it follows from Lemma 3.2 and Lemma 3.3 that for all $t,\ s\in J,\ t\geq s$,

$$T(t-s)f\left(s,\left(\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\right.$$
$$\left.\left(K\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\left(S\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s)\right)$$

is equicontinuous.

So, by virtue of (H2), (3.14) and Lemma 3.1, we have

$$\begin{aligned}
&\alpha\left(\left(Q^{(2,u_0)}(B)\right)(t)\right)\\
&=\alpha\left(T(t)x_0+\int_0^t T(t-s)f\left(s,\left(\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\right.\right.\\
&\qquad\left.\left.\left(K\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\left(S\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s)\right)ds\right)\\
&=\alpha\left(\int_0^t T(t-s)f\left(s,\left(\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\right.\right.\\
&\qquad\left.\left.\left(K\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\left(S\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s)\right)ds\right)\\
&\leq\int_0^t\alpha\left(T(t-s)f\left(s,\left(\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\right.\right.\\
&\qquad\left.\left.\left(K\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s),\left(S\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s)\right)\right)ds\\
&\leq\int_0^t\|T(t-s)\|(L_1+ak_0L_2+ah_0L_3)\alpha\left(\left(\overline{\mathrm{co}}\left\{Q^{(1,u_0)}(B),u_0\right\}\right)(s)\right)ds\\
&\leq\int_0^t M(L_1+ak_0L_2+ah_0L_3)\alpha\left(\left(Q^{(1,u_0)}(B)\right)(s)\right)ds\\
&\leq M\int_0^t(L_1+ak_0L_2+ah_0L_3)^2Ms\alpha(B)ds\\
&=\frac{M^2(L_1+ak_0L_2+ah_0L_3)^2t^2}{2!}\alpha(B).
\end{aligned}\tag{3.15}$$

Suppose that

$$\alpha\left(Q^{(k,u_0)}(B)\right)(s)\quad=\quad\frac{M^k(L_1+ak_0L_2+ah_0L_3)^kt^k}{k!}\alpha(B),\quad\forall t\in J.$$

Then, for any $t \in J$,

$$\begin{aligned}
&\alpha\left(\left(Q^{(k+1,u_0)}(B)\right)(t)\right) \\
&= \alpha\left(T(t)x_0 + \int_0^t T(t-s)f\left(s, \left(\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s),\right.\right. \\
&\qquad \left.\left.\left(K\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s), \left(S\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s)\right) ds\right) \\
&= \alpha\left(\int_0^t T(t-s)f\left(s, \left(\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s),\right.\right. \\
&\qquad \left.\left.\left(K\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s), \left(S\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s)\right) ds\right) \\
&\le \int_0^t \alpha\left(T(t-s)f\left(s, \left(\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s),\right.\right. \\
&\qquad \left.\left.\left(K\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s), \left(S\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s)\right)\right) ds \\
&\le \int_0^t \|T(t-s)\| (L_1 + ak_0L_2 + ah_0L_3)\alpha\left(\left(\overline{\text{co}}\left\{Q^{(k,u_0)}(B), u_0\right\}\right)(s)\right) ds \\
&\le \int_0^t M(L_1 + ak_0L_2 + ah_0L_3)\alpha\left(\left(Q^{(k,u_0)}(B)\right)(s)\right) ds \\
&\le M\int_0^t \frac{(L_1 + ak_0L_2 + ah_0L_3)^{k+1}M^k s^k}{k!}\alpha(B)ds \\
&= \frac{M^{k+1}(L_1 + ak_0L_2 + ah_0L_3)t^{k+1}}{(k+1)!}\alpha(B).
\end{aligned} \tag{3.16}$$

Hence, by the method of mathematical induction, for any positive integer n and $t \in J$, we have

$$\alpha\left(\left(Q^{(n,u_0)}(B)\right)(t)\right) \le \frac{M^n(L_1 + ak_0L_2 + ah_0L_3)^n a^n}{n!}\alpha(B). \tag{3.17}$$

Consequently, by (3.13),

$$\alpha\left(Q^{(n,u_0)}(B)\right) = \max_{t\in J}\alpha\left(\left(Q^{(n,u_0)}(B)\right)(t)\right) \le \frac{M^n(L_1 + ak_0L_2 + ah_0L_3)^n a^n}{n!}\alpha(B).$$

Since

$$\frac{M^n(L_1 + ak_0L_2 + ah_0L_3)^n a^n}{n!} \to 0 \quad (n \to \infty),$$

there exists a positive integer n_0 such that

$$\frac{M^{n_0}(L_1 + ak_0L_2 + ah_0L_3)^{n_0} a^{n_0}}{n_0!} < 1.$$

So,

$$\alpha\left(Q^{(n_0,u_0)}(B)\right) < \alpha(B).$$

Thus, $Q: F \to F$ is a convex-power condensing operator. It follows from Lemma 2.5 that Q has at least one fixed point u^* in F, that is to say, u^* is a mild solution of IVP (1.1) in $F \subset C(J,E)$. Hence, the result is proved. □

Remark 3.5 Noting that many authors applied the famous Sadovskii's fixed point theorem to investigate the similar problems and used the same hypothesis (H2), they required the constants satisfy a strong inequality. For instance, in [14], Liu considered the following IVP of mixed type integro-differential equation,

$$\begin{cases} u' = f(t,u,Ku,Su), \quad t \in J = [0,a], \\ u(t_0) = x_0, \end{cases}$$

where K, S are defined as above, and assumed that the condition (H2) holds and L_1, L_2, L_3 satisfy one of the following conditions:

(a) $ah_0L_3(e^{2a(L_1+ak_0L_2)} - 1) < L_1 + ak_0L_2$;

(b) $a(2L_1 + ak_0L_2 + ah_0L_3) < 1$.

In the present paper, we can see that the condition (a) and (b) are not necessary.

Motivated by [15], we can replace the condition (H2) in Theorem 3.4 by the condition (H3) in the following theorem.

Theorem 3.6 *Let E be a real Banach space, and C_0-semigroup $T(t)(t \geq 0)$ generated by $-A$ be an equicontinuous semigroup. Assume* (H1) *holds and*

(H3) *there exist nonnegative Lebesgue integrable functions $L_i \in L(J,R^+)$ $(i=1,2,3)$ such that for any bounded and equicontinuous sets $D_i \subset C(J,E)$ $(i=1,2,3)$ and $t \in J$,*

$$\alpha(f(t,D_1(t),D_2(t),D_3(t))) \leq L_1(t)\alpha(D_1(t)) + L_2(t)\alpha(D_2(t)) + L_3(t)\alpha(D_3(t)).$$

Then the IVP (1.1) *has at least one mild solution in $C(J,E)$.*

Proof. The proof is similar to that of Theorem 3.4. So, we only demonstrate the differences in the proof.

Define operator $Q : C(J, E) \to C(J, E)$ as in (3.12). From (H3) and Lemma 3.1, we have

$$\begin{aligned}
&\alpha\left(\left(Q^{(1,u_0)}(B)\right)(t)\right) = \alpha((Q(B))(t)) \\
&= \alpha\left(T(t)x_0 + \int_0^t T(t-s)f\left(s, B(s), (KB)(s), (SB)(s)\right) ds\right) \\
&= \alpha\left(\int_0^t T(t-s)f\left(s, B(s), (KB)(s), (SB)(s)\right) ds\right) \\
&\le \int_0^t \alpha\left(T(t-s)f\left(s, B(s), (KB)(s), (SB)(s)\right)\right) ds \\
&\le \int_0^t \|T(t-s)\|\alpha\left(f(s, B(s), (KB)(s), (SB)(s))\right) ds \\
&\le \int_0^t M(L_1(s) + ak_0L_2(s) + ah_0L_3(s))\alpha(B(s))ds \\
&\le M\int_0^t (L_1(s) + ak_0L_2(s) + ah_0L_3(s))\alpha(B)ds = \int_0^t L(s)ds \cdot \alpha(B),
\end{aligned} \tag{3.18}$$

where $L(t) = M(L_1(t) + ak_0L_2(t) + ah_0L_3(t))$.

Due to the fact that there exists a continuous function $\phi : J \to R$ such that for any $\varepsilon \in (0, 1)$,

$$\int_0^a |L(s) - \phi(s)|ds < \varepsilon, \tag{3.19}$$

and taking into account (3.18), (3.19), we have

$$\alpha\left(\left(Q^{(1,u_0)}(B)\right)(t)\right) \le \left[\int_0^t |L(s) - \phi(s)|ds + \int_0^t |\phi(s)|ds\right]\alpha(B) \le (\varepsilon + \lambda t)\alpha(B),$$

where $\lambda = \max\{|\phi(t)| : t \in J\}$. Furthermore, we have

$$\begin{aligned}
&\alpha\left(\left(Q^{(2,u_0)}(B)\right)(t)\right) \\
&= \alpha\left(T(t)x_0 + \int_0^t T(t-s)f\left(s, \left(\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s),\right.\right. \\
&\qquad \left.\left.\left(K\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s), \left(S\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s)\right) ds\right)
\end{aligned} \tag{3.20}$$

$$
\begin{aligned}
&= \alpha\left(\int_0^t T(t-s)f\left(s, \left(\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s),\right.\right.\\
&\qquad \left.\left.\left(K\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s), \left(S\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s)\right) ds\right)\\
&\le \int_0^t \alpha\left(T(t-s)f\left(s, \left(\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s),\right.\right.\\
&\qquad \left.\left.\left(K\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s), \left(S\overline{\text{co}}\left\{Q^{(1,u_0)}(B), u_0\right\}\right)(s)\right)\right) ds\\
&\le \int_0^t M(L_1(s) + ak_0L_2(s) + ah_0L_3(s))\alpha\left(\left(Q^{(1,u_0)}(B)\right)(s)\right) ds\\
&= \int_0^t L(s)\alpha\left(\left(Q^{(1,u_0)}(B)\right)(s)\right) ds\\
&\le \int_0^t \left[|L(s)-\phi(s)| + |\phi(s)|\right]\alpha\left(\left(Q^{(1,u_0)}(B)\right)(s)\right) ds\\
&\le \int_0^t \left[|L(s)-\phi(s)| + |\phi(s)|\right](\varepsilon + \lambda s)\,\alpha(B)ds\\
&= \left(\varepsilon^2 + 2\lambda t\varepsilon + \frac{\lambda^2t^2}{2!}\right)\alpha(B). \qquad (3.21)
\end{aligned}
$$

Hence, by induction, for any positive integer n and $t \in J$, we have, for all $t \in J$,

$$
\alpha\left(\left(Q^{(n,u_0)}(B)\right)(t)\right) \le \left(\varepsilon^n + C_n^1(\lambda t)\varepsilon^{n-1} + \frac{C_n^2(\lambda t)^2\varepsilon^{n-2}}{2!} + \cdots + \frac{(\lambda t)^n}{n!}\right)\alpha(B).
$$

Applying the method used in [15], we obtain that there exist $0 \le k < 1$ and a positive integer n_0 such that (2.7) holds. It follows from Lemma 2.6 that Q has at least one fixed point u^* in F, namely, u^* is a mild solution to IVP (1.1) in $F \subset C(J, E)$. □

Next, we prove the existence of positive mild solutions to IVP (1.1).

Let E be a real partial order Banach space by a cone P of E, i.e., for any $x, y \in E, x \le y$ if and only if $y - x \in P$. For more details of cone theory, we refer the readers to [3, 4, 5].

Let $T(t)(t \ge 0)$ be a C_0-semigroup on E. If for any $x \ge 0$, we have $T(t)x \ge 0$, then $T(t)(t \ge 0)$ is called a positive C_0-semigroup on E.

Theorem 3.7 *Let P be a normal cone of E, and semigroup $T(t)(t \ge 0)$ generated by $-A$ be an equicontinuous and positive C_0-semigroup, $x_0 \ge \theta$. Assume that $f : J \times P \times P \times P \to P$ satisfies*

(H4) *for any $R > 0$, f is uniformly continuous on $J \times B_P(R) \times B_P(R) \times B_P(R)$ with $B_P(R) = \{u \in P : \|u\| \le R\}$, and there are nonnegative continuous functions $N_j(t)(j = 1, 2)$ and $g(t) : J \to P$ such that for any $t \in J$, u, v, $w \in P$, $f(t, u, v, w) \le N_1(t)u + N_2(t)v + g(t)$;*

(H5) *there exist constants $L_i > 0(i = 1, 2, 3)$ such that for any bounded and equicontinuous sets $D_i \subset$*

$C(J,P)(i=1,2,3)$ *and* $t\in J$,

$$\alpha(f(t,D_1(t),D_2(t),D_3(t)))\le L_1\alpha(D_1(t))+L_2\alpha(D_2(t))+L_3\alpha(D_3(t)).$$

Then IVP (1.1) *has at least one positive mild solution in* $C(J,P)$.

Proof. Define the operator Q as in (3.12) and $\widetilde{B}$ by the formula

$$(\widetilde{B}u)(t)=\int_0^t T(t-s)[N_1(s)u(s)+N_2(s)(Ku)(s)]ds,$$

where $(Ku)(s)=\int_0^s k(s,\tau)u(\tau)d\tau$.

Next, we shall prove $r(\widetilde{B})=0$, where $r(\cdot)$ denotes the spectral radius of bounded linear operator. In fact, for any $t\in J$, by the definition of $\widetilde{B}$, we have

$$\begin{aligned}\|(\widetilde{B}u)(t)\| &= \left\|\int_0^t T(t-s)\left(N_1(s)u(s)+N_2(s)\int_0^s k(s,\tau)u(\tau)d\tau\right)ds\right\|\\ &\le MN^*(1+ak_0)t\|u\|_C=\alpha t\|u\|_C,\end{aligned}$$

where $N^*=\max\{\max\limits_{s\in J}N_1(s),\max\limits_{s\in J}N_2(s)\}$, $M=\sup\{\|T(t)\|:t\in J\}$, $\alpha=MN^*(1+ak_0)$. Further,

$$\begin{aligned}\|(\widetilde{B}^2u)(t)\| &\le \int_0^t MN^*\left[\|(\widetilde{B}u)(s)\|+\int_0^s k(s,\tau)\|(\widetilde{B}u)(\tau)\|d\tau\right]ds\\ &\le \int_0^t MN^*\left[\alpha s\|u\|_C+k_0\int_0^s \alpha\tau\|u\|_C d\tau\right]ds\\ &\le MN^*\alpha(1+ak_0)\frac{t^2}{2!}\|u\|_C=\frac{\alpha^2t^2}{2!}\|u\|_C.\end{aligned}$$

By the method of mathematical induction, for any positive integer n and $t\in J$, we have

$$\|(\widetilde{B}^n u)(t)\|\le\frac{\alpha^n t^n}{n!}\|u\|_C.$$

Hence,

$$\|\widetilde{B}^n u\|_C\le\frac{\alpha^n a^n}{n!}\|u\|_C.$$

and thus, $\|\widetilde{B}^n\|\le\frac{\alpha^n a^n}{n!}$. Therefore, $r(\widetilde{B})=\lim\limits_{n\to\infty}\|\widetilde{B}^n\|^{\frac{1}{n}}=0.$

Let $0<\alpha<N^{-1}$, where N is the normal constant of P. Hence, there exists an equivalent norm $\|\cdot\|^*$ in E such that $\|\widetilde{B}\|^*\le r(\widetilde{B})+\alpha=\alpha$, where $\|\widetilde{B}\|^*$ denotes the operator norm of $\widetilde{B}$ with respect to norm $\|\cdot\|^*$ (see [19]).

Let $M^*=\sup\{\|T(t)\|^*:t\in J\}$, $r^*\ge NM^*\left[\|u_0\|^*+a\|g\|_C^*\right](1-N\alpha)^{-1}$, where $\|u\|_C^*=\max_{t\in J}\|u(t)\|^*$, $B_P(r^*)=\{u\in C(J,P):\|u\|_C^*\le r^*\}$. Then for any $u\in B_P(r^*)$, by (H4) and the definition of the operator Q, we have $(Qu)(t) \geq 0$ and

$$\begin{aligned}(Qu)(t) &\leq T(t)x_0 + \int_0^t T(t-s)[N_1(s)u(s) + N_2(s)(Ku)(s) + g(s)]ds \\ &\leq T(t)x_0 + \int_0^t T(t-s)[N_1(s)u(s) + N_2(s)(Ku)(s)]ds + \int_0^a T(t-s)g(s)ds.\end{aligned}$$

Since P is a normal cone, we have

$$\begin{aligned}\|(Qu)(t)\|^* &\leq N\left(\|T(t)x_0\|^* + \|(\widetilde{B}u)(t)\|^* + \|\int_0^t T(t-s)g(s)ds\|^*\right) \\ &\leq N\left(M^*\|x_0\|^* + \|\widetilde{B}\|^*\|u\|_C^* + \int_0^a \|T(t-s)\|^*\|g(s)\|^* ds\right) \\ &\leq NM^*\|x_0\|^* + N\alpha r^* + NaM^*\|g\|_C^* \leq r^*.\end{aligned}$$

Thus, we get $Q : B_P(r^*) \to B_P(r^*)$. Let $\widetilde{F} = \overline{\text{co}}Q(B_P(r^*))$. Then $\widetilde{F}$ is a bounded convex closed set in $C(J,E)$ and $Q : \widetilde{F} \to \widetilde{F}$. Similar to the proof of Theorem 3.4, we can prove that Q is a convex-power condensing operator. Thus, by Lemma 2.5, we get the conclusion. □

Furthermore, we have the following results, and the proof is similar to these of Theorem 3.6 and Theorem 3.7, so we omit it here.

Theorem 3.8 *Let P be a normal cone of E, and semigroup $T(t)(t \geq 0)$ generated by $-A$ be an equicontinuous and positive C_0-semigroup, $x_0 \geq \theta$. Assume that* (H4) *holds and $f : J \times P \times P \times P \to P$ satisfies*

(H6) *there exist nonnegative Lebesgue integral functions $L_i \in L(J, R^+)(i = 1,2,3)$ such that for any bounded and equicontinuous sets $D_i \subset C(J,P)(i = 1,2,3)$ and $t \in J$,*

$$\alpha(f(t, D_1(t), D_2(t), D_3(t))) \leq L_1(t)\alpha(D_1) + L_2(t)\alpha(D_2) + L_3(t)\alpha(D_3).$$

Then IVP (1.1) *has at least one mild solution in $C(J,P)$.*

Remark 3.9 Similarly, we can apply Lemma 2.5 to obtain the existence of solutions to the following IVP for nonlinear second order mixed type integro-differential equation in Banach space E

$$\begin{cases} u'' = f\left(t, u, \int_0^t k(t,s)u(s)ds, \int_0^a h(t,s)u(s)ds\right), t \in J, \\ u(0) = x_0, u'(0) = x_1, \end{cases} \tag{3.22}$$

where $x_0, x_1 \in E$, f, k, h are defined as above. It suffices to note that u is the solution to IVP (3.22) if and only if u is a fixed point of the operator equation $u = \widetilde{Q}u$, where

$$(\widetilde{Q}u)(t) = x_0 + tx_1 + \int_0^t (t-s)f\left(s, u(s), \int_0^t k(s,\tau)u(\tau)d\tau, \int_0^a h(s,\tau)u(\tau)d\tau\right) ds.$$

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References

[1] Chang Y.K.: Controllability of impulsive functional differential systems with infinite delay in Banach spaces, Chaos, Solitons & Fractals, 33, 1601-1609 (2007).

[2] Chang Y.K. and Nieto J.: Some new existence results for fractional differential inclusions with boundary conditions, Math. Computer Modelling, 49, 605-609 (2009).

[3] Deimling K.: Nonlinear Functional Analysis, Springer-Verlag, New York, 1985.

[4] Guo D. and Lakshmikantham V.: Nonlinear Problems in Abstract Cones, Academic press, San Diego, 1988.

[5] Guo D., Lakshmikantham V. and Liu X.: Nonlinear Integral Equations in Abstract Cones, Kluwer Academic Publishers, Dordrecht, 1996.

[6] Li Y.: The positive solutions of abstract semilinear evolution equations and their applications (in Chinese), Acta Math. Sinica, 39, 666-672 (1996).

[7] Li Y.: Periodic solutions of abstract semilinear evolution equations in Banach spaces (in Chinese), Acta Math. Sinica, 41, 629-636 (1998).

[8] Li Y.: Existence of solutions of initial value problems for abstract semilinear evolution equations (in Chinese), Acta Math. Sinica, 48, 1089-1094 (2005).

[9] Li Y.: Existence and uniqueness of periodic solution for a class of semilinear evolution equations, J. Math. Anal. Appl., 349, 226-234 (2009).

[10] Liang J., Liu J. and Xiao T.: Nonlocal Cauchy problems governed by compact operator families, Nonlinear Anal., 57, 183-189 (2004).

[11] Liang J., Liu J. and Xiao T.: Nonlocal impulsive problems for nonlinear differential equations in Banach spaces, Math. Comput. Modelling, 49, 798-804 (2009).

[12] Liang J., Liu J. and Xiao T.: Semilinear integrodifferential equations with nonlocal initial conditions, Comput. Math. Appl., 47, 863-875 (2004).

[13] Lin Y. and Liu J.: Semilinear integrodifferential equations with local Cauchy problem, Nonlinear Anal., 26, 1023-1033 (1996).

[14] Liu L.: Existence of global solutions of initial value problem for nonlinear integro-differential equations of mixed type in Banach spaces (in Chinese), J. System Sci. Math., 20, 112-116 (2000).

[15] Liu L., Guo F., Wu C. and Wu Y.: Existence theorems of global solutions of nonlinear Volterra type integral equations in Banach spaces, J. Math. Anal. Appl., 309, 638-649 (2005).

[16] Pazy A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin, 1983.

[17] Sadovskii B.: A fixed point principle, Functional Anal. Appl., 1, 151-153 (1967).

[18] Sun J. and Zhang X.: The fixed point theorem of convex-power condensing operator and applications to abstract semilinear evolution equations (in Chinese), Acta Math. Sinica, 48, 439-446 (2005).

[19] Vasile I.: Fixed Point Theorem, D. Reidel Publishing Press, Dordrecht, 1981.

[20] Zhang X.: Fixed point theorems for a class of nonlinear operator in Banach spaces and applications, Nonlinear Anal., 69, 536-543 (2008).

[21] Zhang G., Zhang T. and Zhang T.: Fixed point theorems of Rothe and Altman types about convex-power condensing operator and application, Appl. Math. Comput., 214, 618-623 (2009).

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