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Conjugate convolution and characterizations of inner amenable locally compact groups

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Abstract

For locally compact group G, we give some characterizations of inner amenability of G by conjugate convolution operations. Moreover, we study multiples of positive elements in group algebra $L^1(G)$, whenever G is inner amenable.

Key Words: Conjugate convolution; inner amenable; locally compact group; positive element.

1. Introduction

Let G be a locally compact group with identity e and a fixed left Haar measure λ. Let $L^\infty(G)$ and $L^1(G)$ be the usual Lebesgue spaces with respect to λ as defined in [2]. A linear functional m on $L^\infty(G)$ is called a *mean* if it is positive on $L^\infty(G)$ and $m(1) = 1$; this is equivalent to that $\|m\| = m(1) = 1$. A mean m on $L^\infty(G)$ is called *inner invariant* if $m({}_{x^{-1}}f_x) = m(f)$ for all $x \in G$ and $f \in L^\infty(G)$, where ${}_{x^{-1}}f_x(y) = f(x^{-1}yx)$ for all $y \in G$. A locally compact group G is called *inner amenable* if there exists an inner invariant mean on $L^\infty(G)$.

The study of inner amenability is initiated by Effros [1], for discrete groups. For more details on inner amenability of locally compact groups the interested reader is referred to [4], [5], [7], [11], [8], [13], [14], [10], and [6].

For functions $f, g \in L^1(G)$, define the conjugate convolution $\circledast$ in $L^1(G)$ by

$$f \circledast g(x) = \int_G f(y)\Delta(y)g(y^{-1}xy)\, dy,$$

where $x, y \in G$ and Δ is the modular function of G. The concept of conjugate convolution was introduced and studied by C. K. Yuan in [14]. In the same paper he gave several characterizations of inner amenable groups in terms conjugate convolution operations.

In this paper, we give some new characterizations of inner amenability of a locally compact group G in terms of conjugate convolution operations. Moreover, we study multiples of positive elements in group algebra $L^1(G)$ of an inner amenable locally compact group G.

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2. The results

Let G be a locally compact group and $f, g \in L^1(G)$. We note that $f \circledast g \in L^1(G)$ and $\|f \circledast g\|_1 \leq \|f\|_1 \|g\|_1$. Also, we have

$$\int_G f \circledast g(x)\, dx = (\int_G f(x)\, dx)(\int_G g(x)\, dx).$$

This implies that $(f \circledast g)(1) = f(1)g(1)$, where 1 denotes the identity of the W^*-algebra $L^\infty(G)$; see [13].

For any $f \in L^1(G)$ and $\varphi \in L^\infty(G)$ define the function $\varphi \circledcirc f$ on G by

$$\varphi \circledcirc f(x) = \int_G \varphi(y^{-1}xy) f(y)\, dy.$$

We note that $\varphi \circledcirc f \in L^\infty(G)$ for any $f \in L^1(G)$ and $\varphi \in L^\infty(G)$ and $\|\varphi \circledcirc f\|_\infty \leq \|\varphi\|_\infty\, \|f\|_1$. For all $f \in L^1(G)$ and $m \in L^\infty(G)^*$ define the functional $f.m \in L^\infty(G)^*$ by

$$\langle f.m, \varphi\rangle = \langle m, \varphi \circledcirc f\rangle \qquad (\varphi \in L^\infty(G)).$$

We begin this section with the following lemma. Before, let $P(G)$ denote the set all positive functionals $f \in L^1(G)$ with norm one. It is clear that $f \circledast g \in P(G)$ whenever $f, g \in P(G)$.

Lemma 2.1 *A locally compact group G is inner amenable if and only if there is a non-zero element m of $L^\infty(G)^*$ such that $m(\varphi \circledcirc f) = m(\varphi)$ for all $f \in P(G)$ and $\varphi \in L^\infty(G)$.*

Proof. The *"only if"* part is trivial. To prove the converse we may assume that m is self adjoint. So there exist unique positive elements m^+ and m^- on $L^\infty(G)$ such that $m = m^+ - m^-$ and $\|m\| = \|m^+\| + \|m^-\|$ (see [9], 1.14.3). Let $f \in P(G)$, then $f \cdot m = f \cdot m^+ - f \cdot m^-$. Let φ be a positive element of $L^\infty(G)$, then clearly $\varphi \circledcirc f \geq 0$, and so $f \cdot m^+$ and $f \cdot m^-$ are positive. Thus

$$\|f \cdot m^+\| + \|f \cdot m^-\| = (f \cdot m^+)(1) - (f \cdot m^-)(1) = \|m\|.$$

This implies that $f \cdot m^+ = m^+$ and $f \cdot m^- = m^-$ (see [9], 1.14.3). So if $m^+ \neq 0$ (say), then $n = \|m^+\|^{-1} m^+$ is a mean on $L^\infty(G)$ and $n(\varphi \circledcirc f) = n(\varphi)$ for all $f \in P(G)$ and $\varphi \in L^\infty(G)$. Hence G is inner amenable by Proposition 1.10 of [10]. □

Let $N(G)$ be the set of all $\varphi \in L^\infty(G)$ such that

$$\inf\{\|\varphi \circledcirc f\|_\infty : f \in P(G)\} = 0.$$

It is easy to see that $N(G)$ is closed under scalar multiplication. Also, let for $I_1, I_2 \subseteq P(G)$,

$$d(I_1, I_2) = \inf\{\|f_1 - f_2\|_1 : f_1 \in I_1 \text{ and } f_2 \in I_2\}.$$

Let A be a subset of $L^1(G)$. We say that $I \subseteq A$ is a right conjugate ideal of A if $f \circledast g \in I$ for all $f \in I$ and $g \in A$.

Theorem 2.2 *Let G be a locally compact group. Then the following are equivalent:*

(a) *G is inner amenable.*

(b) *For any two right conjugate ideals I_1, I_2 of $P(G)$, $d(I_1,\ I_2) = 0$.*

(c) *$N(G)$ is closed under addition.*

Proof. (a) $\Longrightarrow$ (b). Let G is inner amenable, then by Corollary 1.12 of [10] there exists a net (f_α) in $P(G)$ such that $\|f \circledast f_\alpha - f_\alpha\|_1 \longrightarrow 0$ for all $f \in P(G)$. This implies that $\|f_1 \circledast f_\alpha - f_2 \circledast f_\alpha\|_1 \longrightarrow 0$ for all $f_1 \in I_1$ and $f_2 \in I_2$.

(b) $\Longrightarrow$ (c). Let $\varepsilon > 0$. For any $\varphi_1, \varphi_2 \in N(G)$, there are $f_1, f_2 \in P(G)$ such that $\|\varphi_1 \circledcirc f_1\|_\infty < \varepsilon$ and $\|\varphi_2 \circledcirc f_2\|_\infty < \varepsilon$. Also, there are $g_1, g_2 \in P(G)$ such that $\|f_1 \circledast g_1 - f_2 \circledast g_2\|_\infty < \varepsilon$. Now we have

$$\begin{aligned}\|(\varphi_1 + \varphi_2) \circledcirc (f_1 \circledast g_1)\|_\infty &\le \|\varphi_1 \circledcirc (f_1 \circledast g_1)\|_\infty \\ &+ \|\varphi_2 \circledcirc (f_1 \circledast g_1) - \varphi_2 \circledcirc (f_2 \circledast g_2)\|_\infty \\ &+ \|\varphi_2 \circledcirc (f_2 \circledast g_2)\|_\infty \\ &< \varepsilon(2 + \|\varphi_2\|_\infty).\end{aligned}$$

This proves the validity of (c).

(c) $\Longrightarrow$ (a). Let (c) holds, then $N(G)$ is a subspace of $L^\infty(G)$. We note that $\varphi \circledcirc f - \varphi \in N(G)$ for all $f \in P(G)$ and $\varphi \in L^\infty(G)$. In fact, let $\varphi_n = 1/n\Sigma_{i=1}^n f^i \quad (n \in \mathbb{N})$, where f^i denotes $f * f * ... * f$ (i-times). Clearly $\varphi_n \in P(G)$. Since $(\varphi \circledcirc f) \circledcirc g = \varphi \circledcirc (f * g)$ for all $f, g \in L^1(G)$ and $\varphi \in L^\infty(G)$, it is easy to see that $\|(\varphi \circledcirc f - \varphi) \circledcirc \varphi_n\|_\infty \longrightarrow 0$, and so $\varphi \circledcirc f - \varphi \in N(G)$. Let E be the set of all self -adjoint elements in $L^\infty(G)$. Then E is a real vector subspace of $L^\infty(G)$. Let

$$K = \{x \in E : \inf\{f(x); f \in P(G)\} > 0\}.$$

The K is open in E, $1 \in K$ and clearly $K \cap N(G) = \phi$. By the Hahn-Banach theorem, there exits a continuous real linear functional n on E such that $n(1) = 1$ and $n(x) = 0$ for all $x \in E \cap N(G)$. In particular, $n(\varphi \circledcirc f) = n(\varphi)$ for all $f \in P(G)$ and $x \in E$. Now, define $m \in L^\infty(G)^*$ by

$$m(a + ib) = n(a) + in(b) \qquad (a, b \in E).$$

Clearly $m(1) = 1$ and $m(\varphi \circledcirc f) = m(\varphi)$ for all $f \in P(G)$ and $\varphi \in L^\infty(G)$. Thus G is inner amenable, by Lemma 2.1 and Lemma 1.9 of [10]. □

In the sequel, let

$$I_0(G) = \{f \in L^1(G) : \int_G f(y)\, dy = f(1) = 0\}.$$

It is clear that for $f \in I_0(G)$ and $g \in L^1(G)$ we have $f \circledast g \in I_0(G)$.

Theorem 2.3 *Let G be a locally compact group. Then the following are equivalent:*

(a) *G is inner amenable.*

(b) *There is a net $(e_\alpha) \subseteq P(G)$ such that $\|f \circledast e_\alpha\|_1 \longrightarrow |f(1)|$ for each $f \in L^1(G)$.*

(c) *Let $\varepsilon > 0$. Then for any $f \in I_0(G)$, there is $g \in P(G)$ such that $\|f \circledast g\|_1 < \varepsilon$.*

Proof. Let G be inner amenable, then by Corollary 1.12 of [10] there exists a net (f_α) in $P(G)$ such that $\|f \circledast f_\alpha - f_\alpha\|_1 \longrightarrow 0$ for for all $f \in P(G)$. Let $f \in L^1(G)$, then $f = \sum_{i=1}^n \lambda_i f_i$, where $f_i \in P(G)$ and $\lambda_i \in \mathbb{C}$ $(1 \leq i \leq n)$. We have $|f(1)| = |\sum_{i=1}^n \lambda_i|$. We may assume that $\lambda_i \neq 0$. For $\varepsilon > 0$, there is α_0 such that $\|f_i \circledast f_\alpha - f_\alpha\|_1 < \varepsilon/n|\lambda_i|$ for all $\alpha \geq \alpha_0$. Thus for all $\alpha \geq \alpha_0$

$$\begin{aligned} \|f \circledast f_\alpha\|_1 &\leq \|\sum_{i=1}^n \lambda_i f_i \circledast f_\alpha - \sum_{i=1}^n \lambda_i f_\alpha\|_1 + \|\sum_{i=1}^n \lambda_i\|_1 \\ &\leq \sum_{i=1}^n |\lambda_i| \|f_i \circledast f_\alpha - f_\alpha\|_1 + |\sum_{i=1}^n \lambda_i| \leq \varepsilon + |f(1)|. \end{aligned}$$

On the other hand

$$|f(1)| = |f(1) f_\alpha(1)| = |(f \circledast f_\alpha)(1)| \leq \|f \circledast f_\alpha\|_1.$$

Hence for all $\alpha \geq \alpha_0$

$$|\ |f(1)| - \|f \circledast f_\alpha\|_1\ | = \|f \circledast f_\alpha\|_1 - |f(1)| < \varepsilon.$$

Thus (b) is proved.

Clearly (b)$\Longrightarrow$(c). For (c)$\Longrightarrow$ (a), let $\varepsilon > 0$, and $f_0 \in P(G)$ be fixed, and $T = \{f_1, ..., f_k\}$ be a finite subset of $P(G)$. Since $g_1 = f_1 \circledast f_0 - f_0 \in I_0(G)$, there is $h_1 \in P(G)$ such that $\|g_1 \circledast h_1\|_1 < \varepsilon$. Now, let

$$g_2 = f_2 \circledast (f_0 \circledast h_1) - f_0 \circledast h_1.$$

Clearly $g_2 \in I_0(G)$, and so we may find $h_2 \in P(G)$ such that $\|g_2 \circledast h_2\|_1 < \varepsilon$. Inductively we may fined $h_i \in P(G)$ such that $\|g_i \circledast h_i\|_1 < \varepsilon$, where

$$g_i = f_i \circledast (f_0 \circledast (h_1 \circledast (...h_{i-1})...))) - (h_1 \circledast (...h_{i-1})...))).$$

Let $h_{(T,\varepsilon)} = f_0 \circledast (h_1 \circledast (...h_i)...)))$. Then

$$\|f \circledast h_{(T,\varepsilon)} - h_{(T,\varepsilon)}\|_1 < \varepsilon \quad (f \in T).$$

So we may find a net (f_α) in $P(G)$ such that $\|f \circledast f_\alpha - f_\alpha\|_1 \longrightarrow 0$ for for all $f \in P(G)$. Hence G is inner amenable. □

Corollary 2.4 *Let G be a locally compact group. Then G is inner amenable if and only if for any $f \in L^1(G)$ we have*

$$|f(1)| = \inf\{\|f \circledast g\|_1 : g \in P_1(G)\}.$$

Recall that a locally compact group G is called *amenable* if there is a mean m on $L^\infty(G)$ such that $m({}_x\varphi) = m(\varphi)$ for all $x \in G$ and $\varphi \in L^\infty(G)$.

Corollary 2.5 *Let G be a locally compact group such that*

$$\inf\{\|f \circledast g\|_1 : g \in P(G)\} = \inf\{\|f * g\|_1 : g \in P(G)\} \quad (f \in L^1(G)).$$

Then G is inner amenable if and only if is amenable.

Proof. It is well-known from Corollary 4.8 of [3] that a locally compact G is amenable if and only if $|f(1)| = \inf\{\|f * g\|_1 : g \in P(G)\}$. So the results follows from Corollary 2.4. □

We end this section by a characterization of multiples of positive element in $L^1(G)$ of an inner amenable locally compact group. Before stating the following result, let us recall that for any $f \in L^1(G)$, $|f|$ denote the absolute value of f as an element on $L^\infty(G)$ (see [12], p. 134).

Proposition 2.6 *Let G be an inner amenable group. Then f is a scaler multiple of an element of $L^1(G)$, if $|f \circledast g| = |f| \circledast g$ for all $g \in P(G)$.*

Proof. Since G is inner amenable, from Corollary 2.4, it follows that

$$|f(1)| = \inf\{\|f \circledast g\|_1 : g \in P(G)\}$$

for all $g \in L^1(G)$. Let $\varepsilon > 0$, then there is $g \in P(G)$ such that

$$|f(1)| + \varepsilon > \|f \circledast g\|_1 = \||f| \circledast g\|_1 = \|f\|_1\|g\|_1 = \|f\|_1.$$

So $\|f\|_1 = |f(1)|$. Let $g = f/f(1)$. Then $g(1) = 1$, $\|g\|_1 = 1$ and $f = f(1)g$, as required. □

let $\mathbf{C}P(G)$ denote the set of all scalar multiples of elements in $P(G)$, and note that

$$\mathbf{C}P(G) = \{f \in L^1(G) : |f(1)| = \|f\| \}.$$

Let us remark that clearly for any locally compact group G we have

$$\mathbf{C}P(G) \subseteq \{f \in L^1(G) : |f \circledast g| = |f| \circledast g \text{ for all } g \in P(G) \}$$

This together with Proposition 2.6 imply that

$$\mathbb{C}\, P(G) = \{f \in L^1(G) : |f \circledast g| = |f| \circledast\ g \text{ for all } g \in P(G) \}$$

if G is an inner amenable locally compact group.

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