

A Beurling-type theorem in Bergman spaces

Ali Abkar

Abstract

It is known that Beurling's theorem concerning invariant subspaces is not true in the Bergman space (in contrast to the Hardy space case). However, Aleman, Richter, and Sundberg proved that every cyclic invariant subspace in the Bergman space $L_a^p(\mathbb{D})$, $0 < p < +\infty$, is generated by its extremal function (see [3]). This implies, in particular, that for every zero-based invariant subspace in the Bergman space the Beurling's theorem stands true. Here, we shall supply an alternative proof for this latter statement; our short proof is more direct and closely related to Hedenmalm's original approach to the problem.

Key Words: Bergman space, Beurling's theorem, extremal function, invariant subspace, cyclic subspace, zero-based subspace

1. Introduction

Let \mathbb{D} denote the open unit disk in the complex plane. The Bergman space $L_a^p(\mathbb{D})$ is the space of all holomorphic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ for which

$$\|f\|_{L_a^p(\mathbb{D})}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) < +\infty,$$

where $dA(z) = \pi^{-1} dx dy$ is the normalized area measure. It is well-known that for $1 \leq p < +\infty$, the Bergman space $L_a^p(\mathbb{D})$ is a Banach space, and for $0 < p < 1$, it is a complete metric space.

A closed subspace $M \subset L_a^p(\mathbb{D})$ is said to be *invariant* if $zM \subset M$. A sequence $\Lambda \subset \mathbb{D}$ is said to be a *zero sequence* if there exists a nonzero function $f \in L_a^p(\mathbb{D})$ such that f vanishes precisely on Λ . An invariant subspace of the form

$$M = \{f \in L_a^p(\mathbb{D}) : f(z) = 0, z \in \Lambda\}$$

is called a *zero-based* invariant subspace. For a function $f \in L_a^p(\mathbb{D})$, the closure in $L_a^p(\mathbb{D})$ of all polynomial multiples of f is an invariant subspace which is denoted by $[f]$; this subspace is also known as the invariant subspace generated by f . An invariant subspace M is said to be *cyclic* if $M = [f]$ for some $f \in L_a^p(\mathbb{D})$. It is known that every zero-based invariant subspace is cyclic; this follows from Proposition 5.4 in [3]. For an

invariant subspace M , we consider the extremal problem

$$\sup \left\{ \operatorname{Re} f^{(j)}(0) : f \in M, \|f\|_{L_a^p(\mathbb{D})} \leq 1 \right\},$$

where j is the multiplicity of the common zero at the origin of all the functions in M . The solution to this problem is called the *extremal function* for M and is denoted by G . This problem was first introduced by Hedenmalm in [5], [6] for the case $p = 2$, and subsequently by Peter Duren et al. in [8] and [9] for $0 < p < +\infty$. In the context of the Hardy spaces, Beurling's theorem says that every invariant subspace other than the trivial one, $\{0\}$, is generated by an inner function (which is an extremal function in that context); in other words, every invariant subspace of the Hardy space is cyclic. On the other hand, the invariant subspaces of the Bergman space $L_a^2(\mathbb{D})$ need not be singly generated. Nevertheless, for the Bergman space $L_a^2(\mathbb{D})$, a Beurling-type theorem holds true; every invariant subspace M is generated by $M \ominus zM$; that is,

$$M = [M \ominus zM] = [M \cap (zM)^\perp].$$

A very technical proof for this result was presented in the fundamental paper [3], and the problem is still open for the case $p \neq 2$ (see also [4], page 272). Using this theorem the authors, among other things, deduced that every zero-based invariant subspace in $L_a^2(\mathbb{D})$ is generated by its extremal function; meaning that for zero-based invariant subspaces Beurling's theorem is true (in contrast to Beurling-type theorem for general invariant subspaces). Later on, the current author in [2] supplied an alternative proof for this statement that makes no appeal to the Beurling-type theorem established by the above-named authors. This work was appreciated by Peter Duren and Alexander Schuster in page 273 of their recent monograph [4]. In this way the current author was encouraged to examine the proof already presented for the case $p = 2$, and realized that the argument can be modified to incorporate the other values of $0 < p < +\infty$. We shall prove that every zero-based invariant subspace of $L_a^p(\mathbb{D})$ is generated by its extremal function. Our proof uses the density of the polynomials in some weighted Bergman spaces, a pattern closely related to Hedenmalm's original approach to the problem (see [5], [6] and [10]).

2. Beurling's theorem

Let $\Delta = \frac{1}{4}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ stand for the Laplace operator in the complex plane; so that with this notation we have $\Delta|f|^2 = |f'|^2$ and

$$\Delta|f|^p = \frac{p^2}{4}|f|^{p-2}|f'|^2.$$

Let M be a zero-based invariant subspace in $L_a^p(\mathbb{D})$ and let G be its extremal function. It was shown by Hedenmalm for $p = 2$ and by the authors of [8], [9] and [10] for arbitrary values of $0 < p < +\infty$ that G satisfies the equation

$$\Delta\Phi(z) = |G(z)|^p - 1, \quad z \in \mathbb{D},$$

where Φ is a C^∞ function in \mathbb{D} , it is continuous on $\overline{\mathbb{D}}$ and vanishes on the boundary of the unit disk (in a weak sense). Moreover, Φ satisfies the inequalities

$$0 \leq \Phi(z) \leq 1 - |z|^2.$$

In an attempt to the study of invariant subspaces of the Bergman spaces, H. Hedenmalm then introduced the space

$$\mathcal{A}^p = \left\{ f \in L_a^p(\mathbb{D}) : \int_{\mathbb{D}} \Phi(z) \Delta |f(z)|^p dA(z) < +\infty \right\},$$

for $0 < p < \infty$. For $f \in \mathcal{A}^p$ He defined

$$\|f\|_{\mathcal{A}^p}^p = \|f\|_{L_a^p(\mathbb{D})}^p + \int_{\mathbb{D}} \Phi(z) \Delta |f(z)|^p dA(z).$$

It can be proved that for $1 \leq p < +\infty$, the set \mathcal{A}^p is a vector space and has a norm; moreover for $0 < p < 1$, it enjoys the induced metric

$$d(f, g) = \|f - g\|_{L_a^p(\mathbb{D})}^p + \int_{\mathbb{D}} \Phi(z) \Delta |(f - g)(z)|^p dA(z).$$

Let \mathcal{A}_0^p denote the closure of the polynomials in \mathcal{A}^p (with respect to the norm or metric defined above). It was shown in [6] for $p = 2$ and in [10], Corollary 3, for $p \neq 2$ that $[G] = G \cdot \mathcal{A}_0^p$ and that

$$\|Gf\|_{L_a^p(\mathbb{D})}^p = \|f\|_{L_a^p(\mathbb{D})}^p + \int_{\mathbb{D}} \Phi(z) \Delta |f(z)|^p dA(z), \quad f \in \mathcal{A}_0^p.$$

Moreover, the authors in page 319 of [10] left the following question open: *is $\mathcal{A}^p = \mathcal{A}_0^p$?* (see also the conjecture following the question). It is clear that $[G] \subset M$, and it was already observed that $M \subset G \cdot \mathcal{A}^p$ (see [6], or Corollary 4 in [10]). Therefore, if we know that $\mathcal{A}^p = \mathcal{A}_0^p$, it follows that Beurling's theorem is true for M , because

$$M \subset G \cdot \mathcal{A}^p = G \cdot \mathcal{A}_0^p = [G].$$

It is now time to state and prove the main result of this paper.

Theorem 2.1 *Let M be a zero-based invariant subspace of $L_a^p(\mathbb{D})$, $0 < p < +\infty$. Then M is generated by its extremal function G , that is $M = [G]$.*

Proof. We have already mentioned that it suffices to show $\mathcal{A}^p = \mathcal{A}_0^p$. Let $f \in \mathcal{A}^p$, $0 < r < 1$, and consider the dilated functions $f_r(z) = f(rz)$. Since every f_r can be approximated uniformly by the polynomials, it is enough to show that $\|f_r - f\|_{\mathcal{A}^p} \rightarrow 0$, as $r \rightarrow 1^-$. To do so, we first note that

$$\|f_r\|_{\mathcal{A}^p}^p = \|f_r\|_{L_a^p(\mathbb{D})}^p + \int_{\mathbb{D}} \Phi(z) \Delta |f_r(z)|^p dA(z).$$

But

$$\begin{aligned} \|f_r\|_{L_a^p(\mathbb{D})}^p &= \int_{\mathbb{D}} |f_r(z)|^p dA(z) \\ &= \int_{r\mathbb{D}} |f(z)|^p \frac{dA(z)}{r^2} \\ &= \frac{1}{r^2} \int_{r\mathbb{D}} |f(z)|^p dA(z). \end{aligned}$$

Therefore

$$\lim_{r \rightarrow 1^-} \|f_r\|_{L_a^p(\mathbb{D})}^p = \int_{\mathbb{D}} |f(z)|^p dA(z) = \|f\|_{L_a^p(\mathbb{D})}^p. \tag{2.1}$$

We now manage to show that

$$\lim_{r \rightarrow 1^-} \int_{\mathbb{D}} \Phi(z) \Delta |f_r(z)|^p dA(z) = \int_{\mathbb{D}} \Phi(z) \Delta |f(z)|^p dA(z).$$

Recall that

$$\begin{aligned} \Delta^2 \Phi(z) &= \Delta (|G(z)|^p - 1) \\ &= \frac{p^2}{4} |G(z)|^{p-2} |G'(z)|^2 \\ &\geq 0, \end{aligned}$$

that is, Φ is a superbiharmonic function in the unit disk. Moreover,

$$0 \leq \Phi(z) \leq 1 - |z|^2 \leq 2(1 - |z|), \quad z \in \mathbb{D}.$$

It is now time to recall the following statement from [2].

PROPOSITION: Let $w : \mathbb{D} \rightarrow \mathbb{R}$ be a superbiharmonic function satisfying the condition $0 \leq w(z) \leq C(1 - |z|)$.

Then for $|z| < r < 1$ the function

$$r \mapsto r\Phi\left(\frac{z}{r}\right)$$

is increasing.

PROOF OF PROPOSITION: See Proposition 3.3 in [2].

Note also that

$$\Delta |f_r(z)|^p = r^2 \Delta |f|^p(rz),$$

so that by a change of variables we obtain, in view of the above proposition and the monotone convergence theorem, that

$$\begin{aligned} \lim_{r \rightarrow 1^-} \int_{\mathbb{D}} \Phi(z) \Delta |f_r(z)|^p dA(z) &= \lim_{r \rightarrow 1^-} \frac{1}{r} \int_{|z| < r} r\Phi\left(\frac{z}{r}\right) \Delta |f(z)|^p dA(z) \\ &= \int_{\mathbb{D}} \Phi(z) \Delta |f(z)|^p dA(z). \end{aligned}$$

This together with (2.1) shows that

$$\begin{aligned} \lim_{r \rightarrow 1^-} \|f_r\|_{\mathcal{A}^p}^p &= \|f\|_{L_a^p(\mathbb{D})}^p + \int_{\mathbb{D}} \Phi(z) \Delta |f(z)|^p dA(z) \\ &= \|f\|_{\mathcal{A}^p}^p. \end{aligned}$$

We now have to resort to the following useful lemma from Real Analysis.

LEMMA: Let μ be a finite positive measure on X , and $0 < p < +\infty$. Let f_n and f be measurable functions such that

$$\limsup_{n \rightarrow \infty} \int_X |f_n|^p d\mu \leq \int_X |f|^p d\mu < +\infty,$$

and $f_n \rightarrow f$, $[\mu]$ a.e. Then $f_n \rightarrow f$ in $L^p(X)$.

PROOF OF LEMMA: See page 66 of [7].

Since $f_r \rightarrow f$ pointwise as $r \rightarrow 1^-$, it follows from the above Lemma that

$$\|f_r - f\|_{\mathcal{A}^p} \rightarrow 0, \quad r \rightarrow 1^-,$$

from which the theorem follows. □

Concluding Remark

The stronger result that every cyclic invariant subspace in the Bergman space $L_a^p(\mathbb{D})$ is generated by its extremal function was proved in [3]. The proof essentially consists of two parts. First, the generator of such an invariant subspace lies in the space $G \cdot \mathcal{A}^p$; second, the polynomials are dense in \mathcal{A}^p . In the proof of part one, the authors use a specific technical argument, while for the second part they show that the functions f_r are uniformly norm bounded in the space \mathcal{A}^p . Then arguing separately for the Banach space case $p \geq 1$, and the metric space case $0 < p < 1$, they use some standard functional analysis to settle the problem. In contrast, the short proof presented here can be applied to arbitrary values of $0 < p < +\infty$. In other words, for cyclic invariant subspaces our argument simplifies the second part of the proof given by the authors of [3], while for the zero-based invariant subspaces it is an alternative proof.

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Ali ABKAR
Department of Mathematics,
Imam Khomeini International University,
Qazvin 34194, IRAN
e-mail: abkar@ikiu.ac.ir

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