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Top generalized local cohomology modules

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Abstract

Let $(R,\mathfrak{m})$ be a commutative Noetherian local ring and M, N two non-zero finitely generated R-modules with $\mathrm{pd}(M)=n<\infty$ and $\dim(N)=d$. In this paper, we show that if the top generalized local cohomology module $H^{n+d}_{\mathfrak{m}}(M,N)\neq 0$, then the following statements are equivalent:

(i) $\mathrm{Ann}(0:_{H^d_{\mathfrak{m}}(N)}\mathfrak{p})=\mathfrak{p}$ for all $\mathfrak{p}\in \mathrm{Var}(\mathrm{Ann}(H^d_{\mathfrak{m}}(N)))$;

(ii) $\mathrm{Ann}(0:_{H^{n+d}_{\mathfrak{m}}(M,N)}\mathfrak{p})=\mathfrak{p}$ for all $\mathfrak{p}\in \mathrm{Var}(\mathrm{Ann}(H^{n+d}_{\mathfrak{m}}(M,N)))$.

Key Words: Artinian module, Generalized local cohomology.

1. Introduction

Throughout this paper, we assume that $(R,\mathfrak{m})$ is a commutative Noetherian local ring and M, N two non-zero finitely generated R-modules with $\mathrm{pd}(M)=n<\infty$ and $\dim(N)=d$. For each $i\geq 0$, the generalized local cohomology module

$$H^i_{\mathfrak{m}}(M,N)=\varinjlim_n \mathrm{Ext}^i_R(M/\mathfrak{m}^nM,N),$$

was introduced by Herzog [8] and studied further by Suzuki [16]. With $M=R$, one clearly obtains the ordinary local cohomology module which was introduced by Grothendieck; see for example [4]. There are several well known properties concerning the generalized local cohomology modules. It is well known that $H^{n+d}_{\mathfrak{m}}(M,N)$ is Artinian and $H^i_{\mathfrak{m}}(M,N)=0$ for all $i>n+d$; see for example [12] and [14].

An elementary property of finitely generated modules is that $\mathrm{Ann}(N/\mathfrak{p}N)=\mathfrak{p}$ for all $\mathfrak{p}\in\mathrm{Var}(\mathrm{Ann}(N))$. For any Artinian R-module A, the dual property is as follows:

$$\mathrm{Ann}(0:_A\mathfrak{p})=\mathfrak{p}\quad \text{for all } \mathfrak{p}\in\mathrm{Var}(\mathrm{Ann}(A)). \qquad (*)$$

If R is complete with respect to $\mathfrak{m}$-adic topology, it follows by Matlis duality that the property $(*)$ is satisfied for all Artinian R-modules. However, there are Artinian modules which do not satisfy this property. For example, let R be the Noetherian local domain of dimension 2 constructed by Ferrand and Raynaud [7]

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such that its $\mathfrak{m}$-adic completion $\hat{R}$ has an associated prime $\hat{q}$ of dimension 1. Then the Artinian R-module $A = H^1_{\mathfrak{m}}(R)$ does not satisfy the property $(*)$; cf. Cuong and Nhan [6]. However, it seems to us that the property $(*)$ is an important property of Artinian modules. For example, the property $(*)$ is closely related to some questions on dimension for Artinian modules. In [6], it shown that $\text{N-dim}(A) = \dim(R/\text{Ann}(A))$ provided A satisfies the property $(*)$, where $\text{N-dim}(A)$ is the Noetherian dimension of A defined by Roberts [15] (see also [9]). Note that this equality does not hold in general. Concretely, with the Artinian R-module $A = H^1_{\mathfrak{m}}(R)$ as above, $\text{N-dim}(A) = 1 < 2 = \dim(R/\text{Ann}(A))$ although the top local cohomology module $H^2_{\mathfrak{m}}(R)$ satisfies the property $(*)$. Notice that the property $(*)$ has been studied by many authors (see, for example, [5, 2, 3]).

The purpose of this paper is to prove the following theorem.

Theorem 1.1 *Let the generalized local cohomology module* $H^{n+d}_{\mathfrak{m}}(M,N) \neq 0$. *Then the following statements are equivalent:*

(i) $\text{Ann}(0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{p}) = \mathfrak{p}$ *for all* $\mathfrak{p} \in \text{Var}(\text{Ann}(H^d_{\mathfrak{m}}(N)))$;

(ii) $\text{Ann}(0 :_{H^{n+d}_{\mathfrak{m}}(M,N)} \mathfrak{p}) = \mathfrak{p}$ *for all* $\mathfrak{p} \in \text{Var}(\text{Ann}(H^{n+d}_{\mathfrak{m}}(M,N)))$.

2. The results

Following Macdonald [10], every Artinian module A has a minimal secondary representation $A = A_1 + \ldots + A_n$, where A_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called the set of *attached prime ideals* of A, and denoted by $\text{Att}(A)$. The *cohomological dimension* of M and N with respect to $\mathfrak{m}$ is defined as

$$\text{cd}(\mathfrak{m}, M, N) = \sup\{i : H^i_{\mathfrak{m}}(M,N) \neq 0\}.$$

The following theorem extends [11, Theorem 2.2].

Theorem 2.1 *Let the generalized local cohomology module* $H^{n+d}_{\mathfrak{m}}(M,N) \neq 0$. *Then* $\text{Att}(H^{n+d}_{\mathfrak{m}}(M,N)) = \{\mathfrak{p} \in \text{Ass}(N) : \dim R/\mathfrak{p} = d\}$. *In particular,* $\text{Att}(H^{n+d}_{\mathfrak{m}}(M,N)) = \text{Att}(H^d_{\mathfrak{m}}(N))$.

Proof. We use induction on d. If $d = 0$, the module N has finite length and so is annihilated by some power of $\mathfrak{m}$. Hence, by [13, Lemma 3.2], $H^n_{\mathfrak{m}}(M,N) \cong \text{Ext}^n_R(M,N)$ and so

$$\text{Att}(H^n_{\mathfrak{m}}(M,N)) = \{\mathfrak{m}\} = \text{Ass}(N) = \{\mathfrak{p} \in \text{Ass}(N) : \dim R/\mathfrak{p} = 0\}.$$

The result has been proved in this case. Suppose, inductively, that $d \geq 1$ and that the result has been proved for non-zero, finitely generated R-modules of dimension $d-1$. Let L be a largest submodule of N with $\dim(L) < d$. (Note that, for two submodules N_1 and N_2 of N, and for any positive integer t, if $\dim(N_1) \leq t$ and $\dim(N_2) \leq t$, then $\dim(N_1 + N_2) \leq t$ and so L is well defined.) Thus by the exact sequence $0 \longrightarrow L \longrightarrow N \longrightarrow N/L \longrightarrow 0$ we have $\dim(N) = \dim(N/L)$. It is easy to prove that N/L has no non-zero submodule K with $\dim(K) < d$ and so $\text{Ass}(N/L) \subseteq \{\mathfrak{p} \in \text{Supp}(N/L) : \dim R/\mathfrak{p} = d\}$. In addition, $\min(\text{Supp}(N/L)) \subseteq \text{Ass}(N/L)$ and $\min(\text{Supp}(N)) \subseteq \text{Ass}(N)$. Therefore $\text{Ass}(N/L) = \{\mathfrak{p} \in \text{Supp}(N/L) : \dim R/\mathfrak{p} = d\} \subseteq \{\mathfrak{p} \in \text{Ass}(N) :$

$\dim R/\mathfrak{p} = d\}$. If $\mathfrak{p} \in \operatorname{Ass}(N)$ and $\dim R/\mathfrak{p} = d$, then $\mathfrak{p} \notin \operatorname{Supp}(L)$, otherwise $\dim R/\mathfrak{p} \leq \dim L < d$. Thus $\mathfrak{p} \in \operatorname{Supp}(N/L)$ and so $\{\mathfrak{p} \in \operatorname{Supp}(N/L) : \dim R/\mathfrak{p} = d\} = \{\mathfrak{p} \in \operatorname{Ass}(N) : \dim R/\mathfrak{p} = d\}$. From the exact sequence

$$H^{n+d}_{\mathfrak{m}}(M, L) \longrightarrow H^{n+d}_{\mathfrak{m}}(M, N) \longrightarrow H^{n+d}_{\mathfrak{m}}(M, N/L) \longrightarrow H^{n+d+1}_{\mathfrak{m}}(M, L),$$

we have $H^{n+d}_{\mathfrak{m}}(M, N) \cong H^{n+d}_{\mathfrak{m}}(M, N/L)$. Hence by this assumption our aim is to show that $\operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N)) = \operatorname{Ass}(N)$. Since $d \geq 1$, from the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}}(N) \longrightarrow N \longrightarrow N/\Gamma_{\mathfrak{m}}(N) \longrightarrow 0,$$

and [13, Lemma 3.2], we get that $H^{n+d}_{\mathfrak{m}}(M, N) \cong H^{n+d}_{\mathfrak{m}}(M, N/\Gamma_{\mathfrak{m}}(N))$. Therefore we can assume that N is $\mathfrak{m}$-torsion free, and so $\operatorname{depth}(N) \geq 1$ by [4, Lemma 2.1.1]. Thus, for each $x \in \mathfrak{m}$ which is a non-zero divisor on N, we have $\operatorname{cd}(\mathfrak{m}, M, N/xN) \leq n + d - 1$, so that $H^{n+d}_{\mathfrak{m}}(M, N/xN) = 0$, and the exact sequence

$$0 \longrightarrow N \xrightarrow{x} N \longrightarrow N/xN \longrightarrow 0$$

yields that $H^{n+d}_{\mathfrak{m}}(M, N) = xH^{n+d}_{\mathfrak{m}}(M, N)$. It therefore follows form [4, Proposition 7.2.11] that $\cup_{q \in \operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M,N))} \mathfrak{q} \subseteq \cup_{\mathfrak{p} \in \operatorname{Ass}(N)} \mathfrak{p}$. Let $\mathfrak{q} \in \operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N))$; it follows the above inclusion relation that $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}(N)$. Since $H^{n+d}_{\mathfrak{m}}(-, -)$ is an R-linear functor, it follows that $\operatorname{Ann}(N) \subseteq \operatorname{Ann}(H^{n+d}_{\mathfrak{m}}(M, N)) \subseteq \mathfrak{q} \subseteq \mathfrak{p}$. As $d = \dim R/\operatorname{Ann}(N) = \dim R/\mathfrak{p}$, it follows that $\mathfrak{q} = \mathfrak{p}$. Therefore $\operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N)) \subseteq \operatorname{Ass}(N)$. To establish the reverse inclusion, let $\mathfrak{p} \in \operatorname{Ass}(N)$, so that $\operatorname{cd}(\mathfrak{m}, M, R/\mathfrak{p}) = n + d$. By the theory of primary decomposition, there exists a $\mathfrak{p}$-primary submodule T of N; thus N/T is a non-zero finitely generated R-module with $\operatorname{Ass}(N/T) = \{\mathfrak{p}\}$. By [1, Theorem B], we get that $n + d = \operatorname{cd}(\mathfrak{m}, M, R/\mathfrak{p}) \leq \operatorname{cd}(\mathfrak{m}, M, N/T) \leq \operatorname{cd}(\mathfrak{m}, M, N) = n + d$ and so $H^{n+d}_{\mathfrak{m}}(M, N/T) \neq 0$. Therefore $\emptyset \neq \operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N/T)) \subseteq \operatorname{Ass}(N/T) = \{\mathfrak{p}\}$ and hence $\operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N/T)) = \{\mathfrak{p}\}$. The exact sequence

$$0 \longrightarrow T \longrightarrow N \longrightarrow N/T \longrightarrow 0$$

induces an epimorphism $H^{n+d}_{\mathfrak{m}}(M, N) \longrightarrow H^{n+d}_{\mathfrak{m}}(M, N/T) \longrightarrow 0$. Hence $\{\mathfrak{p}\} \subseteq \operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N))$ and so $\operatorname{Ass}(N) \subseteq \operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N))$. This complete the proof that $\operatorname{Att}(H^{n+d}_{\mathfrak{m}}(M, N)) = \operatorname{Ass}(N)$. □

The following consequence immediately follows by Theorem 2.1 and [4, Proposition 7.2.11].

Corollary 2.2 *Let the situations be as in Theorem* 2.1. *Then*
$\operatorname{Var}(\operatorname{Ann}(H^{d}_{\mathfrak{m}}(N))) = \operatorname{Var}(\operatorname{Ann}(H^{n+d}_{\mathfrak{m}}(M, N)))$.

Following [5], let $U_N(0)$ be the largest submodule of N of dimension less than d. Note that if $0 = \cap_{\mathfrak{p} \in \operatorname{Ass}(N)} N(\mathfrak{p})$ is a reduced primary decomposition of the zero submodule of N, then $U_N(0) = \cap_{\dim R/\mathfrak{p} = d} N(\mathfrak{p})$. Therefore we have $\operatorname{Ass} N/U_N(0) = \{\mathfrak{p} \in \operatorname{Ass}(N) : \dim R/\mathfrak{p} = d\}$. Hence $\operatorname{Supp} N/U_N(0) = \cup_{\mathfrak{p} \in \operatorname{Ass}(N),\, \dim R/\mathfrak{p} = d} \operatorname{Var}(\mathfrak{p})$. The set $\operatorname{Supp} N/U_N(0)$ is called *the unmixed support* of N and denoted by $\operatorname{Usupp}(N)$.

Corollary 2.3 *Let the situations be as in Theorem* 2.1. *Then*
$\operatorname{Usupp}(N) = \operatorname{Var}(\operatorname{Ann}(H^{n+d}_{\mathfrak{m}}(M, N)))$.

Proof. By [10] the set of all minimal prime ideals containing $\mathrm{Ann}(H^{n+d}_{\mathfrak{m}}(M,N))$ and the set of all minimal elements of $\mathrm{Att}(H^{n+d}_{\mathfrak{m}}(M,N))$ are the same. Therefore $\mathrm{Var}(\mathrm{Ann}(H^{n+d}_{\mathfrak{m}}(M,N)) = \cup_{\mathfrak{p}\in \mathrm{Ass}(N), \dim R/\mathfrak{p}=d} \mathrm{Var}(\mathfrak{p}) = \mathrm{Usupp}(N)$. □

Theorem 2.4 *Let the situations be as in Theorem* 2.1*. Then the following statements are equivalent:*

(i) $\mathrm{Usupp}(N)$ *is catenary;*

(ii) $H^d_{\mathfrak{m}}(N)$ *satisfies the property* $(*)$*;*

(iii) $H^{n+d}_{\mathfrak{m}}(M,N)$ *satisfies the property* $(*)$*.*

Proof. By Corollaries 2.2, 2.3, we have $\mathrm{Var}(\mathrm{Ann}(H^d_{\mathfrak{m}}(N)) = \mathrm{Var}(\mathrm{Ann}(H^{n+d}_{\mathfrak{m}}(M,N)) = \mathrm{Usupp}(N)$ and $\mathrm{Var}(\mathrm{Ann}_{\hat{R}}(H^d_{\mathfrak{m}}(N)) = \mathrm{Var}(\mathrm{Ann}_{\hat{R}}(H^{n+d}_{\mathfrak{m}}(M,N)) = \mathrm{Usupp}_{\hat{R}}(\hat{N})$. Hence the equivalence follows by [5, Proposition 2.2 and Theorem 3.4]. □

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