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Combinatorial results for order-preserving and order-decreasing transformations

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Abstract

Let O_n and $\mathcal{C}_n$ be the semigroup of all order-preserving transformations and of all order-preserving and order-decreasing transformations on the finite set $X_n = \{1, 2, \ldots, n\}$, respectively. Let $\text{Fix}(\alpha) = \{x \in X_n : x\alpha = x\}$ for any transformation α. In this paper, for any $Y \subseteq X_n$, we find the cardinalities of the sets $O_{n,Y} = \{\alpha \in O_n : \text{Fix}(\alpha) = Y\}$ and $\mathcal{C}_{n,Y} = \{\alpha \in \mathcal{C}_n : \text{Fix}(\alpha) = Y\}$. Moreover, we find the numbers of transformations of O_n and $\mathcal{C}_n$ with r fixed points.

Key Words: Order-preserving transformations, order-decreasing transformations, nilpotent, Catalan number

1. Introduction

Consider the finite set $X_n = \{1, 2, \ldots, n\}$ ordered in the standard way. Let T_n be the full transformation semigroup on X_n. We shall call a transformation $\alpha : X_n \to X_n$ order-preserving if $x \leq y$ implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and decreasing (increasing) if $x\alpha \leq x$ $(x\alpha \geq x)$ for all $x \in X_n$. Combinatorial properties of the semigroup O_n of order-preserving transformations on X_n, and of its subsemigroup C_n, which consists of all decreasing and order-preserving transformations have been investigated over the last thirty years. (See, for example [2, 3, 4, 5, 6, 7].)

For $\alpha \in T_n$ we denote $\text{Fix}(\alpha) = \{x \in X_n : x\alpha = x\}$. For $Y \subseteq X_n$ we define

$$O_{n,Y} = \{\alpha \in O_n : \text{Fix}(\alpha) = Y\} \quad \text{and} \quad \mathcal{C}_{n,Y} = \{\alpha \in \mathcal{C}_n : \text{Fix}(\alpha) = Y\}.$$

We write $O_{n,m}$ instead of $O_{n,Y}$ when $Y = \{m\}$. The nth Catalan number C_n is $\frac{1}{n+1}\binom{2n}{n}$ (see, for example [3, 9]).

The numbers of transformations of O_n and $\mathcal{C}_n$ with r fixed points have been computed by Higgins, and Laradji and Umar in [3, 7]. In both [3] and [7], there is no information about the cardinalities of the sets $O_{n,Y} = \{\alpha \in O_n : \text{Fix}(\alpha) = Y\}$ and $\mathcal{C}_{n,Y} = \{\alpha \in \mathcal{C}_n : \text{Fix}(\alpha) = Y\}$ for any non-empty subset Y of X_n. The

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aim of this paper we compute these cardinalities as follows:

$$|O_{n,Y}| = C_{m_1-1}\Big(\prod_{j=2}^{r} C_{m_j-m_{j-1}}\Big)C_{n-m_r}$$

for any $Y=\{m_1,m_2,\ldots,m_r\}$ with $m_1<m_2<\cdots<m_r$, and

$$|\mathcal{C}_{n,Y}| = \Big(\prod_{j=2}^{r-1} C_{m_{j+1}-m_j}\Big)C_{n-m_r+1}$$

for any $Y=\{1,m_2,\ldots,m_r\}$ with $m_1=1<m_2<m_3<\cdots<m_r$. Consequently, we also show that there are $\frac{r}{n}\begin{pmatrix} 2n \\ n+r \end{pmatrix}$ order-preserving transformations in O_n with r fixed points as in [3, 7], and that there are $\frac{r}{2n-r}\begin{pmatrix} 2n-r \\ n \end{pmatrix}$ order-preserving and order-decreasing transformations in $\mathcal{C}_n$ with r fixed points, as in [3].

2. Preliminaries

For any $\alpha \in T_n$ the equivalence relation $\equiv$ on X_n, defined by

$$x \equiv y \text{ if and only if } (\exists r,s \geq 0)\, x\alpha^r = y\alpha^s\,,$$

partitions X_n into orbits $\Omega_1,\Omega_2,\ldots,\Omega_k$. The orbits are the connected components of the function graph, and provide valuable information about the structure of the transformation α. Typically, an orbit consists of a cycle with some trees attached. If there are no attached trees, we say that the orbit Ω_i is cyclic, if the cycle consists of a single fixed point and $|\Omega_i|\geq 2$ we say that Ω_i is *acyclic*; if Ω_i consists of a single fixed point, we say that it is *trivial* (see [1, 3]). The following proposition was proved by Higgins in [3, Proposition 1.5]:

Proposition 1 *Each of the cycles of the components of $\alpha \in O_n$ consists of a unique fixed point. Each orbit of $\alpha \in O_n$ is convex in the ordered set X_n.* □

Since the orbits of $\alpha \in O_n$ are either acyclic or trivial, it follows that $\alpha \in O_n$ has a unique orbit if and only if $\alpha \in O_{n,m}$ for some $m \in X_n$.

A proof for the following result can be found in [3]:

Lemma 2 $\sum_{k=1}^{n} C_{k-1}C_{n-k} = C_n$. □

Let $\mathcal{C}_n^+$ be the semigroups of all increasing and order-preserving full transformations on X_n. Then it is a well-known fact that $\mathcal{C}_n$ and $\mathcal{C}_n^+$ are "isomorphic". Moreover, $|\mathcal{C}_n| = |\mathcal{C}_n^+| = C_n$ (see, for example [3, Theorem 3.1]). We denote the set of all nilpotent element of a semigroup S with zero by $N(S)$. The following results were proved in [6, 7].

Lemma 3 $O_{n,1} = N(\mathcal{C}_n)$, $O_{n,n} = N(\mathcal{C}_n^+)$ *and* $|O_{n,1}| = |O_{n,n}| = C_{n-1}$. □

From [8, Ex 16b, p. 169] since

$$\sum_{k=0}^{n} \frac{ac(p+qk)}{(a+bk)(c+bn-bk)} \binom{a+bk}{k} \binom{c+bn-bk}{n-k}$$

$$= \frac{p(a+c)+aqn}{a+c+bn} \binom{a+c+bn}{n},$$

it follows by replacing a, b, c, n, p and q with $2r, 2, 2, n-r-1, 1$ and 0, and with $r, 2, 2, n-r-2, 1$ and 0, respectively that

$$\sum_{k=0}^{n-r-1} \frac{r}{(r+k)(n-r-k)} \binom{2r+2k}{k} \binom{2n-2r-2k}{n-r-1-k} = \frac{r+1}{n} \binom{2n}{n-r-1} \tag{1}$$

and

$$\sum_{k=0}^{n-r-2} \frac{r}{(r+2k)(n-r-1-k)} \binom{r+2k}{k} \binom{2n-2r-2-2k}{n-r-2-k}$$

$$= \frac{r+2}{2n-(r+2)} \binom{2n-(r+2)}{n-(r+2)}. \tag{2}$$

3. Order-preserving with fixed points

Proposition 4 *Let $\alpha \in O_{n,m}$. Then we have*

(i) *if $1 \le x < m \le n$ then $x+1 \le x\alpha$, and*

(ii) *if $1 \le m < x \le n$ then $x\alpha \le x-1$.*

Proof. **(i)** Let $\alpha \in O_{n,m}$. If $1 \le x < m \le n$ then either $x+1 \le x\alpha$ or $x\alpha \le x-1$. If $x=1$, then it is clear that $1\alpha \ne 1$, and so $2 \le 1\alpha$. Now suppose that $1 < x$, and that $x\alpha \le x-1$. Since $(x-1)\alpha \le x\alpha \le x-1$, it follows that $(x-1)\alpha \le x-2$. Similarly if we continue , then we have the following sequence

$$(x-2)\alpha \le x-3,\ (x-3)\alpha \le x-4,\ \dots\ , 2\alpha \le 1.$$

Thus we have $2\alpha = 1$, and so $1\alpha = 1$ which is a contradiction with $\mathrm{Fix}(\alpha) = \{m\} \ne \{1\}$, and hence $x\alpha \ge x+1$.

(ii) Let $1 \le m < x \le n$. If $x = n$, then it is clear that $n\alpha \ne n$, and so $n\alpha \le n-1$. Now suppose that $x < n$, and that $x\alpha \ge x+1$. Similarly, we have the following sequence

$$(x+1)\alpha \ge x+2,\ (x+2)\alpha \ge x+3,\ \dots\ , (n-1)\alpha \ge n.$$

It follows that $(n-1)\alpha = n$, and so $n\alpha = n$ which is a contradiction with $\mathrm{Fix}(\alpha) = \{m\} \ne \{n\}$, and hence $x\alpha \le x-1$. □

We have the following corollary.

Corollary 5 *For $\alpha \in O_{n,m}$ if $m \ne 1$ then $(m-1)\alpha = m$, and if $m \ne n$ then $(m+1)\alpha = m$.* □

Now consider the first special case $|Y| = 1$:

Lemma 6 *For every $m \in X_n$,*

$$|O_{n,m}| = C_{m-1}C_{n-m}.$$

Proof. For each $\alpha \in O_{n,m}$ we fix

$$\begin{aligned}\alpha_1 &= \begin{pmatrix} 1 & \dots & m-2 & m-1 & m \\ 1\alpha & \dots & (m-2)\alpha & m & m \end{pmatrix} \text{ and} \\ \alpha_2 &= \begin{pmatrix} 1 & 2 & 3 & \dots & n-(m-1) \\ 1 & 1 & (m+2)\alpha-(m-1) & \dots & n\alpha-(m-1) \end{pmatrix}.\end{aligned}$$

It follows from Proposition 4 that $\alpha_1 \in O_{m,m}$ and $\alpha_2 \in O_{n-m+1,1}$. Next consider the function

$$f : O_{n,m} \to O_{m,m} \times O_{n-m+1,1}$$

which maps each $\alpha \in O_{n,m}$ to the ordered pair (α_1, α_2). Then it follows from Corollary 5 that f is a bijection. Moreover, it follows from Lemma 3 that

$$|O_{n,m}| = |O_{m,m}| \cdot |O_{n-m+1,1}| = C_{m-1}C_{n-m},$$

as required. □

Next consider the second special case $|Y| = 2$.

Lemma 7 *If $Y = \{m, m+r\} \subseteq X_n$ $(r \geq 1)$ then $|O_{n,Y}| = C_{m-1}C_{n-m-r}C_r$. In particular, $|O_{n,\{1,n\}}| = C_{n-1}$.*

Proof. Let $Y = \{m, m+r\}$, and let $\alpha \in O_{n,Y}$. By Proposition 1 there exists a unique $0 \leq q \leq r-1$ such that

$$s\alpha \leq m+q \qquad \text{and} \qquad t\alpha \geq m+q+1$$

for all $s \leq m+q$, and for all $t \geq m+q+1$. Then we fix

$$\begin{aligned}\alpha_1 &= \begin{pmatrix} 1 & 2 & \dots & m+q \\ 1\alpha & 2\alpha & \dots & (m+q)\alpha \end{pmatrix} \text{ and} \\ \alpha_2 &= \begin{pmatrix} 1 & 2 & \dots & n-m-q \\ (m+q+1)\alpha-m-q & (m+q+2)\alpha-m-q & \dots & n\alpha-m-q \end{pmatrix}\end{aligned}$$

as above. Then it follows from Proposition 4 that $\alpha_1 \in O_{(m+q),m}$ and $\alpha_2 \in O_{(n-m-q),(r-q)}$. Next consider the function

$$f : O_{n,Y} \to \bigcup_{q=0}^{r-1} (O_{(m+q),m} \times O_{(n-m-q),(r-q)})$$

which maps each $\alpha \in O_{n,Y}$ to the ordered pair (α_1, α_2). Since f is a bijection, it follows from Lemmas 6 and 2 that

$$\begin{aligned} |O_{n,Y}| &= \sum_{q=0}^{r-1} |O_{(m+q),m}| \cdot |O_{(n-m-q),(r-q)}| \\ &= \sum_{q=0}^{r-1} \big(C_{m-1}C_q\big)\big(C_{r-q-1}C_{n-m-r}\big) = C_{m-1}C_{n-m-r}\sum_{q=0}^{r-1} C_qC_{r-q-1} \\ &= C_{m-1}C_{n-m-r}\sum_{q=1}^{r} C_{q-1}C_{r-q} = C_{m-1}C_{n-m-r}C_r, \end{aligned}$$

as required. □

Now we have the following theorem.

Theorem 8 *Let $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \cdots < m_r$ be any subset of X_n. Then*

$$|O_{n,Y}| = \prod_{j=1}^{r+1} C_{k_j},$$

where $k_1 = m_1 - 1$, $k_j = m_j - m_{j-1}$ $(2 \le j \le r)$ and $k_{r+1} = n - m_r$.

Proof. By Lemmas 6 and 7 we suppose that $r \ge 3$. Let $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \cdots < m_r$, and let

$$k_1 = m_1 - 1,\ k_j = m_j - m_{j-1}\ (2 \le j \le r) \text{ and } k_{r+1} = n - m_r.$$

Then, for each $\alpha \in O_{n,Y}$, we fix

$$\begin{aligned} \alpha_1 &= \begin{pmatrix} 1 & \cdots & k_1 & k_1+1 \\ 1\alpha & \cdots & k_1\alpha & k_1+1 \end{pmatrix}, \\ \alpha_j &= \begin{pmatrix} 1 & 2 & \cdots & k_j & k_j+1 \\ 1 & (m_{j-1}+1)\alpha - m_{j-1}+1 & \cdots & (m_j-1)\alpha - m_{j-1}+1 & k_j+1 \end{pmatrix}, \\ \alpha_{r+1} &= \begin{pmatrix} 1 & 2 & \cdots & k_{r+1} & k_{r+1}+1 \\ 1 & (m_r+1)\alpha - m_r+1 & \cdots & (n-1)\alpha - m_r+1 & n\alpha - m_r+1 \end{pmatrix}, \end{aligned}$$

where $2 \le j \le r$. Then it follows from Proposition 4 that $\alpha_1 \in O_{k_1+1,k_1+1}$, $\alpha_j \in O_{k_j+1,\{1,k_j+1\}}$ $(2 \le j \le r)$ and $\alpha_{r+1} \in O_{k_{r+1}+1,1}$. Next, define the set

$$\mathcal{O}^*_{n,Y} = O_{k_1+1,k_1+1} \times O_{k_2+1,\{1,k_2+1\}} \times \cdots \times O_{k_r+1,\{1,k_r+1\}} \times O_{k_{r+1}+1,1}.$$

as the cartesian product of the $r+1$ sets. Now consider the function $f : \mathcal{O}_{n,Y} \to \mathcal{O}^*_{n,Y}$ which maps $\alpha \in \mathcal{O}_{n,Y}$ to the ordered $(r+1)$-pair $(\alpha_1, \alpha_2, \ldots, \alpha_{r+1})$. Since f is a bijection, it follows from Lemmas 3 and 7 that

$$\begin{aligned} |\mathcal{O}_{n,Y}| &= |O_{k_1+1,k_1+1}|\big(\prod_{j=2}^{r} |O_{k_j+1,\{1,k_j+1\}}|\big)|O_{k_{r+1}+1,1}| \\ &= \prod_{j=1}^{r+1} C_{k_j}, \end{aligned}$$

as required. $\square$

For any $r \in X_n$ we define

$$F(n,r) = |\{\alpha \in O_n : |\mathrm{Fix}(\alpha)| = r\}|$$

as the number of order-preserving transformations which have exactly r fixed points. Let $Y = \{m_1, m_2, \ldots, m_r\}$ with $m_1 < m_2 < \cdots < m_r$ be any subset of X_n. Now take $k_1 = m_1$, $k_j = m_j - m_{j-1}$ $(2 \le j \le r)$ and $k_{r+1} = n + 1 - m_r$. Then it is clear that $(k_1, k_2, \ldots, k_{r+1})$ is a positive integer solution of the equation

$$x_1 + x_2 + \cdots + x_{r+1} = n + 1. \tag{3}$$

Conversely, every positive integer solution of Equation (3) gives a subset of X_n with $r+1$ elements. If we denote the set of all positive integer solutions of Equation (3) by $P_{r+1}(n+1)$, then we have

$$F(n,r) = \sum_{(k_1,k_2,\ldots,k_{r+1}) \in P_{r+1}(n+1)} C_{k_1-1} C_{k_2} C_{k_3} \cdots C_{k_r} C_{k_{r+1}-1}.$$

Moreover, we have the following result.

Theorem 9 $F(n,r) = \frac{r}{n} \binom{2n}{n+r}$.

Proof. For this we use induction on r. If $r = 1$ then it follows from Lemmas 6 and 2 that

$$F(n,1) = \sum_{m=1}^{n} |O_{n,m}| = \sum_{m=1}^{n} C_{m-1} C_{n-m} = C_n.$$

Suppose that $\alpha \in O_n$ has $r+1$ fixed points, say $m_1 < \cdots < m_r < m_{r+1}$. Then consider the orbit of α which contains m_{r+1}. Since, by Proposition 1, this orbit is convex, there exists a unique $m_r < k \le m_{r+1}$ such that the restricted transformation $\alpha_{|Y_k} : Y_k = \{k, k+1, \ldots, n\} \to Y_k$ of α has unique fixed point, and that the restricted transformation $\alpha_{|X_n - Y_k}$ has r fixed points. Similarly, the transformations $\alpha_{|Y_k} : Y_k \to Y_k$ with a unique fixed point can be put into one-to-one correspondence with $\beta : X_{n-k+1} \to X_{n-k+1}$ with a unique fixed point. Since the number of such transformations is C_{n-k+1}, and since $k \in \{r+1, \ldots, n\}$, it follows from the inductive hypothesis that

$$F(n, r+1) = \sum_{k=r+1}^{n} F(k-1, r) C_{n-k+1} = \sum_{k=0}^{n-r-1} F(r+k, r) C_{n-r-k}.$$

Therefore, it follows from Equation (1) that

$$\begin{aligned} F(n, r+1) &= \sum_{k=0}^{n-r-1} \frac{r}{r+k} \binom{2r+2k}{2r+k} \frac{1}{n-r-k} \binom{2n-2r-2k}{n-r-1-k} \\ &= \frac{r+1}{n} \binom{2n}{n-r-1} = \frac{r+1}{n} \binom{2n}{n+r+1}, \end{aligned}$$

as required. $\square$

4. Order-decreasing with fixed points

Finally, we consider the order-decreasing subsemigroup $\mathcal{C}_n$ of O_n. Recall that $1 \in \mathrm{Fix}(\alpha)$ for all $\alpha \in \mathcal{C}_n$. For any $Y = \{1, m_2, m_3, \ldots, m_r\} \subseteq X_n$ we define

$$\mathcal{C}_{n,Y} = \{\alpha \in \mathcal{C}_n : \mathrm{Fix}(\alpha) = Y\}.$$

Since $\mathcal{C}_{n,\{1\}} = N(\mathcal{C}_n)$, it follows from Lemma 3 that $|\mathcal{C}_{n,\{1\}}| = C_{n-1}$. Next we have the following theorem.

Theorem 10 *Let $Y = \{1, m_2, \ldots, m_r\}$ with $m_1 = 1 < m_2 < \cdots < m_r$ $(r \geq 1)$ be a subset of X_n. Then*

$$|\mathcal{C}_{n,Y}| = \prod_{j=1}^{r} C_{k_j - 1},$$

where $k_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $k_r = n - m_r + 1$.

Proof. Since $|\mathcal{C}_{n,\{1\}}| = C_{n-1}$, we suppose that $r \geq 2$. For each $\alpha \in \mathcal{C}_{n,Y}$, we similarly fix

$$\begin{aligned} \alpha_j &= \begin{pmatrix} 1 & 2 & \cdots & m_{j+1} - m_j \\ 1 & (m_j + 1)\alpha - m_j + 1 & \cdots & (m_{j+1} - 1)\alpha - m_j + 1 \end{pmatrix}, \\ \alpha_r &= \begin{pmatrix} 1 & 2 & \cdots & n - m_r + 1 \\ 1 & (m_r + 1)\alpha - m_r + 1 & \cdots & n\alpha - m_r + 1 \end{pmatrix} \end{aligned}$$

where $1 \leq j \leq r-1)$. Let $k_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $k_r = n - m_r + 1$. Similarly, we have $\alpha_j \in N(\mathcal{C}_{k_j})$ for each $1 \leq j \leq r$. Now consider the function

$$f : \mathcal{C}_{n,Y} \to N(\mathcal{C}_{k_1}) \times N(\mathcal{C}_{k_2}) \times \cdots \times N(\mathcal{C}_{k_r})$$

which maps $\alpha \in \mathcal{C}_{n,Y}$ to the ordered r-pair $(\alpha_1, \alpha_2, \ldots, \alpha_r)$. Since f is a bijection, it follows from Lemma 3 that

$$|\mathcal{C}_{n,Y}| = \prod_{j=1}^{r} C_{k_j - 1},$$

as required. □

For every $r \in X_n$ we define

$$N(n, r) = |\{\alpha \in \mathcal{C}_n : |\mathrm{Fix}(\alpha)| = r\}|$$

as the number of order-decreasing and order-preserving transformations which have exactly r fixed points. Let $Y = \{1, m_2, \ldots, m_r\}$ with $m_1 = 1 < m_2 < \cdots < m_r$ be a subset of X_n. Now take $k_j = m_{j+1} - m_j$ $(1 \leq j \leq r-1)$ and $k_r = n - m_r + 1$. Then it is clear that $(k_1, k_2, \ldots, k_r)$ is a positive integer solution of the equation

$$x_1 + x_2 + \cdots + x_r = n. \tag{4}$$

Conversely, every positive integer solution of Equation (4) gives a subset, which contains 1, of X_n with r elements. If we denote the set of all positive integer solutions of Equation (4) by $P_r(n)$, then we have

$$N(n,r)=\sum_{(k_1,k_2,\ldots,k_r)\in P_r(n)} C_{k_1-1}C_{k_2-1}\cdots C_{k_r-1}.$$

Moreover, we have the the following result.

Theorem 11 $N(n,r)=\frac{r}{2n-r}\binom{2n-r}{n}$.

Proof. We use induction on r as before. From Lemma 3 the equation holds for $r=1$. Suppose that $\alpha\in\mathcal{C}_n$ has $r+1\geq 2$ fixed points, say $1<m_2<\cdots<m_r<m_{r+1}$. Then consider the orbit of α which contains m_{r+1}. Since $\alpha\in\mathcal{C}_n$, the restricted transformation $\alpha_{|Y_{r+1}}:Y_{r+1}=\{m_{r+1},m_{r+1}+1,\ldots,n\}\to Y_{r+1}$ of α has unique fixed point (namely m_{r+1}), and that the restricted transformation $\alpha_{|X_n-Y_{r+1}}$ has r fixed points. Similarly, the transformations $\alpha_{|Y_{r+1}}:Y_{r+1}\to Y_{r+1}$ with a unique fixed point can be put into one-to-one correspondence with $\beta:X_{n-m_{r+1}+1}\to X_{n-m_{r+1}+1}$ with a unique fixed point. Since the number of such transformations is $C_{n-m_{r+1}}$, and since $m_{r+1}\in\{r+1,\ldots,n\}$, it follows from the inductive hypothesis that

$$\begin{aligned}N(n,r+1)&=\sum_{k=r+1}^{n}N(k-1,r)C_{n-k}\\&=\Big(\sum_{k=0}^{n-r-2}N(r+k,r)C_{n-r-1-k}\Big)+N(n-1,r)C_0.\end{aligned}$$

Therefore, it follows from Equation (2) that

$$\begin{aligned}N(n,r+1)&=\sum_{k=0}^{n-r-2}\frac{r}{r+2k}\binom{r+2k}{k}\frac{1}{n-r-1-k}\binom{2n-2r-2-2k}{n-r-2-k}\\&\quad+N(n-1,r)\\&=\frac{r+2}{2n-r-2}\binom{2n-r-2}{n}+\frac{r}{2n-r-2}\binom{2n-r-2}{n-1}\qquad(5)\\&=(r+2)\cdot\frac{(2n-r-3)!}{n!\cdot(n-r-2)!}+r\cdot\frac{(2n-r-3)!}{(n-1)!\cdot(n-r-1)!}\\&=\frac{(2n-r-3)!}{n!\cdot(n-r-1)!}\cdot\Big[(r+2)(n-r-1)+rn\Big]\\&=\frac{(2n-r-3)!}{n!\cdot(n-r-1)!}\cdot\Big[(r+1)(2n-r-2)\Big]\cdot\frac{(2n-r-1)}{(2n-r-1)}\\&=\frac{r+1}{2n-r-1}\binom{2n-r-1}{n},\end{aligned}$$

as required. □

Notice that we also have the recurrence relation

$$N(n, r+1) = N(n, r) - N(n-1, r-1)$$

from Equation (5) as in [3, Equation 3.5].

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