Turk J Math
35 (2011) , 627 – 636.

doi:10.3906/mat-0805-34

Weak-projective dimensions

Mohammad Javad Nikmehr, Zahra Poormahmood and Reza Nikandish

Abstract

In this paper, the notions of weak-projective modules and weak-projective dimension over commutative domain R are given. It is shown that over semisimple rings with weak global dimension 1, these modules are equivalent to weak-injective modules. The weak-projective dimension measures how far away a domain is from being a Prüfer domain. Several properties of these modules are also presented.

Key Words: Semi-Dedekind domain; Weak-injective modules; Weak- projective dimension, projective modules; Prüfer domain

1. Introduction

In this note, R will denote a commutative domain with identity and Q ($\neq R$) will denote its field of quotients. The R-module Q/R will be denoted by K. Lee in [5] studied the structure of weak-injective modules. An R-module M is called *weak-injective* if $Ext^1_R(N, M) = 0$ for all R-modules N of weak dimension ≤ 1. In section 2, we introduce a class of R-modules under the name of weak-projective R-modules. We show that weak-projective R-modules are identical to projective R-modules if and only if R is semisimple. Recall that R is called *Prüfer domain* if every finitely generated ideal of R is projective. There are numerous characterizations of Prüfer domains, which can be found in [3]. We show that each weak-projective R-module is FP-projective when R is a Noetherian ring. The domain R is called *semi-Dedekind* if every h-divisible R-module is pure-injective. For more details of these domains, we refer the reader to [4].

In section 3, we introduce the concept weak-projective dimension $wpd(M)$ of an R-module M and give some results. We show that this dimension has the properties that we expect of a "dimension" when the domain is semi-Dedekind.

Throughout this paper, M is an R-module. The notation $(w.)D(R)$ stands for the (weak) global dimension of R. Also, $pd(M)$ and $id(M)$ denote the projective and injective dimension of M, respectively. The character module $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of an R-module M will be denoted by M^b.

2000 *AMS Mathematics Subject Classification:* 16E10, 18G20.

2. Weak-projective modules

Recall that an R-module M is called *weak-injective* if $Ext^1_R(N, M) = 0$, for all R-modules N of weak dimension ≤ 1.

Definition 2.1 *An R-module M is called* weak-projective *if $Ext^1_R(M, N) = 0$, for every weak-injective R-module N.*

Evidently, direct products and summands of weak-projective R-modules are again weak-projective. All projective R-modules are trivially weak-projective, but the converse is not true. For example, $\mathbb{Q}/\mathbb{Z}$ as a $\mathbb{Z}$-module is weak-projective, but is not projective. Over a semisimple ring R, weak-projective R-modules are projective. It is obvious that if R is a semisimple ring with $w.D(R) = 1$, then every R-module M is weak-projective if and only if M is weak-injective. Also, if R is semisimple and M is a weak-projective R-module, then $Ext^1_R(M, R) = 0$.

A well-known result states that an R-module F is flat if and only if its character module F^b is injective. The following lemma is an analog of this equivalence.

Lemma 2.2 *(Lee [5, Lemma 3.1]) An R-module A is torsion-free if and only if A^b is weak-injective.*

An R-module M is called *FP-injective* if $Ext^1_R(N, M) = 0$ for all finitely presented R-modules N.

Lemma 2.3 *(Lee[5, Lemma 3.2]) For a domain R, the following are equivalent:*
(a) R is Prüfer;
(b) Every weak-injective R-module is FP-injective;
(c) Every weak-injective R-module is injective.

We may obtain some elementary results on the notion of the weak-projective modules.

Recall that the R-module M is called *FP-projective* [6] if $Ext^1_R(M, N) = 0$, for every FP-injective R-module N.

Lemma 2.4 *If R is a Noetherian ring and M a weak-projective R-module, then M is FP-projective.*

Proof. Let M be a weak-projective R-module. We must prove that $Ext^1_R(M, N) = 0$, for any FP-injective R-module N. Since R is a Noetherian ring, N is an injective R-module, and therefore N is weak-injective. □

The converse is an easy application of Lemma 2.3.

Lemma 2.5 *Let R be a semi-Dedekind domain and M an R-module. Then the following are equivalent:*
(a) M is weak-projective;
(b) $Tor^R_1(M, A) = 0$, for all torsion-free R-modules A;
(c) $pd(M) \leq 1$.

Proof. (a) $\Rightarrow$ (b) The isomorphism $Ext^1_R(M, A^b) \cong Hom_{\mathbb{Z}}(Tor^R_1(M, A), \mathbb{Q}/\mathbb{Z})$, together with Lemma 2.2, proves the result.

(b) $\Rightarrow$(a) This follows from [4, Lemma 4.1].
(b) $\Leftrightarrow$ (c) See [4, Lemma 4.9]. □

It is easy to check that the quotient $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ is weak-projective.

Combining Lemma 2.5, with the simple fact that an R-module D is divisible if and only if D^b is torsion-free gives the next corollary.

Corollary 2.6 *Let R be a semi-Dedekind domain and M an R-module. Then M is weak-projective if and only if $Tor_1^R(M, D^b) = 0$, for all divisible R-modules D.*

The following fact can be easily verified, so we omit its proof.

Lemma 2.7 *If R is a Prüfer domain, then every R-module is weak-projective.*

Lemma 2.8 *Let $0 \to A \to B \to C \to 0$ be an exact sequence such that A and C are weak-projective R-modules. Then B is weak-projective.*

Proof. Let N be a weak-injective R-module. From the induced exact sequence

$$Ext_R^1(C, N) \to Ext_R^1(B, N) \to Ext_R^1(A, N),$$

we have $Ext_R^1(B, N) = 0$, since $Ext_R^1(C, N) = Ext_R^1(A, N) = 0$. □

Corollary 2.9 *If every submodule and quotient of an R-module M is weak-projective, then M is weak-projective.*

From the previous corollary we have the following example.

Example 2.10 *The $\mathbb{Z}$-module $\mathbb{Q}$ is weak-projective.*

Recall that R is called a *Matlis domain* if the projective dimension of Q (or, equivalently, K) is 1. The R-module C is called *Matlis cotorsion* if $Ext_R^1(Q, C) = 0$, and M is called *strongly flat* if $Ext_R^1(M, C) = 0$ for every Matlis cotorsion R-module C.

The next result gives a relationship between weak-projective R-modules and strongly flat R-modules.

Lemma 2.11 *If R is a Matlis domain and M a strongly flat R-module, then M is weak-projective.*

Proof. If M is a strongly flat R-module, then $Ext_R^1(M, N) = 0$, for all Matlis cotorsion R-modules N. It is easy to see that if R is a Matlis domain, then every weak-injective R-module is Matlis cotorsion. □

Lemma 2.12 *Let R be a semi-Dedekind domain. If M is a projective R-module and N a weak-projective R-module, then $M \otimes_R N$ is weak-projective.*

Proof. The isomorphism $Tor_n^R(M \otimes N, A) \cong M \otimes Tor_n^R(N, A)$, together with Lemma 2.5, proves the result. □

The converse is true when R is a local semi-Dedekind domain.

In what follows, $\sigma_M : M \to E(M)$ denotes the injective envelope of an R-module M. Recall that an injective envelope $\sigma_M : M \to E(M)$ has the unique mapping property (see [1]) if for any homomorphism $f : M \to N$ with N injective, there exists a unique homomorphism $g : E(M) \to N$ such that $g\sigma_M = f$.

Corollary 2.13 *The following statements are equivalent:*
(a) R is a Prüfer domain;
(b) Every R-module is weak-projective;
(c) $Ext^1_R(M, N) = 0$, for all weak-injective R-modules N;
(d) Every weak-injective R-module has an injective envelope with the unique mapping property.

Proof. It is enough to show that (d)⇒ (a).
(d)⇒ (a) Let M be any weak-injective R-module. We have the following exact commutative diagram:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow{\sigma_M} & E(M) & \xrightarrow{\gamma} & L & \longrightarrow & 0 \\ & & \downarrow^{0} & & \downarrow^{\sigma_L\gamma} & & \downarrow^{\sigma_L} & & \\ & & E(L) & \cong & E(L) & \cong & E(L). & & \end{array}$$

Note that $\sigma_L\gamma\sigma_M = 0 = 0\sigma_M$, so $\sigma_L\gamma = 0$ by (d). Therefore $L = im(\gamma) \subseteq \ker(\sigma_L) = 0$, and hence M is injective. Thus (a) follows. □

We end this section with the following characterizations of weak-projective R-modules.

Let ℓ be a class of R-modules and M an R-module. A homomorphism $\phi \in Hom_R(N, M)$ with $N \in \ell$ is called an *ℓ-precover* of M if the induced map

$$Hom_R(1_{N'}, \phi) : Hom_R(N', N) \to Hom_R(N', M)$$

is surjective for all $N' \in \ell$. An ℓ-precover $\phi \in Hom_R(N, M)$ is called an *ℓ-cover* if each $\gamma \in Hom_R(N, N)$ satisfying $\phi = \phi\gamma$ is an automorphism of N. The class ℓ is called a *precover(cover) class* if every R-module has an ℓ-precover(ℓ-cover).

The *ℓ-preenvelope, ℓ-envelope, preenvelope* and *envelope classes* are defined dually (see [9]). In particular, if ℓ is the class of weak-injective R-modules, an ℓ-envelope is called a *weak-injective envelope.*

Proposition 2.14 *If M is an R-module, then the following are equivalent:*
(a) M is weak-projective;
(b) M is projective with respect to every exact sequence $0 \to A \to B \to C \to 0$, where A is weak-injective;
(c) For every exact sequence $0 \to K \to F \to M \to 0$, where F is weak-injective, $K \to F$ is a weak-injective preenvelope of K;
(d) M is cokernel of a weak-injective preenvelope $K \to F$ with F projective.

Proof. (a) $\Rightarrow$ (b) Let $0 \to A \to B \to C \to 0$ be an exact sequence, where A is weak-injective. Then $Ext^1_R(M, A) = 0$ by (a). Thus $Hom_R(M, B) \to Hom_R(M, C) \to 0$ is exact, and (b) holds.

(b)$\Rightarrow$ (a) For every weak-injective R-module N, there is a short exact sequence $o \to N \to E \to L \to 0$ with E injective, which induces an exact sequence $Hom_R(M, E) \to Hom_R(M, L) \to Ext^1_R(M, N) \to 0$. Since $Hom_R(M, E) \to Hom_R(M, L) \to 0$ is exact by (b), we have $Ext^1_R(M, N) = 0$, and (a) follows.

(a) $\Rightarrow$ (c) is easy to verify.

(c) $\Rightarrow$ (d) Let $0 \to K \to P \to M \to 0$ be an exact sequence with P projective. Note that P is weak-injective by hypothesis, thus $K \to P$ is a weak-injective preenvelope.

(d) $\Rightarrow$ (a) By (d), there is an exact sequence $0 \to K \to P \to M \to 0$, where $K \to P$ is a weak-injective preenvelope with P projective. It gives rise to the exactness of $Hom_R(P, N) \to Hom_R(K, N) \to Ex^1_R(M, N) \to 0$, for each weak-injective R-module N. Note that $Hom_R(P, N) \to Hom_R(K, N) \to 0$ is exact by (d). Hence $Ext^1_R(M, N) = 0$, as desired. □

3. The weak-projective dimension over semi-Dedekind domains

We begin this section with the definition of weak-injective dimension.

Definition 3.1 *(a) For any R-module M, let* weak-injective dimension $wid(M)$ *of M, denote the smallest integer $n \geq 0$ such that $Ext^{n+1}_R(N, M) = 0$ for every R-module N of weak dimension ≤ 1. (If no such n exists, set $wid(M) = \infty$).*
(b) $wiD(R) = sup\{wid(M) : M$ is an R-module$\}$.

Lemma 3.2 *Let R be a semi-Dedekind domain. For an R-module M, the following statements are equivalent:*
(a) $wid(M) \leq n$;
(b) $Ext^{n+1}_R(N, M) = 0$ for all R-modules N of weak dimension ≤ 1;
(c) If the sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_n \to 0$ is exact with $E_0, E_1, \cdots, E_{n-1}$ weak-injective, then also E_n is weak-injective.

Proof. (a) $\Rightarrow$ (b) Use induction on n. Clear if $wid(M) = n$. If $wid(M) \leq n - 1$ resolve N by $0 \to K \to P \to N \to 0$ with K and P flat. K have weak dimension ≤ 1 by [4, Corollary 4.4], and $Ext^{n+1}_R(N, M) \cong Ext^n_R(K, M) = 0$ by induction hypothesis.

(b) $\Leftrightarrow$ (c) follows from the isomorphism $Ext^{n+1}_R(N, M) \cong Ext^1_R(N, E_n)$.

(b) $\Rightarrow$ (a) are trivial. □

Definition 3.3 *For an R-module M, let $wpd(M)$ denotes the smallest integer $n \geq 0$ such that $Ext^{n+1}_R(M, N) = 0$ for every weak-injective R-module N and call $wpd(M)$ the weak-projective dimension of M. If no such n exists, set $wpd(M) = \infty$.*

Put $rwpD(R)= sup\{wpd(M) : M$ is a right R-module$\}$ and call $rwpD(R)$ the right weak-projective dimension of R. Similarly, we have $lwpD(R)$ (we drop the unneeded letters r and l, because R is commutative).

M is called weak-projective if $wpd(M) = 0$, i.e., $Ext^1_R(M,N) = 0$ for every weak-injective R-module N.

Remark 3.4 *For every ring R and every R-module M, the inequalities $wpD(R) \leq D(R)$ and $wpd(M) \leq pd(M)$ are valid. It is easy to see that $wpd(M) = pd(M)$ for any R-module M if and only if every weak-projective R-module is projective.*

Proposition 3.5 *Let R be a semi-Dedekind domain. For any R-module M and an integer $n \geq 0$, the following are equivalent:*
(a) $wpd(M) \leq n$;
(b) $Ext^{n+1}_R(M,N) = 0$ for any weak-injective R-module N;
(c) $Ext^{n+j}_R(M,N) = 0$ for any weak-injective R-module N and $j \geq 1$;
(d) There exists an exact sequence $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, where each P_i is weak-projective.

Proof. (c)$\Rightarrow$ (a) is obvious.

(b)$\Rightarrow$ (c) For any weak-injective R-module N, there is a short exact sequence $0 \to N \to E \to L \to 0$, where E is injective. Then the sequence $Ext^{n+1}_R(M,L) \to Ext^{n+2}_R(M,N) \to Ext^{n+2}_R(M,E) = 0$ is exact. Note that L is weak-injective by Lemma 3.2, so $Ext^{n+1}_R(M,L) = 0$ by (b). Hence $Ext^{n+2}_R(M,N) = 0$, and (c) follows by induction.

The proof of (a)$\Rightarrow$ (b) is similar to that of (b)$\Rightarrow$ (c).

(a) $\Leftrightarrow$ (d) is straightforward. □

Proposition 3.6 *For an R-module M, the following are equivalent:*
(a) $wpD(R) = 0$;
(b) $Tor^R_1(M,A) = 0$, for all torsion-free R-modules A;
(c) M has weak dimension $\leq$ 1;
(d) R is Prüfer;
(e) Every R-module is weak-projective.

Proof. (a) $\Rightarrow$ (b) The isomorphism $Ex^1_R(M,A^b) \cong Hom_{\mathbb{Z}}(Tor^R_1(M,A), \mathbb{Q}/\mathbb{Z})$, together with Lemma 2.2, proves the result.

(b) $\Rightarrow$ (c) see [5, Corollary 2.4].

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (e) see Lemma 2.7.

(e) $\Rightarrow$ (a) is trivial. □

Remark 3.7 *(a) By Proposition 3.6, $wpD(R)$ measures how far away a domain R is from being a Prüfer domain.*
(b) It is well known that R is semihereditary domain if and only if R is Prüfer domain.

The proof of the next proposition is standard homological algebra.

Proposition 3.8 *Let R be a semi-Dedekind domain, $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. If two of $wpd(A)$, $wpd(B)$, and $wpd(C)$ are finite, so is the third. Moreover,*
(a) $wpd(B) \le max\{wpd(A), wpd(C)\}$.
(b) $wpd(A) \le max\{wpd(B), wpd(C) - 1\}$.
(c) $wpd(C) \le max\{wpd(B), wpd(A) + 1\}$.

Corollary 3.9 *Let R be a semi-Dedekind domain.*
(a) If $0 \to A \to B \to C \to 0$ is an exact sequence of R-modules, where $0 < wpd(A) < \infty$ and B is weak-projective, then $wpd(C) = wpd(A) + 1$.
(b) $wpD(R) = n$ if and only if $sup\{wpd(I)$:I is any ideal of $R\}= n - 1$ for any integer $n \ge 2$.

Proof. (a) is true by Proposition 3.8.

(b) For an ideal of R, consider the exact sequence $0 \to I \to R \to R/I \to 0$. Then (b) follows from (a). □

Theorem 3.10 *Let R be a semi-Dedekind domain. Then The following values are identical:*
(a) $wpD(R)$;
(b) $sup\{wpd(M)$: M is a cyclic R-module$\}$;
(c) $sup\{wpd(M)$: M is any R-module$\}$;
(d) $sup\{id(F)$: F is a weak-injective R-module$\}$.

Proof. (b) $\le$ (a) $\le$ (c) are obvious.

(c) $\le$ (d) We may assume sup $\{id(F)$: F is a weak-injective R-module$\}=m < \infty$. Let M be any R-module and N any weak-injective R-module. Since $id(N) \le m$, it follows that $Ext_R^{m+1}(M, N) = 0$. Hence $wpd(M) \le m$.

(d) $\le$ (b) We may assume sup$\{wpd(M)$: M is a cyclic R-module$\}=n < \infty$. Let N be a weak-injective R-module and I any ideal, then $wpd(R/I) \le n$. By Proposition 3.5, $Ext_R^{n+1}(R/I, N) = 0$, and so $id(N) \le n$. □

Proposition 3.11 *Let R be a semi-Dedekind domain. Then the following are equivalent:*
(a) $wpD(R) \le 1$;
(b) Every submodule of a (weak-)projective R-module is weak-projective;
(c) Every ideal of R is weak-projective.

Proof. (a) $\Rightarrow$ (b) Let N be a submodule of a weak-projective R-module M. Then, for any weak-injective R-module L, we get an exact sequence

$$0 = Ext_R^1(M, L) \to Ext_R^1(N, L) \to Ext_R^2(M/N, L).$$

Note that the last term is zero by (a), hence $Ext^1_R(N,L)=0$, and (b) follows.
(b) $\Rightarrow$ (c) is trivial.
(c)$\Rightarrow$ (a) Let I be an ideal of R. The exact sequence $0\to I\to R\to R/I\to 0$ implies $wpd(R/I)\leq 1$ by Proposition 3.5. So (a) follows from Theorem 3.10 (b). □

It is well known that if M is finitely generated projective R-module, then $Hom_R(M,R)$ is finitely generated projective R-module. Here we have the following corollary.

Corollary 3.12 *If R is a semi-Dedekind domain with $wpD(R)\leq 1$, then the dual module $Hom_R(M,R)$ of any finitely generated R-module M is weak-projective.*

In addition, if $w.D(R)=1$, then the following are equivalent:
(a) Every torsion-free R-module is weak-projective;
(b) M^b is weak-projective for every injective R-module M;
(c) N^{bb} is weak-projective for every torsion-free R-module N.

Proof. Let M be a finitely generated R-module. Then there exists an exact sequence $P\to M\to 0$ with P finitely generated projective. So we have an R-module exact sequence $0\to Hom_R(M,R)\to Hom_R(P,R)$. Note that $Hom_R(P,R)$ is projective, therefore $Hom_R(M,R)$ is weak-projective by Proposition 3.11.

Also, if $w.D(R)=1$, then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

(c)$\Rightarrow$ (a) Let N be any torsion-free R-module. There exists an exact sequence $0\to N\to N^{bb}$. Since $wpD(R)\leq 1$ and N^{bb} is weak-projective by (c), we have that N is weak-projective by Proposition 3.11. □

A ring R is called semi-Artinian if every nonzero cyclic R-module has a nonzero socle. The following proposition shows that we may compute the weak-projective dimension of semi-Artinian ring using just the weak-projective dimension of simple modules.

Proposition 3.13 *If R is a semi-Artinian semi-Dedekind domain, then $wpD(R)=sup\{wpd(M)\colon M$ is a simple R-module$\}$.*

Proof. It suffices to show that $wpD(R)\leq sup\{wpd(M)\colon M$ is a simple R-module$\}$. We may assume that $sup\{wpd(M)\colon M$ is a simple R-module$\}=n<\infty$. Let N be a weak-injective R-module and I a maximal ideal of R. Consider the injective resolution of N

$$0\to N\to E^0\to E^1\to E^2\to\cdots\to E^{n-1}\to E^n\to\cdots.$$

Write $L=coker(E^{n-2}\to E^{n-1})$. Then $Ext^1_R(R/I,L)=Ext^{n+1}_R(R/I,N)=0$ by Proposition 3.5. Therefore L is injective by [8, Lemma 4], since R is semi-Artinian. So $id(N)\leq n$, and hence $wpD(R)\leq n$ by Theorem 3.10. □

Proposition 3.14 *Let R be a semi-Dedekind domain. Then $sup\{pd(M)\colon M$ is a weak-projective R-module$\}\leq wiD(R)$.*

Proof. Let M be a weak-projective R-module. It is enough to show that $pd(M) \leq wiD(R)$. We may assume that $wiD(R) = n < \infty$. M admits a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to \cdots P_1 \to P_0 \to M \to 0.$$

Let N be any R-module. We have $wid(N) \leq n$, thus by Lemma 3.2, there is an exact sequence

$$0 \to N \to E^0 \to E^1 \to \cdots \to E^{n-1} \to E^n \to 0,$$

where $E^0, E^1, \cdots, E^n$ are weak-injective. Therefore we form a double complex

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & 0 & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & Hom_R(M,E^n) & \to & Hom_R(P_0,E^n) & \to & \cdots & \to & Hom_R(P_n,E^n) & \to & \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & Hom_R(M,E^1) & \to & Hom_R(P_0,E^1) & \to & \cdots & \to & Hom_R(P_n,E^1) & \to & \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
0 & \to & Hom_R(M,E^0) & \to & Hom_R(P_0,E^0) & \to & \cdots & \to & Hom_R(P_n,E^0) & \to & \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \\
 & & 0 & \to & Hom_R(P_0,N) & \to & \cdots & \to & Hom_R(P_n,N) & \to & \cdots \\
 & & & & \uparrow & & & & \uparrow & & \\
 & & & & 0 & & & & 0. & &
\end{array}$$

Note that all rows are exact except for the bottom row, since M is weak-projective and all E^i are weak-injective; also note that all columns are exact except for the left column since all P_i are projective.

Using a spectral sequence argument, we know that the two complexes

$$0 \to Hom_R(P_0, N) \to Hom_R(P_1, N) \to \cdots \to Hom_R(P_n, N) \to \cdots$$

and

$$0 \to Hom_R(M, E^0) \to Hom_R(M, E^1) \to \cdots \to Hom_R(M, E^n) \to 0$$

have isomorphic homology groups. Thus $Ext_R^{n+j}(M, N) = 0$ for all $j \geq 1$. Hence $pd(M) \leq n$. □

It is known that $D(R)$=sup$\{pd(M)$: M is a weak-projective R-module$\}$ if R is a Prüfer domain, and it is easy to see that $D(R)=$ $wpD(R)$ if R is a semisimple ring. In general, we have

Proposition 3.15 *Let R be a semi-Dedekind domain and M be an R-module. Then $D(R) \leq$ sup$\{pd(M)$: M is a weak-projective R-module$\}$ + $wpD(R)$.*

Proof. We may assume without loss of generality that $wpD(R)$ is finite. Let $wpD(R) = m < \infty$ and Sup$\{pd(M)$: M is a weak-projective R-module$\}$=$n < \infty$. If M is an R-module, then $wpd(M) \leq m$ by Theorem 3.10. So M admits a weak-projective resolution

$$0 \to P_m \to P_{m-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$

where each P_i is weak-projective, $i = 0, 1, 2, \cdots, m$. Let $K_i = Ker(P_i \to P_{i-1})$, $i = 0, 1, 2, \cdots, m-1$, $P_{-1} = M$, $K_{m-1} = P_m$. Then we have the following short exact sequence

$$0 \to P_m \to P_{m-1} \to K_{m-2} \to 0,$$

$$0 \to K_{m-2} \to P_{m-2} \to K_{m-3} \to 0,$$

$$\vdots$$

$$0 \to K_0 \to P_0 \to M \to 0.$$

It follows that $pd(K_{m-2}) \leq 1 + n$, $pd(K_{m-3}) \leq 2 + n, \cdots$, $pd(M) \leq m + n$, and hence $D(R) \leq m + n$. This completes the proof. □

Acknowledgement

The authors express their thank to the referee for his careful reading, and for helpful suggestions.

References

[1] Ding, N.: *On envelopes with the unique mapping property.* Comm. Alg. 24(4), 1459-1470(1996).

[2] Fuchs, L. and Salce, L.: *Modules Over Non-Noetherian Domains.* Math. Surveys and Monographs. 84. Providence R.I. Amer. Math. Society(2001).

[3] Gilmer, R.: *Multiplicative Ideal Theory.* Queens Papers Pure Appl. Math. 90. Kingston: Queens University(1972).

[4] Lee, S. B.: *h-divisible modules.* Comm. Alg. 31, 513-525(2003).

[5] Lee, S. B.: *Weak-injective modules.* Comm. Alg. 34, 361-370(2006).

[6] Mao, L. and Ding N. Q.: *Notes on FP-projective modules and FP-injective modules.* Advances in ring theory. 151-166(2005).

[7] Rotman, J. J.: *An Introduction to Homological Algebra.* New York: Academic Press(1979).

[8] Smith, P. F.: *Injective modules and Prime ideals.* Comm. Alg. 9, 989-999(1981).

[9] Trlifaj, J.: *Covers, Envelopes and Cotorsion Theories.* Lecture Notes for the Workshop Homological Methods in Modules Theory. Cortona(2000).

Mohammad Javad NIKMEHR
Zahra POORMAHMOOD and Reza NIKANDISH
Department of Mathematics,
K. N. Toosi University of Technology,
P. O. Box 16315-1618, Tehran-IRAN
e-mail: nikmehr@kntu.ac.ir,
e-mail: poormahmood@sina.kntu.ac.ir
e-mail: r_nikandish@sina.kntu.ac.ir

Received: 27.05.2008