Turk J Math
35 (2011) , 637 – 648.

doi:10.3906/mat-0912-57

# GPQ modules and generalized Armendariz modules

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**Abstract**

Let $M_R$ be a right $R$-module. We introduce the concept of right generalized p.q.-Baer modules (or simply, right GPQ modules) to extend the notion of right p.q.-Baer modules. We study on the relationship between the GPQ property of a module $M_R$ and various quasi-Armendariz properties. We prove that every right GPQ module is a quasi-Armendariz module. As a sequence, we obtain a general form of some known results considering the p.q.Baer property of a ring, some known results are extended. Moreover, we prove that for the formal triangular ring $R$ constructed from a pair of rings $S, T$ and a bimodule ${}_SM_T$, $R$ is weak Armendariz if and only if (1) $S$ and $T$ are weak Armendariz rings. (2) ${}_SM$ and $M_T$ are weak Armendariz as a left $S$-module and right $R$-module. (3) If $s(x)s^{'}(x) = t(x)t^{'}(x) = 0$, then $s(x)M[x] \cap M[x]t^{'}(x) = 0$. This gives the relationship of weak Armendarizness between $R$ and $S, T, {}_S M_T$, which plays a very important role in ring theory.

**Key Words:** GPQ modules; quasi-Armendariz modules; p.q.-Baer modules; weak Armendariz modules

## 1. Introduction

Throughout this paper, all rings are associative with identity and modules are unital right modules and $\alpha : R \to R$ is an endomorphism of the ring $R$. Clark defined quasi-Baer rings in [9] and use them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. A ring $R$ is called quasi-Baer if the right annihilator of each right ideal of $R$ is generated by an idempotent. As a generalization of quasi-Baer rings, Birkenmeier [5] introduced the concept of principally quasi-Baer rings. A ring $R$ is called right principally quasi-Baer (or simply right p.q.-Baer) if the right annihilator of a principal right ideal of $R$ is generated by an idempotent. Similarly, left p.q.-Baer rings can be defined. A ring $R$ is called *p.q.*-Baer if it is both right and left p.q.-Baer. Another generalization of Baer rings is a p.p.-ring. A ring $R$ is called a right (resp. left) p.p.-ring [6] if the right (resp. left) annihilator of every element of $R$ is generated by an idempotent. $R$ is called a *p.p.*-ring if it is both right and left p.p.

An ideal $I$ of $R$ is said to be right (resp. left) s-unital [18] if, for each $a \in I$ there exists an element $x \in I$ such that $ax = a$ (resp. $xa = a$). Note that if $I$ and $J$ are right s-unital ideal of $R$, then so is $I \cap J$ (if $a \in I \cap J$, then $a \in aIJ \subseteq a(I \cap J)$). It is well known that $I$ is right s-unital if and only if $R/I$ is flat as a left $R$-module if and only if $I$ is pure as a left ideal of $R$.

2000 *AMS Mathematics Subject Classification:* 16U80,16S34.

For a subset $X$ of a module $M_R$, let $r_R(X) = \{r \in R : Xr = 0\}$. In [15], Lee-Zhou introduced Baer modules, quasi-Baer modules, p.p.-modules and reduced modules as follows: (1) $M_R$ is called *Baer* if, for any subset $X$ of $M$, $r_R(X) = eR$ where $e^2 = e \in R$. (2) $M_R$ is called *quasi-Baer* if, for any submodule $N$ of $M$, $r_R(N) = eR$ where $e^2 = e \in R$. (3) $M_R$ is called *p.p.* if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$. (4) $M_R$ is said to be reduced if, for any $m \in M$ and $a \in R$, $ma = 0$ implies $mR \cap Ma = 0$. It is clear that $R$ is reduced if and only if $R_R$ is a reduced module. Recently, Baser et al. introduced the notion of *principally quasi-Baer* modules. A module $M_R$ is called *principally quasi-Baer* [3] (or simply *p.q.-Baer*) module if, for any $m \in M$, $r_R(mR) = eR$, where $e^2 = e \in R$. It is clear that $R$ is a right *p.q.*-Baer ring if and only if $R_R$ is a *p.q.-Baer* module. Moreover, every quasi-Baer module is p.q.-Baer and every Baer module is quasi-Baer.

We introduce the concept of right generalized p.q.-Baer modules (or simply right GPQ modules) to extend the notion of right p.q.-Baer modules. We prove that $N_R$ is a right GPQ module if and only if $M_n(N)$ is a right GPQ module and $M_R$ is a right GPQ module if and only if $M[x]_{R[x]}$ is a right GPQ module. We study the relationship between the GPQ property of a module $M_R$ and various quasi-Armendariz properties (including skew power series, skew Laurent polynomials and skew polynomials). It is shown that every right GPQ module is a quasi-Armendariz module. As an immediate consequence of these facts, we obtain a unified form of some well-known results considering the p.q.Baer property of a ring. We show that if $R$ is an $\alpha$-compatible ring, then $R$ is a right p.q.-Baer ring if and only if $R[x;\alpha]$ is a right p.q.-Baer-ring. We prove, among others, that the trivial extension $T(R,R)$ of $R$ by $R$ is a weak Armendariz ring if and only if the following two conditions are satisfied: (1) $R$ is a weak Armendariz ring. (2) If $f(x)g(x) = 0$ in $R[x]$, then $f(x)R[x] \cap R[x]g(x) = 0$.

## 2. GPQ modules and quasi-Armendariz modules

Following [4], $M_R$ is called *quasi-Armendariz* if, whenever $m(x)R[x]f(x) = 0$ where $m(x) = \sum_{i=0}^{n} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{s} a_j x^j \in R[x]$, then $m_i R a_j = 0$ for all $i$ and $j$. It is clear that $R$ is a quasi-Armendariz ring if and only if $R_R$ is a quasi-Armendariz right $R$-module. Note that every reduced module is a quasi-Armendariz module.

Our focus in this section is to introduce the concepts of right GPQ modules and quasi-Armendariz modules relative to skew power series modules, skew Laurent polynomial modules and skew polynomial modules, respectively. Moreover, we study on the relationship between the GPQ property of a module $M_R$ and those of various quasi-Armendariz properties.

We first give the notion of a right GPQ module which is a generalization of right p.q.-Baer modules. We begin with the following definition.

**Definition 2.1.** *A module $M_R$ is called right GPQ if the right annihilator $r_R(mR)$ is left s-unital as an ideal of $R$ for any $m \in M$.*

The left version for a left $R$-module can be defined similarly. It is obvious that every right *p.q.*-Baer module is a right GPQ module. Moreover, if $M$ is a bimodule ${}_RM_R$, then every left *p.p.* module is right GPQ by [10, Proposition 1]. The following example shows that there exists a right GPQ module which is neither p.p. nor p.q.-Baer.

**Example 2.1** *(see 17, Example 2.5) Let $\mathbb{Z}$ be the ring of integers. We consider the ring*

$$S = (\prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z})/(\bigoplus_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}).$$

*It is clear that $S$ is a Boolean ring. Let $R = S[[x]]$, then $R_R$ is a right GPQ module by* [17*, Example* 2.5]*, but it is neither p.p. nor p.q.-Baer.*

Recall that a module $M_R$ is called semicommutative if $r_R(m)$ is an ideal of $R$ for all $m \in M$, or equivalently, if for any $m \in M$ and $a \in R$, then $ma = 0$ implies that $mRa = 0$. It was shown in [1] that if $M_R$ is a semicommutative module, then $M_R$ is a $p.q.$-Baer module if and only if the right annihilator of every finitely generated submodule is generated (as a right ideal) by an idempotent. Similarly, we have the following

**Lemma 2.1** *The following conditions are equivalent for a module $M_R$:*

(1) *$M_R$ is a right GPQ module.*

(2) *If $N$ is a finitely generated submodule of $M_R$ then for all $a \in r_R(N), a \in r_R(N)a$.*

**Proof.** The implication (2) ⇒ (1) is straightforward. Now suppose that $M_R$ is a right GPQ module. Let $N = m_1R + m_2R + \cdots + m_nR$ be a finitely generated submodule of $M_R$, then $r_R(N) = \cap_{i=1}^{n} r_R(m_iR)$. If $a \in r_R(N)$, then $a \in r_R(m_iR)$ for each $i$. Since $M_R$ is a right GPQ module, there exists $t_i \in r_R(m_iR)$ such that $a = t_ia$ for each $i$. So we have $ta = a$, where $t = t_nt_{n-1}\cdots t_1 \in r_R(N)$. This yields desired result.

Let $n$ be a positive integer and let $M_n(R)$ be the ring of $n \times n$ matrixes over $R$. For a module $N_R$, we denote $M_n(N)$ the formal $n \times n$ matrixes over $N$. Then $M_n(N)$ is an Abelian group under obvious addition operation. Moreover, $M_n(N)$ becomes a module over $M_n(R)$ under the usual scalar product operation. The next result shows one way to build new $GPQ$-modules from old ones. □

**Proposition 2.1** *$N_R$ is a right GPQ module if and only if $M_n(N)$ is a right GPQ module.*

**Proof.** Suppose that $N_R$ is a right GPQ module and $\tilde{N} = (n_{ij}) \in M_n(N)$. Let $A = (a_{ij}) \in M_n(R)$ is such that $A \in r_{M_n(R)}(\tilde{N}M_n(R))$, then we have $\tilde{N}M_n(R)A = 0$. Let $E_{ij}$ denote the $(i,j)$-matrix unit. Then $(\sum_{p,q} n_{pq}E_{pq})rE_{ij}(\sum_{s,t} a_{st}E_{st}) = 0$ for any $r \in R$ and any $i$ and $j$, where $n_{pq}$ is the element of $\tilde{N}$ in $(p,q)$ and $a_{st}$ is the element of $A$ in $(s,t)$. It is easy to see that $\sum_{p,t} n_{pi}ra_{jt}E_{pt} = 0$, this shows that $n_{pi}ra_{jt}$=0 for any $p$ and $t$. Hence $a_{jt} \in r_R(n_{pi}R)$ for all $i, j, t$ and $p$. Then $a_{st} \in r_R(\sum_{i,j} n_{ij}R)$ for all $s, t$, and so there exists $c \in r_R(\sum_{i,j} n_{ij}R)$ such that $a_{st} = ca_{st}$ for all $s, t$ by Lemma 2.1. Now we have the following equation (†).

$$A = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix} A \quad (\dagger), \qquad \tilde{N}M_n(R) \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix} = 0. \quad (\ddagger)$$

It is straightforward to verify that (‡) is also true. This implies that $M_n(N)$ is a right $GPQ$ module. Conversely, if $M_n(N)$ is a right $GPQ$ module. Let $n \in N$ and $a \in R$ such that $a \in r_R(nR)$. Let

$$\tilde{N} = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

It is clear that $\tilde{N}M_n(R)A$=0, so $A \in r_{M_n(R)}(\tilde{N}M_n(R))$. Since $M_n(N)$ is a right $GPQ$ module, there exists $B = (b_{ij}) \in M_n(R)$ such that $B \in r_{M_n(R)}(\tilde{N}M_n(R))$ and $A = BA$. Since $\tilde{N}M_n(R)B$=0, it follows that $nRb_{11} = 0$. This implies that $b_{11} \in r_R(nR)$. It is easy to see that $a = b_{11}a$ and so $N_R$ is a right $GPQ$ module. □

**Lemma 2.2** *Let $M_R$ be a right $R$-module. If $M_R$ is a right GPQ module, then $M_R$ is a quasi-Armendariz module.*

**Proof.** Assume that $M_R$ is a right GPQ module. Let $m(x) = m_0 + m_1x + \cdots + m_nx^n \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_sx^s \in R[x]$ such that $m(x)R[x]f(x) = 0$ with $m_i \in M$ and $a_j \in R$. We shall prove that $m_iRa_j = 0$ for all $i, j$. Let $c$ be an arbitrary element of $R$. Then we have the following equation:

$$\begin{aligned} 0 \;=\; & m(x)cf(x) = m_0ca_0 + \cdots + (m_nca_{s-2} + m_{n-1}ca_{s-1} + m_{n-2}ca_s)x^{n+s-2} \\ & +(m_nca_{s-1} + m_{n-1}ca_s)x^{n+s-1} + m_nca_sx^{n+s}. \end{aligned} \qquad (*)$$

It follows that $m_nca_s = 0$, and so $a_s \in r_R(m_nR)$. Since $r_R(m_nR)$ is left s-unital by hypothesis, there exists $t_n \in r_R(m_nR)$ such that $t_na_s = a_s$. Replacing $c$ by $ct_n$ in equation $(*)$, we obtain

$$m_0ct_na_0 + \cdots + (m_{n-1}ct_na_{s-1} + m_{n-2}ct_na_s)x^{n+s-2} + m_{n-1}ct_na_sx^{n+s-1} = 0$$

.

Then we have $m_{n-1}ca_s = m_{n-1}ct_na_s = 0$, so $a_s \in r_R(m_nR + m_{n-1}R)$. Since $r_R(m_{n-1}R)$ is left s-unital, there exists $h \in r_R(m_{n-1}R)$ such that $ha_s = a_s$. If we put $t_{n-1} = ht_n$, then $t_{n-1}a_s = a_s$ and $t_{n-1} \in r_R(m_nR + m_{n-1}R)$. Next, replacing $c$ by $ct_{n-1}$ in equation $(*)$, we obtain $m_{n-2}ca_s = 0$ in the same way as above. Hence we have $a_s \in r_R(m_nR+m_{n-1}R+m_{n-2}R)$. Continuing this process, we obtain $m_iRa_s = 0$ for all $i = 1, 2, \cdots, n$. Thus we get

$$(m_0 + m_1x + \cdots + m_nx^n)R[x](a_0 + a_1x + \cdots + a_{s-1}x^{s-1}) = 0.$$

Using induction on $m+n$, we obtain $m_iRa_j = 0$ for all $i, j$. This implies that $M_R$ is a quasi-Armendariz module, as desired. □

The following example shows that there exists a quasi-Armendariz module $M_R$ which is not right GPQ.

**Example 2.2** *(see 7, Example 2.3) For a given field $F$. Let*

$$S = \{(a_n)_{n=1}^{\infty} \in \prod F | a_n \text{ is eventually constant}\},$$

*which is a subring of the countably infinite direct product $\prod F$. Then the ring $S$ is a commutative von Neumann regular ring. Let $R = S[[x]]$. It is clear that $S$ is a reduced ring, it follows from* [17*, Example* 2.4] *that $R$ is a* *reduced ring and so $R$ is Armendariz as a right $R$-module. This implies that $R$ is quasi-Armendariz as a right $R$-module. But $R$ is neither right p.q.Baer by* [8, *Example* 3.6] *nor GPQ as a right $R$-module by* [17, *Example* 2.4].

**Lemma 2.3.** *Let $M_R$ be a right $R$-module. If $M[x]_{R[x]}$ is a right GPQ module, then $M_R$ is a right GPQ module.*

**Proof.** Let $m$ be any element of $M$. Suppose that $M[x]_{R[x]}$ is a right GPQ module, then $r_{R[x]}(mR[x])$ is left s-unital. Hence for any $a \in r_R(mR)$, there exists a polynomial $f(x) \in R[x]$ such that $f(x)a = a$. Let $b_0$ be the constant term of $f(x)$. Then $b_0 \in r_R(mR)$ and $b_0 a = a$. This implies that $r_R(mR)$ is left s-unital. □

In view of the foregoing lemma, we are now in a position to give the following characterization of GPQ modules.

**Proposition 2.2** *Let $M_R$ be a right $R$-module. Then $M_R$ is a right GPQ module if and only if $M[x]_{R[x]}$ is a right GPQ module.*

**Proof.** This follows directly from Lemma 2.2, Lemma 2.3 and [18, Theorem 1]. □

Note that if $M_R$ is a p.q.-Baer module and let $m \in M$. Then $r_R(mR) = eR$ for some idempotent $e^2 = e \in R$, and so $R/r_R(mR) = R/eR \cong (1-e)R$ is projective. Therefore a p.q.-Baer module satisfies the hypothesis of Proposition 2.2, hence we have the following corollary.

**Corollary 2.1** *[3, Theorem 11] Let $M_R$ be a right $R$-module. Then $M_R$ is a p.q.Baer-module if and only if $M[x]_{R[x]}$ is a p.q.Baer-module.*

Based on the fact that if $R$ is a commutative ring then $M_R$ is a p.p.-module if and only if $M_R$ is a p.q.Baer-module. We have the following corollaries.

**Corollary 2.2** *Assume that $R$ is a commutative ring. Then $M_R$ is a p.p.-module if and only if $M[x]_{R[x]}$ is a p.p.-module.*

**Corollary 2.3** *[8, Theorem 3.1] A ring $R$ is a right p.q.Baer-ring if and only if $R[x]$ is a right p.q.Baer-ring.*

In [15], Lee-Zhou introduced the following notation. For a module $M_R$, we consider $M[[x;\alpha]] = \{\sum_{i=0}^{\infty} m_i x^i : m_i \in M\}$. Then $M[[x;\alpha]]$ becomes a module over $R[[x;\alpha]]$ with the usual addition and the following scalar product operation: For $m(x) = \sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x) = \sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$, $m(x)f(x) = \sum_k (\sum_{i+j=k} m_i \alpha^i(a_j))x^k$. The module $M[[x;\alpha]]$ is called the *skew power series extension* of $M$.

Following [11], a ring $R$ is called *$\alpha$-compatible* if for each $a, b \in R, ab = 0 \Leftrightarrow a\alpha(b) = 0$. According to Krempa [13], an endomorphism $\alpha$ of a ring $R$ is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring $R$ is said to be *$\alpha$-rigid* if there exists a rigid endomorphism $\alpha$ of $R$. It was shown in [11, Lemma 2.2] that $R$ is $\alpha$-rigid if and only if $R$ is $\alpha$-compatible and reduced. Thus the $\alpha$-compatible ring is a generalization of *$\alpha$-rigid* rings to the more general case where $R$ is not assumed to be reduced. We extend the definition of an $\alpha$-compatible ring to the version of modules as follows.

**Definition 2.2** *A module $M_R$ is called $\alpha$-compatible if, for any $m \in M$ and any $a \in R$, $ma = 0$ if and only if $m\alpha(a) = 0$.*

The left version for a left $R$-module can be defined similarly. Motivated by the results in Baser [4], Lee and Zhou [15], we introduce the concept of *power series* $\alpha$-*quasi-Armendariz* modules which is the power series version of quasi-Armendariz modules.

**Definition 2.3** *$M_R$ is called a power series $\alpha$-quasi-Armendariz module if the following conditions are satisfied:*

(1) *$M_R$ is $\alpha$-compatible.*

(2) *For any $m(x)=\sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x)=\sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$, $m(x)R[[x;\alpha]]f(x)=0$ implies that $m_iRa_j=0$ for all $i$ and $j$.*

**Proposition 2.3** *Let $M_R$ be a $\alpha$-compatible module. Then we have the following:*

(1) *If $M_R$ is a right GPQ module, then $M_R$ is a power series $\alpha$-quasi-Armendariz module.*

(2) *If $M[[x;\alpha]]_{R[[x;\alpha]]}$ is a right GPQ module, then $M_R$ is right GPQ.*

**Proof.** (1) Assume that $m(x)=\sum_{i=0}^{\infty} m_i x^i \in M[[x;\alpha]]$ and $f(x)=\sum_{j=0}^{\infty} a_j x^j \in R[[x;\alpha]]$ such that $(\sum_{i=0}^{\infty} m_i x^i)R[[x;\alpha]](\sum_{j=0}^{\infty} a_j x^j)=0$ with $m_i\in M$, $a_j\in R$. Let $c$ be an arbitrary element of $R$. Then we have the following equation:

$$\sum_{k=0}^{\infty}\Big(\sum_{i+j=k} m_i x^i c a_j x^j\Big)=\sum_{k=0}^{\infty}\Big(\sum_{i+j=k} m_i \alpha^i(ca_j)x^{i+j}\Big)=0. \qquad (*)$$

We will show that $m_iRa_j=0$ for all $i$ and $j$. We proceed by induction on $i+j$. It is true for $i+j=0$ since $m_0Rb_0=0$ by $(*)$. Suppose that $m_iRa_j=0$ is true for $i+j\leqslant n-1$. Then $a_j\in r_R(m_iR)$ for $j=0,1,\cdots,n-1$ and $i=0,1,\cdots,n-1-j$. Since $M_R$ is a right GPQ module, there exists $t_{ij}\in r_R(m_iR)$ such that $t_{ij}a_j=a_j$ for $j=0,1,\cdots,n-1$ and $i=0,1,\cdots,n-1-j$. From $(*)$, we have

$$\sum_{i+j=k} m_i\alpha^i(ca_j)=0 \text{ for all } k\geq 0. \qquad (\dagger)$$

Let $f_j=t_{n-1-j,\,j}\cdots t_{1,j}$ for $j=0,1,\cdots,n-1$. It is clear that $f_ja_j=a_j$, and so $f_j\in r_R(m_0R)\cap r_R(m_1R)\cap\cdots\cap r_R(m_{n-1-j}R)$. If $k=n$, then the equation $(\dagger)$ becomes

$$m_0ca_n+m_1\alpha(ca_{n-1})+\cdots+m_n\alpha^n(ca_0)=0. \qquad (\sharp)$$

Replacing $c$ by $cf_0$ in $(\sharp)$, we obtain $m_0cf_0a_n+m_1\alpha(cf_0a_{n-1})+\cdots+m_n\alpha^n(cf_0a_0)=0$. Since $M_R$ is $\alpha$-compatible and $m_0Rf_j=m_1Rf_j=\cdots=m_{n-1-j}Rf_j=0$ for $j=0,1,\cdots,n-1$, it follows that $m_n\alpha^n(cf_0a_0)=m_ncf_0a_0=m_nca_0=0$. Hence $m_nRa_0=0$. Continuing this process by replacing $c$ by $cf_j$ in $(\dagger)$ and using $\alpha$-compatibility of $M_R$, we obtain $m_iRa_j=0$ for all $i+j=n$. This shows that $M_R$ is power series $\alpha$-quasi-Armendariz.

(2) The proof is similar to that of Lemma 2.3. □

**Corollary 2.4** *Let $R$ be an $\alpha$-compatible ring. If $R$ is right GPQ as a module, then $R$ is a power series $\alpha$-quasi-Armendariz ring.*

For a module $M_R$, let $M[x;\alpha] = \{\sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M\}$. Then $M[x;\alpha]$ becomes a module over $R[x;\alpha]$. Recall that $M_R$ is called *$\alpha$-quasi-Armendariz* module if the following conditions are satisfied:

(1) $M_R$ is *$\alpha$-compatible*.

(2) $m(x)R[x;\alpha]f(x) = 0$ with $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x;\alpha]$ and $f(x) = \sum_{j=0}^{t} a_j x^j \in R[x;\alpha]$ implies that $m_i R a_j = 0$ for all $i$ and $j$.

By analogy with the case of Lemma 2.3 and the proof of Proposition 2.3 we give the following proposition.

**Proposition 2.4** *Let $M_R$ be an $\alpha$-compatible module. Then $M_R$ is a right GPQ module if and only if $M[x;\alpha]_{R[x;\alpha]}$ is a right GPQ module. In this case, $M_R$ is $\alpha$-quasi-Armendariz.*

**Corollary 2.5** *Let $R$ be an $\alpha$-compatible ring. Then $R$ is a right p.q.-Baer ring if and only if $R[x;\alpha]$ is a right p.q.-Baer-ring.*

For a module $M_R$, consider $M[x,x^{-1};\alpha] = \{\sum_{i=-s}^{t} m_i x^i : s \geq 0, t \geq 0, m_i \in M\}$. Then $M[x,x^{-1};\alpha]$ becomes a module over $R[x,x^{-1};\alpha]$. We give the following definition by considering the definition of a quasi-Armendariz module.

**Definition 2.4** *A module $M_R$ is called Laurent $\alpha$-quasi-Armendariz if the following conditions are satisfied:*

*(1) $M_R$ is $\alpha$-compatible.*

*(2) For any $m(x) = \sum_{i=-s}^{t} m_i x^i \in M[x,x^{-1};\alpha]$ and $f(x) = \sum_{j=-\alpha}^{\beta} a_j x^j \in R[x,x^{-1};\alpha]$, $m(x)R[x,x^{-1};\alpha]f(x) = 0$ implies that $m_i R a_j = 0$ for all $i$ and $j$.*

**Proposition 2.5** *Let $\alpha$ be an automorphism of a ring $R$ and let $M_R$ be an $\alpha$-compatible module. Then $M_R$ is a right GPQ module if and only if $M[x,x^{-1};\alpha]_{R[x,x^{-1};\alpha]}$ is a right GPQ module. In this case, $M_R$ is Laurent $\alpha$-quasi-Armendariz.*

**Corollary 2.6** *Let $R$ be an $\alpha$-compatible ring. Then $R$ is a right p.q.-Baer ring if and only if $R[x,x^{-1};\alpha]$ is a right p.q.-Baer-ring.*

**Corollary 2.7** *Let $R$ be an $\alpha$-rigid ring. Then $R$ is a right p.q.-Baer ring if and only if $R[x,x^{-1};\alpha]$ is a right p.q.-Baer-ring.*

## 3. Related topics

In this section we relate the problem on the weak Armendariz property of a module to the formal triangular matrix ring constructed from a pair of rings $S, T$ and a bimodule ${}_SM_T$. Due to Lee and Wong [14], a ring $R$ is called weak Armendariz if for given $f(x) = a_0 + a_1 x$ and $g(x) = b_0 + b_1 x \in R[x]$, $f(x)g(x) = 0$ implies that $a_i b_j = 0$ for each $i, j$ (the converse is obviously true).

We say a module $M_R$ is a weak Armendariz module if whenever $m(x)f(x) = 0$ where $m(x) = m_0 + m_1 x \in M[x]$ and $f(x) = a_0 + a_1 x \in R[x]$, then $m_i a_j = 0$ for each $i, j$. It is obvious that Armendariz modules are weak Armendariz. Note that there exists a weak Armenariz module $M_R$ which is not right GPQ by Example 2.2. The following example shows that there exists a weak Armendariz module which is not Armendariz.

**Example 3.1** *Let $R = \mathbb{Z}_3[x,y]/(x^3, x^2y^2, y^3)$, where $\mathbb{Z}_3$ is the Galois field of order 3. $\mathbb{Z}_3[x,y]$ is the polynomial* *ring with two indeterminates $x, y$ over $\mathbb{Z}_3$, and $(x^3, x^2y^2, y^3)$ is the ideal of $\mathbb{Z}_3[x, y]$ generated by $x^3, x^2y^2, y^3$. Let $R[t]$ be the polynomial ring with an indeterminate $t$ over $R$. Since $(\bar{x}+\bar{y}t)^3 = (\bar{x}+\bar{y}t)(\bar{x}^2+2\bar{x}\bar{y}t+\bar{y}^2t^2) = 0$, but $\bar{x}\bar{y}^2 \neq 0$. Then $R_R$ is not Armendariz, but it is weak Armendariz by* [14, *Example* 3.2].

**Proposition 3.1 .** *If $M_R$ be a reduced module and $R$ is a reduced ring. Then $M_R$ is a weak Armendariz module if and only if its torsion submodule $T(M)$ is weak Armendariz as a right $R$-module.*

**Proof.** If $T(M)$ is weak Armendariz. Let $m(x) = m_0 + m_1x \in M[x]$ and $f(x) = a_0 + a_1x \in R[x]$ such that $m(x)f(x) = 0$. Then we have $m_0a_0 = 0$, $m_0a_1 + m_1a_0 = 0$, $m_1a_1 = 0$. we can assume $a_0 \neq 0$. If we multiply the second equation by $a_0$ from the right, we can obtain that $m(x) \in T(M)[x]$ by the hypothesis. Since $T(M)$ is weak Armendariz, it follows that $m_ia_j = 0$ for each $i, j$. The other implication is trivial. □

Given a pair of rings $S, T$ and a bimodule ${}_SM_T$, let $R = \begin{pmatrix} S & M \\ 0 & T \end{pmatrix}$ denote the set of all symbols $\begin{pmatrix} s & m \\ 0 & t \end{pmatrix}$, where $s \in S, t \in T$, $m \in M$. It is straightforward to verify that $R$ is a ring with the usual rules for addition and the following multiplication of matrices:

$$\begin{pmatrix} s & m \\ 0 & t \end{pmatrix}\begin{pmatrix} s' & m' \\ 0 & t' \end{pmatrix} = \begin{pmatrix} ss' & sm' + mt' \\ 0 & tt' \end{pmatrix}.$$

The ring $R$ above constructed from $S, T$ and ${}_SM_T$ is called the formal triangular matrix ring. Note that if $M$ is an $(S, T)$-bimodule, then $M[x]$ is an $(S[x], T[x])$-bimodule.

The rest of this section is devoted to a discussion of some basic facts concerning the foregoing formal triangular matrix ring. The following proposition gives the relationship of weak Armendariz property between $R, S, T$ and ${}_SM$, $M_T$.

**Proposition 3.2** *Suppose that $S$ and $T$ are two rings, $M$ is an $(S, T)$-bimodule and $R$ is the formal triangular matrix ring constructed from $S, T$ and ${}_SM_T$. Then $R$ is a weak Armendariz ring if and only if the following three conditions hold:*

(1) *$S$ and $T$ are weak Armendariz rings.*

(2) *${}_SM$ and $M_T$ are weak Armendariz as a left $S$-module and right $R$-module.*

(3) *If $s(x)s'(x) = t(x)t'(x) = 0$, then $s(x)M[x] \cap M[x]t'(x) = 0$.*

**Proof.** First we shall prove that $R$ is a weak Armendariz ring if the given three conditions are satisfied. Suppose that $f(x)g(x) = 0$ with

$$f(x) = \begin{pmatrix} s_0 & m_0 \\ 0 & t_0 \end{pmatrix} + \begin{pmatrix} s_1 & m_1 \\ 0 & t_1 \end{pmatrix}x, \;\; g(x) = \begin{pmatrix} s_0' & m_0' \\ 0 & t_0' \end{pmatrix} + \begin{pmatrix} s_1' & m_1' \\ 0 & t_1' \end{pmatrix}x \in R[x].$$

Let $s(x) = s_0 + s_1x$, $s'(x) = s_0' + s_1'x$, $m(x) = m_0 + m_1x$, $m'(x) = m_0' + m_1'x$ and $t(x) = t_0 + t_1x$, $t'(x) = t_0' + t_1'x$. Then $s(x), s'(x) \in S[x]$, $m(x), m'(x) \in M[x]$ and $t(x), t'(x) \in T[x]$. It is easy to see:

$$\left[\begin{pmatrix} s_0 & m_0 \\ 0 & t_0 \end{pmatrix} + \begin{pmatrix} s_1 & m_1 \\ 0 & t_1 \end{pmatrix}x\right]\left[\begin{pmatrix} s_0' & m_0' \\ 0 & t_0' \end{pmatrix} + \begin{pmatrix} s_1' & m_1' \\ 0 & t_1' \end{pmatrix}x\right]$$

$$=\begin{pmatrix} s(x)s'(x) & s(x)m'(x)+m(x)t'(x) \\ 0 & t(x)t'(x) \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus $s(x)s'(x) = 0, t(x)t'(x) = 0$ and $s(x)m'(x) + m(x)t'(x) = 0$. Since $S$ and $T$ are both weak Armendariz rings, we have $s_i s'_j = 0$ and $t_i t'_j = 0$ for each $i, j$. Moreover, $s(x)m'(x) = -m(x)t'(x) \in s(x)M[x] \cap M[x]t'(x) = 0$ by (3). It follows that $s(x)m'(x) = m(x)t'(x) = 0$. Since $_SM$ and $M_T$ are weak Armendariz as a left $S$-module and right $R$-module by (2), we have $s_i m'_j = 0$ and $m_i t'_j = 0$ for each $i, j$. Therefore

$$\begin{pmatrix} s_i & m_i \\ 0 & t_i \end{pmatrix}\begin{pmatrix} s'_j & m'_j \\ 0 & t'_j \end{pmatrix}=\begin{pmatrix} s_i s'_j & s_i m'_j + m_i t'_j \\ 0 & t_i t'_j \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

for all $i$ and $j$. This shows that the desired implication is established.

Conversely, if $R$ is a weak Armendariz ring. We shall prove that the other implication is true.

(1) This is because

$$S \cong \left\{\begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} | s \in S\right\}, T \cong \left\{\begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} | t \in T\right\}.$$

(2) Let $s(x) = s_0 + s_1 x \in S[x]$, $t(x) = t_0 + t_1 x \in T[x]$ and $m(x) = m_0 + m_1 x$, $m'(x) = m'_0 + m'_1 x \in M[x]$. Suppose $s(x)m(x) = m'(x)t(x) = 0$. Then

$$\left[\begin{pmatrix} s_0 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} s_1 & 0 \\ 0 & 0 \end{pmatrix}x\right]\left[\begin{pmatrix} 0 & m_0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & m_1 \\ 0 & 0 \end{pmatrix}x\right]=\begin{pmatrix} s(x) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 0 & s(x)m(x) \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since $R$ is a weak Armendariz ring, we have $\begin{pmatrix} s_i & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & m_j \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for each $i, j$. This implies that $s_i m_j = 0$ for each $i, j$. Therefore $_SM$ is weak Armendariz as a left $S$-module. The argument that $M_T$ is weak Armendariz as a right $R$-module is similar.

(3) Note that if $R$ is weak Armendariz, we can prove that $R[x]$ is weak Armendariz by a similar way in [2, Theorem 2]. Assume that $s(x)s'(x) = t(x)t'(x) = 0$ and $s(x)m(x) = -m'(x)t'(x) \neq 0$ with $m(x), m'(x) \in M[x]$. Then

$$\left[\begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix}+\begin{pmatrix} 0 & m'(x) \\ 0 & 0 \end{pmatrix}y\right]\cdot\left[\begin{pmatrix} s'(x) & 0 \\ 0 & t'(x) \end{pmatrix}+\begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix}y\right]=0,$$

but

$$\begin{pmatrix} s(x) & 0 \\ 0 & t(x) \end{pmatrix}\begin{pmatrix} 0 & m(x) \\ 0 & 0 \end{pmatrix}\neq\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a contradiction. It follows that (3) is true, as desired. □

Given a ring $R$ and a bimodule ${}_RM_R$, the trivial extension of $R$ by $M$ is the ring $T(R,M)=R\bigoplus M$ with the usual addition and the multiplication

$$(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+m_1r_2).$$

This is isomorphic to the ring of all matrix $\begin{pmatrix} r & m \\ 0 & r \end{pmatrix}$, where $r\in R$, $m\in M$ and the usual matrix operations are used. Note that if $M$ is an $(R,R)$-bimodule, then $M[x]$ is an $(R[x],R[x])$-bimodule and $T(R[x],M[x])$=$T(R,M)[x]$.

As an immediate consequence of the foreging proposition we have the following characterization considering the trivial extension for a given ring $R$ and a module ${}_RM_R$.

**Corollary 3.1** *Let $M$ be an $(R,R)$-bimodule. Then the trivial extension $T(R,M)$ is a weak Armendariz ring if and only if the following three conditions hold:*

(1) *$R$ is a weak Armendariz ring.*

(2) *$M$ is a left and right weak Armendariz $R$-module.*

(3) *If $f(x)g(x)=0$ in $R[x]$, then $f(x)M[x]\cap M[x]g(x)=0$.*

As an application, we consider the case when the trivial extension $T(R,R)$ of $R$ by $R$ is weak Armendariz if $R$ is a weak Armendariz ring.

**Corollary 3.2** *The trivial extension $T(R,R)$ is a weak Armendariz ring if and only if the following two conditions are satisfied:*

(1) *$R$ is a weak Armendariz ring.*

(2) *If $f(x)g(x)=0$ in $R[x]$, then $f(x)R[x]\cap R[x]g(x)=0$.*

The following example shows that the condition If $f(x)g(x)=0$ in $R[x]$, then $f(x)M[x]\cap M[x]g(x)=0$ in Corollary 3.2 is not superfluous.

**Example 3.2** *Let $S$ be a reduced ring. Then the trivial extension $T(S,S)$ is an Armendariz ring by* [14, *Theorem* 2.3], *and hence $T(S,S)$ is weak Armendariz. Let $R=T(S,S)$, we prove that $T(R,R)$ is not weak Armendariz. In fact, let*

$$f(x)=\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}+\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}x$$

*and*

$$g(x)=\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}+\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix}x$$

*be two polynomials in $T(R,R)$. Then $f(x)g(x)=0$, but*

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\right)\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \neq 0.$$

*This shows that $T(R,R)$ is not weak Armendariz.*

It was shown in [15] that a module $M_R$ is Armendariz if and only if $M[x,x^{-1}]_{R[x,x^{-1}]}$ is Armendariz. Similar to the proof of [15, Theorem 1.12], we can get the following

**Proposition 3.3** *$M_R$ is a weak Armendariz module if and only if $M[x,x^{-1}]_{R[x,x^{-1}]}$ is weak Armendariz.*

**Acknowledgements**

The authors would like to thank the referees for the valuable suggestions and many careful comments that improved the present paper. The authors also wish to thank Professor Gary F. Birkenmeier for his helpful communications. This work was supported by the Program Sponsored for Scientific Innovation Research of College Graduate in Jiangsu Province (No.CX10B_001z) and was partially supported by the National Natural Science Foundation of China (No.10971090) and the Youth Fund of Jiangxi Provincial Education Department (No.GJJ10155).

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Received: 22.12.2009