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The cyclicity of the period annulus of a quadratic reversible system with one center of genus one*

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Abstract

This paper is concerned with a quadratic reversible and non-Hamiltonian system with one center of genus one. By using the properties of related elliptic integrals and the geometry of some planar curves defined by them, we prove that the cyclicity of the period annulus of the considered system under small quadratic perturbations is two. This verifies Gautier's conjecture about the cyclicity of the related period annulus.

Key Words: Cyclicity, bifurcation of limit cycles, quadratic perturbations, period annulus, a quadratic reversible system with one center of genus one

1. Introduction and statement of the main result

It is well known that the weak Hilbert 16th problem asks for the least upper bound of the number of zeros of the associated Abelian integral. This problem in the quadratic Hamiltonian case has already been solved, that is, the least upper bound of the number of zeros of the Abelian integrals associated with quadratic Hamiltonian systems under quadratic perturbations is two; see [9, 19, 7, 2, 14, 3] and the references therein.

The next natural step is to consider quadratic reversible but non-Hamiltonian systems. Form [10], the quadratic reversible systems can be written in the real form

$$\begin{aligned} \dot{x} &= y + (a+b+2)x^2 - (a+b-2)y^2, \\ \dot{y} &= -x[1-2(a-b)y], \end{aligned} \tag{1.1}$$

where $a, b \in R$. If $c = a - b \neq 0$, we can make the transformation $(x, y, t) \to (x/c, y/c, -t)$, and let $\overline{a} = -(a+b+2)/(a-b)$, $\overline{b} = (a+b-2)/(a-b)$, then system (1.1) becomes the following

$$\begin{aligned} \dot{x} &= -y + \overline{a}x^2 + \overline{b}y^2, \\ \dot{y} &= x(1-2y). \end{aligned} \tag{1.2}$$

Studies show that the orbital topological properties of quadratic reversible systems under quadratic perturbations are very rich. Most mathematicians working in this field believe that the weak Hilbert 16th problem for quadratic reversible systems is very interesting and difficult.

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The bifurcation of limit cycles from system (1.2) under quadratic perturbations has been studied in recent years. Such as [1] for the isochronous centers; [17] for the unbounded heteroclinic loop; [6, 11, 16, 18, 12] for $\bar{a}=-3$ with different $\bar{b}$; [4] for $\bar{a}=-4$ and $\bar{a}=2$ with $0<\bar{b}<2$; [5] for $\bar{a}=-1/2$ with $0<\bar{b}<2$ and [15] for $\bar{a}=-3/2$ with $b\in(-\infty,0]\cup\{2\}$. However, to our knowledge, known results are very limited.

Recently, reference [8] lists all quadratic centers of genus one, including 18 reversible cases (r1)–(r18), 6 reversible Lotka-Volterra cases (rlv1)-(rlv6) and 5 generic (i.e. non-reversible) Lotka-Volterra cases (lv1)–(lv5), and gives the conjecture about the cyclicity of the period annuli of quadratic centers of genus one. In this paper, we will study the case $a=-8, b=-2$ in (1.1), which is (r9) from [8], and verify Conjecture 1 about the cyclicity of the period annulus of the system

$$\begin{aligned} \dot{x} &= y-8x^2+12y^2, \\ \dot{y} &= -x(1+12y). \end{aligned} \tag{1.3}$$

See [8] for more details.

In order to have $y=0$ as a symmetric axis, we make the change $(x,y)\to(y,-x)$, and get

$$\begin{aligned} \dot{x} &= y(1-12x), \\ \dot{y} &= -x+12x^2-8y^2, \end{aligned} \tag{1.4}$$

which has a first integral of the form

$$H^*(x,y)=(1-12x)^{-\frac{4}{3}}(\frac{1}{2}y^2+\frac{1}{96}(1-12x)^2+\frac{1}{48}(1-12x))=h$$

with the integrating factor $(1-12x)^{-7/3}$.

If we make the transformation $(x,y,t)\to((1-X)/12, Y, -\tau/12)$ followed by $(X,Y,\tau)\to(x,y,t)$, then system (1.4) becomes

$$\begin{aligned} \dot{x} &= xy, \\ \dot{y} &= \tfrac{2}{3}y^2+\tfrac{1}{144}x-\tfrac{1}{144}x^2 \end{aligned} \tag{1.5}$$

with a first integral of the form

$$H(x,y)=x^{-\frac{4}{3}}(\frac{1}{2}y^2+\frac{1}{96}x^2+\frac{1}{48}x)=h \tag{1.6}$$

and the corresponding integrating factor $\mu(x)=x^{-7/3}$.

System (1.5) has two singularities $(0,0)$ and $(1,0)$. The former is a degenerate singularity, and the latter is a center. The closed orbits of system (1.5) are

$$\Gamma_h=\{(x,y): x^{-\frac{4}{3}}(\frac{1}{2}y^2+\frac{1}{96}x^2+\frac{1}{48}x)=h\},\quad h\in(\frac{1}{2^5},+\infty).$$

The orientation of Γ_h is clockwise, and $1/2^5$ corresponds to the critical value of H at the center $(1,0)$. As $h\to+\infty$, Γ_h expands to the hemicycle H formed by the invariant line $\{x=0\}$ and the half of the equator and surrounding the center.

The phase portrait of system (1.5) in the Poincaré disk is shown in Figure 1.

The main result of this paper is the following theorem.

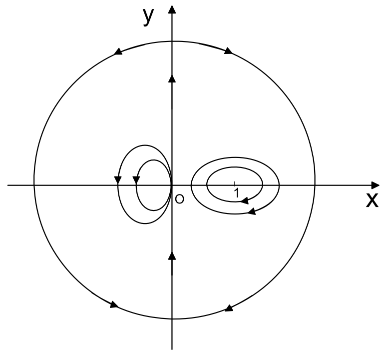

Figure 1. The phase portrait of system (1.5) in the Poincaré disk.

Theorem 1.1 *The cyclicity of the period annulus of system* (1.5) *under small quadratic perturbations is two.*

Theorem 1.1 here is in fact Conjecture 1 for system (1.3) in [8].

As is well known, the problem concerning the cyclicity of the period annulus can be reduced to counting the number of zeros of the associated Abelian integral. From [10], we know that under small quadratic perturbations, the cyclicity of the period annulus of system (1.5) is equal to the least upper bound of the number of zeros of the following Abelian integral

$$\begin{aligned} I(h) &= \int\int_{H(x,y)<h} x^{-\frac{7}{3}}(\alpha+\beta x+\gamma x^{-1})dxdy \\ &= \alpha I_0(h)+\beta I_1(h)+\gamma I_{-1}(h), \end{aligned} \tag{1.7}$$

where $I_k(h)=\oint_{\Gamma_h} x^{k-7/3}ydx>0$ for $h\in(1/2^5,+\infty)$ and $I_k(1/2^5)=0$ with $k=0,1$ and -1, α,β and γ are any constants.

Hence Theorem 1.1 is equivalent to the following theorem.

Theorem 1.2 *For* $h\in(\frac{1}{2^5},+\infty)$*, the least upper bound of zeros(counting multiplicity) of the Abelian integral* $I(h)$ *related to system (1.5) under small quadratic perturbations is equal to two.*

In the following sections, we will prove Theorem 1.2 instead of Theorem 1.1.

The remainder of this paper is organized as follows. In Section 2, we derive the closed Picard-Fuchs equations satisfied by $I_k(h)$ for $h\in(1/2^5,+\infty)$ and the expansions of $I_k(h)$ in terms of h as $h\to+\infty$ with $k=0,1,-1$ and $-1/3$. The properties of two planar curves called the auxiliary curve, and centroid curve such as the monotonicity and convexity and so on, are studied respectively in Sections 3 and 4. In the last section, we first estimate the number of zeros of the associated Abelian integral, and then prove Theorem 1.2.

2. Some preliminary results

The aim of this section is to derive the Picard-Fuchs equation satisfied by $I_k(h)$ for $h\in(1/2^5,+\infty)$ and the expansions of $I_k(h)$ in terms of h as $h\to+\infty$ with $k=0,1,-1$ and $-1/3$.

Lemma 2.1 *For system* (1.5), *the vector function* $U(h) = (I_0(h), I_1(h), I_{-1}(h), I_{-1/3}(h))^T$ *satisfies the Picard-Fuchs equation*

$$U(h) = A(h)U'(h), \tag{2.1}$$

where

$$A(h) = \begin{pmatrix} -2h & 0 & 0 & \frac{1}{16} \\ -6h & h & 0 & \frac{5}{32} \\ 0 & 0 & \frac{2}{11}h & -\frac{1}{176} \\ -\frac{1}{3584h} & 0 & 0 & \frac{2}{7}h \end{pmatrix}.$$

Proof. Differentiating (1.6) with respect to h and x respectively, we get

$$\frac{\partial y}{\partial h} = \frac{x^{\frac{4}{3}}}{y}, \tag{2.2}$$

$$\frac{\partial y}{\partial x} = \frac{\frac{4}{3}hx^{\frac{1}{3}} - \frac{1}{48}x - \frac{1}{48}}{y}. \tag{2.3}$$

Noting the orientation of Γ_h and (2.2), we obtain that

$$I_k'(h) = \oint_{\Gamma_h} x^{k-\frac{7}{3}} \frac{\partial y}{\partial h}\,\mathrm{d}x = \oint_{\Gamma_h} \frac{x^{k-1}}{y}\,\mathrm{d}x > 0, \quad h \in (\frac{1}{2^5}, +\infty). \tag{2.4}$$

From (1.6) and (2.4), we know

$$\begin{aligned} I_k(h) &= \oint_{\Gamma_h} \frac{x^{k-\frac{7}{3}}y^2}{y}\,dx = \oint_{\Gamma_h} \frac{x^{k-\frac{7}{3}}(2hx^{\frac{4}{3}} - \frac{1}{48}x^2 - \frac{1}{24}x)}{y}\,dx \\ &= 2hI_k'(h) - \frac{1}{48}I'_{k+\frac{2}{3}}(h) - \frac{1}{24}I'_{k-\frac{1}{3}}(h). \end{aligned} \tag{2.5}$$

Meanwhile, if $k \neq \frac{4}{3}$, by integrating by parts and using (2.3) and (2.4), we express $I_k(h)$ as follows

$$\begin{aligned} I_k(h) &= \oint_{\Gamma_h} x^{k-\frac{7}{3}} y\,dx = \frac{1}{k-\frac{4}{3}} \oint_{\Gamma_h} y\,dx^{k-\frac{4}{3}} \\ &= -\frac{1}{k-\frac{4}{3}} \oint_{\Gamma_h} x^{k-\frac{4}{3}} \frac{\frac{4}{3}hx^{\frac{1}{3}} - \frac{1}{48}x - \frac{1}{48}}{y}\,dx \\ &= -\frac{4}{3k-4}hI_k'(h) + \frac{1}{16(3k-4)}I'_{k+\frac{2}{3}}(h) + \frac{1}{16(3k-4)}I'_{k-\frac{1}{3}}(h). \end{aligned} \tag{2.6}$$

Removing $I'_{k+2/3}(h)$, $I'_{k-1/3}(h)$ and $I_k(h)$ from (2.5) and (2.6), respectively, we have

$$I_k(h) = \frac{2}{3k-1}hI_k'(h) - \frac{1}{16(3k-1)}I'_{k-\frac{1}{3}}(h), \tag{2.7}$$

$$I_k(h) = -\frac{2}{6k-5}hI_k'(h) + \frac{1}{16(6k-5)}I'_{k+\frac{2}{3}}(h), \tag{2.8}$$

and

$$96(3k-2)hI_k'(h)-(6k-5)I_{k-\frac{1}{3}}'(h)-(3k-1)I_{k+\frac{2}{3}}'(h)=0,\qquad k\neq\frac{4}{3}. \tag{2.9}$$

Taking $k=0, k=1$ in (2.7) and $k=-1, k=-1/3$ in (2.8) respectively, we get

$$\begin{aligned}
I_0(h) &= -2hI_0'(h)+\frac{1}{16}I_{-\frac{1}{3}}'(h),\\
I_1(h) &= hI_1'(h)-\frac{1}{32}I_{\frac{2}{3}}'(h),\\
I_{-1}(h) &= \frac{2}{11}hI_{-1}'(h)-\frac{1}{176}I_{-\frac{1}{3}}'(h),\\
I_{-\frac{1}{3}}(h) &= \frac{2}{7}hI_{-\frac{1}{3}}'(h)-\frac{1}{112}I_{\frac{1}{3}}'(h).
\end{aligned} \tag{2.10}$$

To get the closed Picard-Fuchs equation of $I_0(h)$, $I_1(h)$, $I_{-1}(h)$ and $I_{-1/3}(h)$, we take $k=0, k=1/3$ in (2.9), then obtain

$$\begin{aligned}
I_{\frac{2}{3}}'(h) &= 192hI_0'(h)-5I_{-\frac{1}{3}}'(h),\\
I_{\frac{1}{3}}'(h) &= \frac{1}{32h}I_0'(h).
\end{aligned} \tag{2.11}$$

Substituting (2.11) into (2.10), and taking $U(h)=(I_0(h),I_1(h),I_{-1}(h),I_{-1/3}(h))^T$, we finally get the Picard-Fuchs equation (2.1). □

By direct calculation, we have

$$G(h)U''(h)=A_1(h)U'(h), \tag{2.12}$$

and

$$G(h)U'(h)=A_2(h)U(h), \tag{2.13}$$

where

$$G(h)=2h(1-32768h^3),$$

$$A_1(h)=\begin{pmatrix} 2+98304h^3 & 0 & 0 & -5120h^2\\ -1+196608h^3 & 0 & 0 & -5120h^2\\ 5 & 0 & 9-294912h^3 & -5120h^2\\ 160h & 0 & 0 & -163840h^3 \end{pmatrix},$$

and

$$A_2(h)=\begin{pmatrix} 32768h^3 & 0 & 0 & -7168h^2\\ -5+196608h^3 & 2-65536h^3 & 0 & -7168h^2\\ 1 & 0 & 11-360448h^3 & -7168h^2\\ 32h & 0 & 0 & -229376h^3 \end{pmatrix}.$$

Lemma 2.2 *When $h \to +\infty$, we have the following expansions*

$$\begin{aligned} I_0(h) &= a_0 h^{\frac{5}{2}} + a_1 h^{-\frac{1}{2}} \ln h + O(h^{-\frac{1}{2}}), \\ I_1(h) &= b_0 h + O(h^{-\frac{1}{2}}), \\ I_{-1}(h) &= c_0 h^{\frac{11}{2}} + O(h^{\frac{5}{2}}), \\ I_{-\frac{1}{3}}(h) &= d_0 h^{\frac{7}{2}} + O(h^{\frac{1}{2}}), \end{aligned} \tag{2.14}$$

where

$$a_0 = 2^{\frac{19}{2}} 3^3 B(2, \frac{3}{2}) > 0, \quad a_1 = -2^{-\frac{11}{2}} 3 < 0, \quad b_0 = 2^3 3^{\frac{3}{2}} B(\frac{1}{2}, \frac{3}{2}) > 0,$$
$$c_0 = 2^{\frac{43}{2}} 3^6 B(5, \frac{3}{2}) > 0, \quad d_0 = 2^{\frac{27}{2}} 3^4 B(3, \frac{3}{2}) > 0,$$

and $B(\alpha, \beta)$ is the following Beta-function

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{1}{\alpha} + \sum_{n=1}^{+\infty} \frac{(-1)^n(\beta-1)(\beta-2)\cdots(\beta-n)}{n!(\alpha+n)}.$$

Proof. In the following, we deduce the expansion of $I_0(h)$ in terms of h as $h \to +\infty$. Others are similar. We omit them here.

Since Γ_h is symmetric with respect to x-axis, we only need to consider the case $y > 0$.

We know that the equation of Γ_h is

$$\frac{1}{2}y^2 + \frac{1}{96}x^2 + \frac{1}{48}x - x^{\frac{4}{3}}h = 0.$$

Assume that Γ_h has two intersection points with x-axis, denoted respectively by $(\xi_h, 0)$ and $(\eta_h, 0)$ with $0 < \xi_h < 1 < \eta_h < +\infty$, see Figure 2.

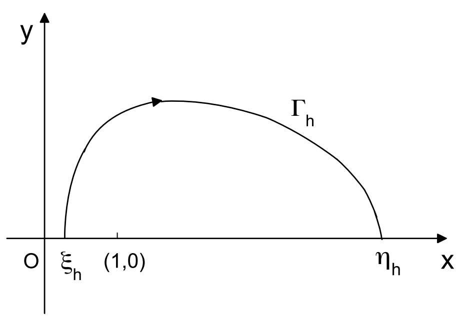

Figure 2. The behavior of Γ_h above x-axis.

It is easy to know

$$\xi_h \sim (48h)^{-3}, \quad \eta_h \sim (96h)^{\frac{3}{2}}, \quad h \to +\infty.$$

By the definition of $I_0(h)$, we get

$$I_0(h) = \oint_{\Gamma_h} x^{-\frac{7}{3}} y \,\mathrm{d}x = 2\int_{\xi_h}^{\eta_h} x^{-\frac{7}{3}} \sqrt{2hx^{\frac{4}{3}} - \frac{1}{48}x^2 - \frac{1}{24}x}\, dx$$
$$= 2\int_{\xi_h}^{\eta_h} x^{-\frac{7}{3}} (2hx^{\frac{4}{3}})^{\frac{1}{2}} \sqrt{1 + \frac{1}{h}(-\frac{1}{96}x^{\frac{2}{3}} - \frac{1}{48}x^{-\frac{1}{3}})}\, dx$$
$$= 2^{\frac{3}{2}} h^{\frac{1}{2}} \int_{\xi_h}^{\eta_h} x^{-\frac{5}{3}} [1 + \sum_{n=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} (-\frac{1}{96}x^{\frac{2}{3}} - \frac{1}{48}x^{-\frac{1}{3}})^n h^{-n}]\, dx$$
$$= 3^3 2^{\frac{17}{2}} h^{\frac{5}{2}} - 2^{-\frac{9}{2}} h^{-\frac{1}{2}} + M,$$

where

$$M = 2^{\frac{3}{2}} h^{\frac{1}{2}} \int_{\xi_h}^{\eta_h} x^{-\frac{5}{3}} \sum_{n=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} (-\frac{1}{96}x^{\frac{2}{3}} - \frac{1}{48}x^{-\frac{1}{3}})^n h^{-n}\, dx$$
$$= 2^{\frac{3}{2}} \sum_{n=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} (-\frac{1}{96})^n h^{\frac{1}{2}-n} \sum_{k=0}^{n} C_n^k 2^{n-k} \int_{\xi_h}^{\eta_h} x^{-\frac{1}{3}n+k-\frac{5}{3}}\, dx.$$

When $-\frac{1}{3}n + k - \frac{5}{3} \neq -1$, we have

$$M = 2^{\frac{3}{2}} [\sum_{n=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} (-\frac{1}{96})^n$$
$$\cdot (\sum_{k=0}^{n} C_n^k 2^{n-k} \frac{1}{-\frac{1}{3}n + k - \frac{2}{3}} (96)^{-\frac{1}{2}n+\frac{3}{2}k-1} h^{-\frac{3}{2}n+\frac{3}{2}k-\frac{1}{2}}$$
$$- \sum_{k=0}^{n} C_n^k 2^{n-k} \frac{1}{-\frac{1}{3}n + k - \frac{2}{3}} (48)^{n-3k+2} h^{\frac{5}{2}-3k})].$$

Taking $k = 0$, we get the leading term of M, that is

$$3^3 2^{\frac{17}{2}} h^{\frac{5}{2}} + 2^{\frac{3}{2}} \sum_{n=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} (-\frac{1}{96})^n 2^n \frac{3}{n+2} (48)^{n+2} h^{\frac{5}{2}}$$
$$= 3^3 2^{\frac{19}{2}} B(2, \frac{3}{2}) h^{\frac{5}{2}}.$$

When $-\frac{1}{3}n + k - \frac{5}{3} = -1$, i.e. $k = \frac{1}{3}n + \frac{2}{3}$,

$$M = 2^{\frac{3}{2}} \sum_{n=1}^{+\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} (-\frac{1}{96})^n$$
$$\cdot (C_n^{\frac{1}{3}n+\frac{2}{3}} 2^{\frac{2}{3}n-\frac{2}{3}} \frac{9}{2} h^{\frac{1}{2}-n} \ln h + C_n^{\frac{1}{3}n+\frac{2}{3}} 2^{\frac{2}{3}n-\frac{2}{3}} \ln(3^{\frac{9}{2}} 2^{\frac{39}{2}}) h^{\frac{1}{2}-n}).$$

In this case, taking $n = 1$, then $k = 1$, we obtain the following term

$$2^{\frac{3}{2}} \frac{1}{2} (-\frac{1}{96}) (\frac{9}{2} h^{-\frac{1}{2}} \ln h) = -2^{-\frac{11}{2}} 3 h^{-\frac{1}{2}} \ln h,$$

and $O(h^{-\frac{1}{2}})$.

Hence we have

$$I_0(h) = a_0 h^{\frac{5}{2}} + a_1 h^{-\frac{1}{2}} \ln h + O(h^{-\frac{1}{2}}),$$

where $a_0 = 2^{19/2}3^3 B(2, 3/2) > 0$ and $a_1 = -2^{-11/2}3 < 0$. □

Remark 2.3

1. Integrals $I_k'(h) = \oint_{\Gamma_h} x^{k-1}/y \, \mathrm{d}x > 0$ for $h \in (1/2^5, +\infty)$ with $k = 0, 1, -1, -\frac{1}{3}$.
2. In Lemma 2.2, the terms $O(h^{-1/2}), O(h^{5/2})$ and $O(h^{1/2})$ have no contribution to our following calculation. Hence we omit their exact expressions.

Definition 2.4 *For $h \in (1/2^5, +\infty)$, we define the functions*

$$\nu(h) = \frac{I_0'(h)}{I_{-\frac{1}{3}}'(h)}, \quad \omega(s) = h\nu(h),$$

where $s = h^3$.

For $s \in (1/2^{15}, +\infty)$, we obtain a curve

$$C_\omega = \{(s, \omega) : s \in (\frac{1}{2^{15}}, +\infty), \omega = \omega(s)\}$$

in (s, ω)-plane, which is called the auxiliary curve.

3. The properties of the auxiliary curve C_ω

Lemma 3.1 *For $s \in (1/2^{15}, +\infty)$, $\omega(s)$ satisfies the following Riccati equation*

$$3s(1 - 32768s)\omega' = -80\omega^2 + (2 + 98304s)\omega - 2560s,$$

which is equivalent to

$$\begin{aligned} \dot{s} &= 3s(1 - 32768s), \\ \dot{\omega} &= -80\omega^2 + (2 + 98304s)\omega - 2560s. \end{aligned} \tag{3.1}$$

Proof. Differentiating $\nu(h) = \frac{I_0'(h)}{I_{-1/3}'(h)}$ with respect to h, we get

$$\nu'(h) = \frac{I_0''(h) I_{-\frac{1}{3}}'(h) - I_0'(h) I_{-\frac{1}{3}}''(h)}{(I_{-\frac{1}{3}}'(h))^2}.$$

Removing $I_0''(h)$ and $I_{-\frac{1}{3}}''(h)$ by (2.12), we get the following Riccati equation

$$\nu' = \frac{-80h\nu^2 + (1 + 131072h^3)\nu - 2560h^2}{h(1 - 32768h^3)}. \tag{3.2}$$

Since

$$\frac{d\omega}{ds} = \frac{d\omega}{dh}\frac{dh}{ds} = (\nu + h\nu')\frac{1}{3h^2},$$

Removing ν' by (3.2), we get

$$\begin{aligned}\frac{d\omega}{ds} &= \frac{-80h^2\nu^2 + (98304h^3 + 2)h\nu - 2560h^3}{3h^3(1 - 32768h^3)} \\ &= \frac{-80\omega^2 + (2 + 98304s)\omega - 2560s}{3s(1 - 32768s)}.\end{aligned}$$

Lemma 3.1 is proved. □

Since $\omega \to 1/2^5$ as $s \to 1/2^{15}$, if we define the value of $\omega(s)$ at $1/2^{15}$ by its limit as $s \to 1/2^{15}$, then the domain of $\omega(s)$ can be extended to $[1/2^{15}, +\infty)$. It is easy to know $(1/2^{15}, 1/2^5)$ is the saddle-node of system (3.1), hence curve C_ω is a trajectory of system (3.1) passing through saddle-node $(1/2^{15}, 1/2^5)$.

In the following, we study the convexities of C_ω near the two endpoints.

Lemma 3.2 *We have*

$$\omega'(\frac{1}{2^{15}}) = -\frac{512}{3}, \qquad \omega''(\frac{1}{2^{15}}) = \frac{322961408}{27}.$$

Proof. From Lemma 3.1, we know that $\omega(s)$ satisfies equation (3.1).

Since $I_0(h)$ and $I_{-1/3}(h)$ are analytical at $h = 1/2^5$, we suppose that $\omega(s)$ has the following expansion near the point $(1/2^{15}, 1/2^5)$

$$\omega(s) = \frac{1}{2^5} + \omega_1(s - \frac{1}{2^{15}}) + \frac{\omega_2}{2!}(s - \frac{1}{2^{15}})^2 + \cdots. \tag{3.3}$$

Then substituting (3.3) into the following equality

$$\dot{\omega} - \dot{s}\frac{d\omega}{ds} = 0,$$

and comparing the coefficients on both side, we have

$$\omega_1 = -\frac{512}{3}, \quad \omega_2 = \frac{322961408}{27}.$$

□

Lemma 3.3 *When $0 < 1/s \ll 1$, we have that $\omega'(s) < 0$, $\omega''(s) > 0$.*

Proof. From (2.14), we know that when $h \to +\infty$,

$$(h\frac{I_0'(h)}{I'_{-\frac{1}{3}}(h)})' \sim \frac{3a_1 \ln h}{7d_0 h^4}.$$

Hence when $s \to +\infty$, we have that

$$\omega'(s) \sim \frac{a_1 \ln s}{21 d_0 s^2} < 0,$$

$$\omega''(s) \sim -\frac{2a_1 \ln s}{21 d_0 s^3} > 0.$$

This implies the result of the lemma. □

Lemma 3.4 *The auxiliary curve C_ω is strictly decreasing for $s \in (1/2^{15}, +\infty)$.*

Proof. From Lemmas 3.2 and 3.3, we know that $\omega(s)$ is strictly decreasing near two endpoints, and $\omega(s) \to 5a_0/(7d_0) < \omega(1/2^{15}) = 1/2^5$ as $s \to +\infty$, if C_ω is not monotonically decreasing for $s \in (1/2^{15}, +\infty)$, then it must have at least one minimum and one maximum points, and we would find a value c such that the straight line $l_c = \{(s, \omega) : \omega = c\}$ cuts C_ω at least at three points, which implies that there are at least two points on this line for $s \in (1/2^{15}, +\infty)$ where the vector field (3.1) is horizontal.

However, we know from the Riccati equation

$$\dot{\omega}|_{\omega=c} = (98304c - 2560)s - 80c^2 + 2c. \tag{3.4}$$

Obviously (3.4) has at most one root, which is a contradiction.

Hence C_ω is strictly decreasing for $s \in (1/2^{15}, +\infty)$. □

Lemma 3.5 *The auxiliary curve C_ω is globally convex for $s \in (1/2^{15}, +\infty)$.*

Proof. Assume that the auxiliary curve C_ω is not globally convex for $s \in (1/2^{15}, +\infty)$. From Lemmas 3.2 and 3.3, it must have even inflection points. Without loss of generality, we suppose there are two inflection points on the auxiliary curve C_ω.

Since we have proved that C_ω is monotonically decreasing for $s \in (1/2^{15}, +\infty)$ and convex at two endpoints, there exists a straight line $l_{a,b} = \{(s, \omega) : \omega = as + b, \ a < 0\}$ on (s, ω)-plane which has three intersection points with C_ω, denoted by A, B and C respectively, and $l_{a,b}$ cuts the line $s = 1/2^{15}$ at point F below the saddle node $E(1/2^{15}, 1/2^5)$, see Figure 3.

Since $\omega(s) \to 5a_0/(7d_0) > 0$ and $l_{a,b} \to -\infty$ as $s \to +\infty$, there must be another intersection point D of the straight line $l_{a,b}$ with the auxiliary curve C_ω for some s.

Hence the straight line $l_{a,b}$ has at least four intersection points with the curve C_ω. Then there are at least three tangent points with system (3.1) on the straight line $l_{a,b}$.

On the other hand, from the Riccati equation we have

$$\begin{aligned}&(\dot{\omega} - a\dot{s})|_{\omega(s)=as+b} \\ =&(-80a^2 + 196608a)s^2 + (-160ab - a + 98304b - 2560)s - 80b^2 + 2b.\end{aligned} \tag{3.5}$$

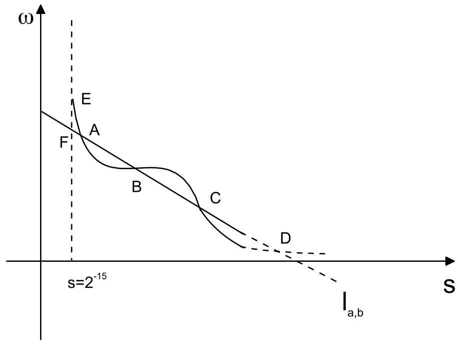

Figure 3. The behaviors of the straight line $l_{a,b}$ and auxiliary curve C_ω.

Obviously (3.5) has at most two zeros. This leads to a contradiction.

Hence the auxiliary curve C_ω is globally convex for $s \in (\frac{1}{2^{15}}, +\infty)$. □

Definition 3.6 *Define three functions*

$$P(h) = \frac{I_1(h)}{I_0(h)}, \quad Q(h) = \frac{I_{-1}(h)}{I_0(h)}, \quad R(h) = \frac{I_{-\frac{1}{3}}(h)}{I_0(h)}$$

for $h \in (1/2^5, +\infty)$*. Then we obtain a curve*

$$\Lambda = \{(Q, P):\ Q = Q(h),\ P = P(h),\ h \in (\frac{1}{2^5}, +\infty)\}$$

in (Q, P)*-plane, which is called the centroid curve.*

Remark 3.7 *In fact, the definition of the centroid curve here is not standard, see [9]. For convenience, we also call it the centroid curve.*

Since $I_0(h) \neq 0$ for $h \in (1/2^5, +\infty)$, we can rewrite $I(h)$ in the following form

$$I(h) = I_0(h)(\alpha + \beta\frac{I_1(h)}{I_0(h)} + \gamma\frac{I_{-1}(h)}{I_0(h)}),$$

then the number of zeros of $I(h)$ is equal to the number of the intersection points of the straight line

$$L_{\alpha\beta\gamma} = \{(Q, P):\ \alpha + \beta P + \gamma Q = 0\}$$

and the centroid curve Λ. Hence we need to study the properties of the centroid curve.

4. The properties of the centroid curve Λ

Since $I_k(h) > 0$ with $k = 0, 1, -1$ and $-1/3$, we easily know that $P(h) > 0, Q(h) > 0$ and $R(h) > 0$ for $h \in (1/2^5, +\infty)$. Moreover, When $h \to 1/2^5$, we have that $(P, Q)(h) \to (1, 1)$ and $R(h) \to 1$. Hence we can extend the domain of functions $P(h), Q(h)$ and $R(h)$ from $(1/2^5, +\infty)$ to $[1/2^5, +\infty)$. Besides these, $P(h), Q(h)$ and $R(h)$ also satisfy the following properties.

Lemma 4.1 *For* $h \in (1/2^5, +\infty)$, $P(h)$, $Q(h)$ *and* $R(h)$ *satisfy the following 4-dimension system*

$$\begin{aligned}
&\dot{h} = G(h),\\
&\dot{P} = -5 + 196608h^3 + (2 - 98304h^3)P - 7168h^2R + 7168h^2PR,\\
&\dot{Q} = 1 + (11 - 393216h^3)Q - 7168h^2R + 7168h^2QR,\\
&\dot{R} = 32h - 262144h^3R + 7168h^2R^2,
\end{aligned} \tag{4.1}$$

where $G(h) = 2h(1 - 32768h^3)$.

Proof. Differentiating the function $P(h) = \frac{I_1(h)}{I_0(h)}$ with respect to h, we have

$$P'(h) = \frac{I_1'(h)I_0(h) - I_0'(h)I_1(h)}{(I_0(h))^2}.$$

Removing $I_0'(h)$ and $I_1'(h)$ by (2.13) from $P'(h)$, we get

$$G(h)P'(h) = -5 + 196608h^3 + (2 - 98304h^3)P - 7168h^2R + 7168h^2PR,$$

which is equivalent to

$$\begin{aligned}
&\dot{h} = G(h) = 2h(1 - 32768h^3),\\
&\dot{P} = -5 + 196608h^3 + (2 - 98304h^3)P - 7168h^2R + 7168h^2PR.
\end{aligned}$$

We can compute $Q'(h)$ and $R'(h)$ similarly, then it is easy to obtain the 4-dimension system (4.1).

□

Lemma 4.2 *We have that*

$$\begin{aligned}
&P'(\frac{1}{2^5}) = -24,\ Q'(\frac{1}{2^5}) = 96,\\
&P''(\frac{1}{2^5}) = \frac{2624}{3},\ Q''(\frac{1}{2^5}) = \frac{17152}{3}.
\end{aligned}$$

Proof. At the singularity $(1/2^5, 1, 1, 1)$, the linear matrix of system (4.1) is

$$6\begin{pmatrix} -1 & 0 & 0 & 0\\ 48 & 1 & 0 & 0\\ -192 & 0 & 1 & 0\\ -48 & 0 & 0 & 1 \end{pmatrix}.$$

From (4.1), we know that the function $(h, P(h), Q(h), R(h))$ is given by the one dimensional stable manifold at this singularity.

Noting $(P(h), Q(h), R(h)) \to (1, 1, 1)$ as $h \to 1/2^5$ and the analyticities of $I_0(h)$, $I_1(h)$, $I_{-1}(h)$ and $I_{-1/3}(h)$ at $h = 1/2^5$, we suppose that near the singularity $(1/2^5, 1, 1, 1)$, $P(h)$, $Q(h)$ and $R(h)$ have the following expansions

$$\begin{aligned} P &= 1 + p_1(h - \frac{1}{2^5}) + \frac{p_2}{2!}(h - \frac{1}{2^5})^2 + \cdots, \\ Q &= 1 + q_1(h - \frac{1}{2^5}) + \frac{q_2}{2!}(h - \frac{1}{2^5})^2 + \cdots, \\ R &= 1 + r_1(h - \frac{1}{2^5}) + \frac{r_2}{2!}(h - \frac{1}{2^5})^2 + \cdots. \end{aligned} \tag{4.2}$$

Substituting (4.2) into the following equations

$$\begin{aligned} \dot{P} - \dot{h}\frac{dP}{dh} &= 0, \\ \dot{Q} - \dot{h}\frac{dQ}{dh} &= 0, \\ \dot{R} - \dot{h}\frac{dR}{dh} &= 0, \end{aligned}$$

and comparing the coefficients on both sides, we get

$$\begin{aligned} p_1 &= -24, \qquad p_2 = \frac{2624}{3}, \\ q_1 &= 96, \qquad q_2 = \frac{17152}{3}. \end{aligned}$$

□

From Lemma 2.2, we easily obtain the following lemma.

Lemma 4.3 *When $h \to +\infty$, we have that*

$$\begin{aligned} P(h) &\sim \frac{b_0}{a_0}h^{-\frac{3}{2}}, \quad Q(h) \sim \frac{c_0}{a_0}h^3, \\ P'(h) &\sim -\frac{3b_0}{2a_0}h^{-\frac{5}{2}}, \quad Q'(h) \sim \frac{3c_0}{a_0}h^2, \\ P''(h) &\sim \frac{15b_0}{4a_0}h^{-\frac{7}{2}}, \quad Q''(h) \sim \frac{6c_0}{a_0}h. \end{aligned}$$

Lemma 4.4 *For $h \in (1/2^5, +\infty)$, we have $P'(h) < 0$ and $Q'(h) > 0$.*

Proof. According to Theorem 2 of [13], we will prove this lemma by using the criterion function

$$\zeta(x) = \frac{f_2(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_2(\tilde{x})\sqrt{\phi(x)}\Phi'(x)}{f_1(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x}) - f_1(\tilde{x})\sqrt{\phi(x)}\Phi'(x)},$$

where $\tilde{x} = \tilde{x}(x)$ is defined by $\Phi(x) = \Phi(\tilde{x})$ for $0 < x < 1 < \tilde{x} < +\infty$.

In our case, since

$$P(h)=\frac{I_1(h)}{I_0(h)}=\frac{\oint_{\Gamma_h} x^{-\frac{4}{3}}y\,\mathrm{d}x}{\oint_{\Gamma_h} x^{-\frac{7}{3}}y\,\mathrm{d}x},$$

and

$$H(x,y)=x^{-\frac{4}{3}}(\frac{1}{2}y^2+\frac{1}{96}x^2+\frac{1}{48}x)=\frac{1}{2}x^{-\frac{4}{3}}y^2+\frac{1}{96}x^{\frac{2}{3}}+\frac{1}{48}x^{-\frac{1}{3}},$$

we take

$$f_1(x)=x^{-\frac{7}{3}},\quad f_2(x)=x^{-\frac{4}{3}},$$
$$\phi(x)=\frac{1}{2}x^{-\frac{4}{3}},\quad \Phi(x)=\frac{1}{96}x^{\frac{2}{3}}+\frac{1}{48}x^{-\frac{1}{3}},$$

then

$$\zeta(x)=\frac{x^{-\frac{4}{3}}\tilde{x}^{-2}(\tilde{x}-1)-\tilde{x}^{-\frac{4}{3}}x^{-2}(x-1)}{x^{-\frac{7}{3}}\tilde{x}^{-2}(\tilde{x}-1)-\tilde{x}^{-\frac{7}{3}}x^{-2}(x-1)}=\frac{x\tilde{x}^{\frac{1}{3}}(\tilde{x}-1)-\tilde{x}x^{\frac{1}{3}}(x-1)}{\tilde{x}^{\frac{1}{3}}(\tilde{x}-1)-x^{\frac{1}{3}}(x-1)}.$$

Let $\psi(x)=x^{1/3}(x-1)$, thus the criterion function $\zeta(x)$ is simplified as

$$\zeta(x)=\frac{x\psi(\tilde{x})-\tilde{x}\psi(x)}{\psi(\tilde{x})-\psi(x)}.$$

Differentiating $\zeta(x)$ with respect to x, we have

$$\zeta'(x)=\zeta_x+\zeta_{\tilde{x}}\frac{d\tilde{x}}{dx},\tag{4.3}$$

where

$$\zeta_x=\frac{\psi(\tilde{x})[(\psi(\tilde{x})-\psi(x))+\psi'(x)(x-\tilde{x})]}{\left(\psi(\tilde{x})-\psi(x)\right)^2},$$
$$\zeta_{\tilde{x}}=\frac{\psi(x)[(\psi(x)-\psi(\tilde{x}))+\psi'(\tilde{x})(\tilde{x}-x)]}{\left(\psi(\tilde{x})-\psi(x)\right)^2}.$$

By using the mean value theorem, we get

$$\zeta_x=\frac{\psi(\tilde{x})(\tilde{x}-x)(\psi'(\xi)-\psi'(x))}{\left(\psi(\tilde{x})-\psi(x)\right)^2}\quad (x<\xi<\tilde{x}),$$

and

$$\zeta_{\tilde{x}}=\frac{\psi(x)(\tilde{x}-x)(\psi'(\tilde{x})-\psi'(\eta))}{\left(\psi(\tilde{x})-\psi(x)\right)^2}\quad (x<\eta<\tilde{x}).$$

Since $\psi''(x)=2/9x^{-5/3}(2x+1)>0$, we get that

$$\psi'(\xi)>\psi'(x),\quad \psi'(\tilde{x})>\psi'(\eta).$$

Recalling $\psi(x)<0,\ \psi(\tilde{x})>0,\ \tilde{x}-x>0$, hence we have

$$\zeta_x>0,\quad \zeta_{\tilde{x}}<0.\tag{4.4}$$

On the other hand, by $\Phi'(x)=1/144x^{-4/3}(x-1)$, we easily get

$$\frac{d\tilde{x}}{dx}=\frac{\Phi'(x)}{\Phi'(\tilde{x})}=\frac{x^{-\frac{4}{3}}(x-1)}{\tilde{x}^{-\frac{4}{3}}(\tilde{x}-1)}<0, \tag{4.5}$$

(4.3), (4.4) and (4.5) yield

$$\zeta'(x)>0.$$

By Theorem 2 of [13], we have

$$P'(h)<0.$$

Similarly we can get $Q'(h)>0$. □

Remark 4.5 *Lemma 4.4 implies that we may treat P as a function of Q, then we rewrite the centroid curve as follows*

$$\Lambda=\{(Q,P):P=\widetilde{P}(Q)=P(h(Q)),Q\in[1,+\infty)\},$$

where $h=h(Q)$ is the inverse function of $Q=Q(h)$.

Thus we have that for $h\in[1/2^5,+\infty)$,

$$\frac{dP}{dQ}=\frac{P'(h)}{Q'(h)}<0,$$

which implies the centroid curve Λ is strictly decreasing. Moreover, we have

Corollary 4.6

$$\lim_{h\to\frac{1}{2^5}}\frac{dP}{dQ}=-\frac{1}{4},\quad \lim_{h\to\frac{1}{2^5}}\frac{d^2P}{dQ^2}=\frac{1}{4};$$

When $h\to+\infty$,

$$\frac{dP}{dQ}\sim\frac{-b_0}{2c_0}h^{-\frac{9}{2}},\quad \frac{d^2P}{dQ^2}\sim\frac{3a_0b_0}{4c_0^2}h^{-\frac{15}{2}}.$$

Corollary 4.7 *The centroid curve has the same convexity near the two endpoints.*

In the following, we study the number of zeros of $I(h)$ for $h\in(1/2^5,+\infty)$.

5. Proof of Theorem 1.2

Removing $I_k(h)$ $(k=0,1,-1)$ from (1.7) by using Picard-Fuchs equation (2.1), we get

$$\begin{aligned}I(h)=&(-2\alpha h-6\beta h)I_0'(h)+\beta hI_1'(h)+\frac{2}{11}\gamma hI_{-1}'(h)\\&+(\frac{1}{16}\alpha+\frac{5}{32}\beta-\frac{1}{176}\gamma)I_{-\frac{1}{3}}'(h).\end{aligned}\tag{5.1}$$

Differentiating (1.7) with respect to h, we have

$$I'(h) = \alpha I_0'(h) + \beta I_1'(h) + \gamma I_{-1}'(h). \tag{5.2}$$

Using equalities (5.1) and (5.2), we get

$$\begin{aligned}\varphi(h) :&= -\frac{2}{11}h^{\frac{13}{2}}(h^{-\frac{11}{2}}I(h))' = I(h) - \frac{2}{11}hI'(h)\\ &= \bar{\alpha}hI_0'(h) + \bar{\beta}hI_1'(h) + \bar{\gamma}I_{-\frac{1}{3}}'(h),\end{aligned} \tag{5.3}$$

where

$$\bar{\alpha} = (-\frac{24}{11}\alpha - 6\beta), \quad \bar{\beta} = \frac{9}{11}\beta, \quad \bar{\gamma} = \frac{1}{16}\alpha + \frac{5}{32}\beta - \frac{1}{176}\gamma,$$

then we have

$$(\frac{\varphi(h)}{h})' = \bar{\alpha}I_0''(h) + \bar{\beta}I_1''(h) + \frac{\bar{\gamma}}{h^2}(I_{-\frac{1}{3}}''(h)h - I_{-\frac{1}{3}}'(h)). \tag{5.4}$$

Removing $I_k''(h)$ $(k = 0, 1, -1/3)$ by using (2.12) from (5.4), we can get

$$\begin{aligned}\tilde{\varphi}(h) :&= 2(1 - 32768h^3)h^2(\frac{\varphi(h)}{h})'\\ &= (m_1 + n_1h^3)hI_0'(h) + (m_2 + n_2h^3)I_{-\frac{1}{3}}'(h),\end{aligned} \tag{5.5}$$

where

$$m_1 = 2\bar{\alpha} - \bar{\beta} + 160\bar{\gamma}, \quad n_1 = 98304\bar{\alpha} + 196608\bar{\beta},$$

$$m_2 = -2\bar{\gamma}, \quad n_2 = -5120\bar{\alpha} - 5120\bar{\beta} - 98304\bar{\gamma}.$$

Lemma 5.1 *The function $\varphi(h)$ has at most three zeros for $h \in (1/2^5, +\infty)$, taking into account the multiplicity.*

Proof. Since $\tilde{\varphi}(h) = 2(1 - 32768h^3)h^2(\varphi(h)/h)'$, we can study the zeros of $\tilde{\varphi}(h)$ instead of $\varphi(h)$ for $h \in (1/2^5, +\infty)$.

Noting $I_{-1/3}'(h) \neq 0$ for $h \in (1/2^5, +\infty)$, from (5.5), we know that

$$\tilde{\varphi}(h) = I_{-\frac{1}{3}}'(h)((m_1 + n_1s)\omega(s) + (m_2 + n_2s)), \tag{5.6}$$

where s and $\omega(s)$ is defined as before.

If $m_1n_2 - n_1m_2 = 0$, then there exists a constant λ, such that

$$\tilde{\varphi}(h) = I_{-\frac{1}{3}}'(h)(m_1 + n_1s)(\omega(s) + \lambda).$$

Since $\omega(s)$ is strictly decreasing for $s \in [1/2^{15}, +\infty)$, it has at most one intersection point with the line $\{\omega = -\lambda\}$, then $\tilde{\varphi}(h)$ has at most two zeros for $h \in (1/2^5, +\infty)$.

From now on, we suppose $m_1n_2 - n_1m_2 \neq 0$.

If there exists $s_1 \in (1/2^{15}, +\infty)$ such that $m_1 + n_1 s_1 = 0$, then $\tilde{\varphi}(h_1) = I'_{-1/3}(h_1)(m_2 + n_2 s_1) \neq 0$, where $s_1 = h_1^3$. Without loss of generality, we assume $m_1 + n_1 s \neq 0$ for $s \in (1/2^{15}, +\infty)$, then (5.6) can be rewritten as

$$\begin{aligned}\tilde{\varphi}\ (h) &= I'_{-\frac{1}{3}}(h)(m_1 + n_1 s)(\omega(s) + \frac{m_2 + n_2 s}{m_1 + n_1 s}) \\ &= I'_{-\frac{1}{3}}(h)(m_1 + n_1 s)(\omega(s) - \rho(s)),\end{aligned}$$

where

$$\rho(s) = -\frac{m_2 + n_2 s}{m_1 + n_1 s},$$

and m_1, n_1, m_2, and n_2 are defined in (5.5).

For convenience, we define

$$C_\rho = \{(s, \rho(s)) : \rho(s) = -\frac{m_2 + n_2 s}{m_1 + n_1 s}, s \in (\frac{1}{2^{15}}, +\infty)\}.$$

Since $I'_{-1/3}(h) \neq 0$ for $h \in (1/2^5, +\infty)$, the number of zeros of $\tilde{\varphi}(h)$ is equal to the number of intersection points of C_ω and C_ρ.

If $n_1 = 0$, then C_ρ is a straight line. Since C_ω is globally decreasing and convex for $h \in [1/2^5, +\infty)$, there are at most two intersection points for C_ω and C_ρ.

Now we suppose $n_1 \neq 0$, then C_ρ are two branches of the hyperbola with asymptotes $s = -m_1/n_1$, $\rho(s) = -n_2/n_1$, see Figure 4.

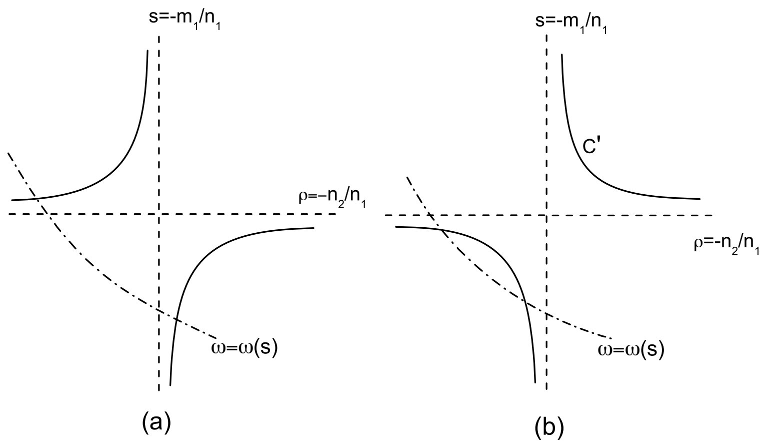

Figure 4. The behaviors of curve C_ω and curve C_ρ.

If the hyperbola is increasing, see Figure 4(a), we know that the curve C_ρ has at most two intersection points with C_ω, because C_ω is monotonically decreasing for $s \in [1/2^{15}, +\infty)$.

If the hyperbola is decreasing, see Figure 4(b), C_ω can only intersect with one branch of the hyperbola. If C_ω intersects with the left-lower branch, then they have at most two intersection points, because the left-lower branch is concave and C_ω is globally convex.

Now we only need to consider the case that C_ω intersects with the right-upper branch of the hyperbola denoted by C'.

From the Riccati equation of $\omega(s)$ and the definition of $\rho(s)$, we know for $s \in (1/2^{15}, +\infty)$,

$$(\frac{\dot{\omega}}{\dot{s}} - \rho'(s))|_{\omega=\rho(s)} = \frac{-4}{3s(160\bar{\gamma} + 2\bar{\alpha} - \bar{\beta} + 98304\alpha s + 196608\beta s)^2}(\alpha_1 s^2 + \beta_1 s + \gamma_1),$$

where α_1, β_1 and γ_1 are constants depending on $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$. This implies C_ω and C' have at most three intersection points for $s \in [1/2^{15}, +\infty)$.

Note the fact that $\omega(1/2^{15}) = \lim_{s\to 1/2^{15}} \rho(s) = 1/2^5$, hence for $s \in (1/2^{15}, +\infty)$, C_ω and C' have at most two intersection points.

Thus we know that for $m_1 n_2 - n_1 m_2 \neq 0$, C_ω and C_ρ have at most two intersection points.

Summing up the discussions above, the function $\tilde{\varphi}(h)$ has at most two zeros for $h \in (1/2^5, +\infty)$, taking into account the multiplicity, which, by (5.5), implies that the function $\varphi(h)$ has at most three zeros. □

Now we begin to prove Theorem 1.2.

Proof. From Lemma 5.1 and the definition of the function $\varphi(h)$, we know that the associated Abelian integral $I(h)$ has at most four zeros for $h \in [1/2^5, +\infty)$. Since $I(1/2^5) = 0$ for any constants α, β and γ, $I(h)$ has at most three zeros for $h \in (1/2^5, +\infty)$.

In the following, we will prove that the centroid curve is globally convex without zero curvature, which yields that the associated Abelian integral $I(h)$ has at most two zeros for $h \in (1/2^5, +\infty)$, taking into account the multiplicities.

From Corollary 4.7, we know that the centroid curve Λ is convex near the two endpoints, which implies that its infection points (if exists) must appear in pair. Without loss of generality, suppose there exist two inflection points on Λ.

Since $P \sim (a_0^{-3/2} b_0 c_0^{1/2}) Q^{-1/2}$ as $h \to +\infty$ and $P'(Q) < 0$ for $Q > 1$, using the same arguments as in the proof of Lemma 3.5, we can find a straight line which has at least four intersection points with the centroid curve Λ. That is, there exist some $\alpha_0, \beta_0, \gamma_0$ such that Abelian integral $I(h) = \alpha_0 I_0(h) + \beta_0 I_1(h) + \gamma_0 I_{-1}(h)$ has at least four zeros for $h \in (1/2^5, +\infty)$, which contradicts the above result. Hence for $h \in (1/2^5, +\infty)$, the centroid curve Λ is globally convex without zero curvature. Otherwise, there exists one point with at least quadruple tangency, which also implies that $I(h)$ has four zero points for $h \in (1/2^5, +\infty)$, leading to the same contradiction.

Since the centroid curve Λ is strictly convex for $h \in (1/2^5, +\infty)$, we can find some constants $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$, such that $I(h) = \tilde{\alpha} I_0(h) + \tilde{\beta} I_1(h) + \tilde{\gamma} I_{-1}(h)$ has exactly two zeros for $h \in (1/2^5, +\infty)$.

Therefore we conclude that two is the least upper bound of the zeros of $I(h)$ for $h \in (1/2^5, +\infty)$.

This finishes the proof of Theorem 1.2. □

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