

Multi-dimensional Weiss operators

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Abstract

We present a solution of the Weiss operator family generalized for the case of \mathbb{R}^d and formulate a d -dimensional analogue of the Weiss Theorem. Most importantly, the generalization of the Weiss Theorem allows us to find a subset of null class functions for a partial differential equation with the generalized Weiss operators. We illustrate the significance of our approach through several examples of both linear and non-linear partial differential equations.

Key Words: Partial differential equations, Weiss operators

1. Introduction

To investigate the integrability of nonlinear partial differential equations, many different methods have been developed; among them is Painlevé test [1]. In the framework of this approach, the class of one-dimensional ('scalar') differential operators was defined first in [2], and in [3] has been applied to solitonic-type PDEs.

The family of Weiss operators performs a special kind of ordinary derivative operators of integer order n ($n > 0$ and for each order only one such an operator exists), and the general solution for each Weiss operator can be found in a very simple form. Following [2], let's define for a differentiable scalar function $\phi(x)$ the class of factorized differential operators as

$$L_{n+1} = \prod_{j=0}^n \left[\frac{d}{dx} + \left(j - \frac{n}{2} \right) V \right] = \left(\frac{d}{dx} - \frac{n}{2} V \right) \cdot \dots \cdot \left(\frac{d}{dx} + \frac{n}{2} V \right), \quad (1)$$

where $V = \phi_{xx}/\phi_x$ is the so-called pre-Schwarzian. For $n = 0$ equation (1) produces the ordinary derivative $L_1 = d/dx$; for $n = 1$, we get the Schrödinger operator

$$L_2 = \left(\frac{d}{dx} - \frac{1}{2} V \right) \left(\frac{d}{dx} + \frac{1}{2} V \right) = \left(\frac{d}{dx} \right)^2 + \frac{1}{2} S;$$

for $n = 2$, we get the the Lenard operator

$$L_3 = \left(\frac{d}{dx} - V \right) \frac{d}{dx} \left(\frac{d}{dx} + V \right) = \left(\frac{d}{dx} \right)^3 + 2S \frac{d}{dx} + S_x ,$$

where by S we denote the Schwarzian of the function $\phi(x)$

$$S = V_x - \frac{1}{2}V^2 = \left(\frac{\phi_{xx}}{\phi_x} \right)_x - \frac{1}{2} \left(\frac{\phi_{xx}}{\phi_x} \right)^2 . \tag{2}$$

The important result from [2] is related to the null space of Weiss operators:

Theorem 1.1 *The null space of the operator family (1) is given by the set of linearly independent $n+1$ solutions $\{\phi_x^{-n/2}\phi^k\}$; $k = 0, 1, 2, \dots, n$, i.e. for every k*

$$L_{n+1}[\phi_x^{-n/2}\phi^k] = 0 . \tag{3}$$

In this paper we propose the generalization of Weiss family L_{n+1} for multi-dimensional case of \mathbb{R}^d . We formulate a d -dimensional analogue of Theorem 1.1 for a special class of partial differential operators that provides us with a sub-set of null class functions for a partial differential equation in terms of the generalized Weiss operators. Our generalization therefore extends the role played by the Weiss operators in the construction of some special class of ordinary differential equations to partial differential equations as well. We illustrate the significance of our approach through several examples of both linear and non-linear partial differential equations.

2. Generalized Weiss operators in \mathbb{R}^d

Let's define for the vector $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ the linear differential operator

$$D = \sum_{i=1}^d a_i(\mathbf{x}) \frac{\partial}{\partial x_i} \tag{4}$$

with differentiable scalar functions $a_i(\mathbf{x})$. Note that, for $a_i(\mathbf{x}) = x_i$, the definition (4) becomes the so-called homogeneous operator.

Definition (4) implies for the power m of some scalar function $A(\mathbf{x})$

$$D(A^m(\mathbf{x})) = \sum_{i=1}^d a_i(\mathbf{x}) \frac{\partial A^m(\mathbf{x})}{\partial x_i} = mA^{m-1}(\mathbf{x})DA(\mathbf{x}) . \tag{5}$$

Also, for a given differentiable function $\phi(\mathbf{x})$, we define with (4) a generalized pre-Schwarzian:

$$V = \frac{D^2\phi(\mathbf{x})}{D\phi(\mathbf{x})} . \tag{6}$$

The class of differential operators in d -dimensional space, given by:

$$L_{n+1} = \prod_{j=0}^n \left[D + \left(j - \frac{n}{2} \right) V \right] = \left(D - \frac{n}{2} V \right) \cdot \dots \cdot \left(D + \frac{n}{2} V \right) , \quad (7)$$

includes Weiss' operators (1) as a particular case for $d = 1$ and $a_1 = 1$. The index $n + 1$ in (7) denotes the order of the differential operator L_{n+1} .

If, for instance, for $d = 1$, we put $x \equiv x$, $a(x) \equiv a_1(x_1)$, then $V = a_x + a\phi_{xx}/\phi_x$ and

$$L_2 = a^2 \frac{d^2}{dx^2} + aa_x \frac{d}{dx} + \frac{1}{2} a V_x - \frac{1}{4} V^2 .$$

Also, if we suggest the particular case

$$a_i(\mathbf{x}) = F(\mathbf{x})c_i ,$$

where $c_i \neq 0$ are constants, then defining

$$\xi = \sum_{i=1}^d \frac{x_i}{c_i} ,$$

we get

$$\frac{\partial}{\partial \xi} = \sum_{i=1}^d c_i \frac{\partial}{\partial x_i}$$

and $D = F(\mathbf{x})\partial/\partial\xi$, that allows to present the operators in one-dimensional form, i.e. $D\phi = F\phi_\xi$,

$$V = F_\xi + F \frac{\phi_{\xi\xi}}{\phi_\xi} .$$

The “degenerated” case $F = 1$ corresponds to (1).

Now we can formulate the following theorem.

Theorem 2.1 *A set of linearly-independent functions*

$$\{ (D\phi(\mathbf{x}))^{-n/2} \phi^k(\mathbf{x}) , k = 0, 1, 2, \dots, n \} , \quad (8)$$

forms a null function

$$f_{n+1}(\mathbf{x}) = [D\phi(\mathbf{x})]^{-n/2} \sum_{k=0}^n c_k \phi^k(\mathbf{x}) , \quad (9)$$

where $\{c_k\}$ is a set of constants, such that $f_{n+1}(\mathbf{x})$ satisfies the equation

$$L_{n+1} f_{n+1}(\mathbf{x}) = 0 . \quad (10)$$

Proof. Let's check how the operator L_{n+1} acts on $D\phi^{-n/2}\phi^k$. Its first (rightmost) bracket using (5) and (6) produces

$$\left(D + \frac{n}{2} V \right) (D\phi)^{-n/2} \phi^k = (D\phi)^{-n/2} \phi^k \left[-\frac{n}{2} \frac{D^2\phi}{D\phi} + k \frac{D\phi}{\phi} + \frac{n}{2} V \right] = k (D\phi)^{-n/2+1} \phi^{k-1} .$$

The next bracket produces

$$\left(D + \left(n - 1 - \frac{n}{2}\right)V\right) k(D\phi)^{-n/2+1}\phi^{k-1} = k(k-1)(D\phi)^{-n/2+2}\phi^{k-2},$$

and so on. Each operator bracket from (7) increases the power of $D\phi$ and decreases the power of ϕ by one. After the application of all $n + 1$ brackets we end up with

$$L_{n+1}(D\phi)^{-n/2}\phi^k = k(k-1)(k-2) \cdots (k-n)(D\phi)^{-n/2+n+1}\phi^{k-n-1}.$$

But $k = 0, 1, 2, \dots, n$, and one of the factors $k, k-1, \dots, k-n$ is zero, thus,

$$L_{n+1}(D\phi)^{-n/2}\phi^k = 0. \tag{11}$$

As each operator bracket $(D + mV)$ is a linear differential operator, (11) can be applied to each term of the linearly independent set $\sum_k (D\phi)^{-n/2}\phi^k$, that is

$$L_{n+1} \sum_{k=0}^n (D\phi)^{-n/2}\phi^k = 0.$$

Then we end up with (10). □

Thus, in our approach the function $\phi(\mathbf{x})$ plays the role of producing function. Choosing first $\phi(\mathbf{x})$ and then using its pre-Schwarzian (6), we define the operator (7) and immediately get the solution (9) following from Theorem 2.1.

Let's give an example of our method. We consider the following 2-dimensional partial differential equation for $\psi(x, y)$:

$$\psi_{xx} + \psi_{yy} - 2\psi_{xy} = 0. \tag{12}$$

Now let's choose for the simple producing function

$$\phi(x, y) = \frac{x}{y} \tag{13}$$

the operator

$$D = \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, \tag{14}$$

i.e. $a_1 = 1, a_2 = -1, x_1 = x$, and $x_2 = y$. Then

$$D\left(\frac{x}{y}\right) = \frac{x+y}{y^2}; \quad D^2\left(\frac{x}{y}\right) = \frac{2(x+y)}{y^3}$$

and

$$V = \frac{2}{y}.$$

The generalized Weiss operator for (14) is

$$\begin{aligned} L_2 &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} - \frac{1}{y} \right) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y} + \frac{1}{y} \right) = \\ &= \frac{\partial^2}{\partial x^2} - 2 \frac{\partial}{\partial x} \frac{\partial}{\partial y} + \frac{\partial^2}{\partial y^2} \equiv D^2. \end{aligned} \quad (15)$$

Note it reproduces the differential operator in the left hand side of (12).

The solution of (12) by Theorem 2.1 is given by

$$\psi(x, y) = \frac{c_0 + c_1 \phi}{(D\phi)^{1/2}} = \frac{c_0 y + c_1 x}{\sqrt{x + y}}. \quad (16)$$

2.1. Linear and non-linear examples

Whether or not the partial differential equation under consideration is non-linear, the use of generalized Weiss operators can be seen to be an effective method in the construction of some class of partial differential equations. In order to make it clear, we provide the reader with one more linear example and two non-linear examples of partial differential equations.

In general, the generalized Weiss operator

$$L_2 = \left(D - \frac{V}{2} \right) \left(D + \frac{V}{2} \right)$$

acting on a continuously differentiable function $\psi(x, y)$ yields a partial differential equation in the form

$$D^2\psi(x, y) + Q(x, y)\psi(x, y) = 0, \quad (17)$$

where

$$Q = \frac{1}{2} \left(DV - \frac{1}{2}V^2 \right) = \frac{1}{2} \left[D \left(\frac{D^2\phi}{D\phi} \right) - \frac{1}{2} \left(\frac{D^2\phi}{D\phi} \right)^2 \right]. \quad (18)$$

In the following, as a linear example that corresponds to (17), we consider the partial differential equation

$$\psi_{xx} + x^4\psi_{yy} + 2x^2\psi_{xy} + 2x\psi_y + \frac{1 - 2x^2}{(1 + x^2)^2}\psi = 0. \quad (19)$$

Choosing for the producing function $\phi = x + y$ the operator

$$D = \frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y},$$

we get $D\phi = 1 + x^2$ and $D^2\phi = 2x$. Then we obtain

$$V = \frac{2x}{1 + x^2}$$

which yields

$$Q(x, y) = \frac{1 - 2x^2}{(1 + x^2)^2}.$$

Thus the generalized Weiss operator

$$L_2 = \left(\frac{\partial}{\partial x} + x^2 \frac{\partial}{\partial y} \right)^2 + \frac{1 - 2x^2}{(1 + x^2)^2}$$

reproduces (19) whose solution according to Theorem 2.1 is given by

$$\psi(x, y) = \frac{c_0 + c_1(x + y)}{\sqrt{1 + x^2}}.$$

Let us now discuss the following partial differential equation in the form of (17) which, however, is non-linear:

$$\psi\psi_{xx} + \psi\psi_{yy} - 2\psi\psi_{xy} + \frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_y^2 - \psi_x\psi_y = 0. \quad (20)$$

Next, we consider

$$D = -\psi \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y}$$

which leads to

$$D\psi = -\psi\psi_x + \psi\psi_y$$

and

$$D^2\psi = \psi^2\psi_{xx} - 2\psi^2(\psi_{xy}) + \psi^2\psi_{yy} + \psi\psi_x^2 + \psi\psi_y^2 - 2\psi\psi_x\psi_y$$

for $\psi_{xy} = \psi_{yx}$. Thus (17) can be written as

$$\psi^2\psi_{xx} - 2\psi^2\psi_{xy} + \psi^2\psi_{yy} + \psi\psi_x^2 + \psi\psi_y^2 - 2\psi\psi_x\psi_y + Q\psi = 0. \quad (21)$$

Here, Q can be specified once we define the producing function $\phi(x, y)$. Next, we choose $\phi(x, y) = y - x$ to end up with $D\phi = 2\psi$ and $D^2\phi = -2\psi\psi_x + 2\psi\psi_y$ which together yield $V = \psi_y - \psi_x$ for a non-trivial solution (i.e., $\psi \neq 0$). According to (18), we obtain

$$Q(x, y) = \frac{1}{2}\psi\psi_{xx} + \frac{1}{2}\psi\psi_{yy} - \psi\psi_{xy} - \frac{1}{4}\psi_y^2 + \frac{1}{2}\psi_x\psi_y - \frac{1}{4}\psi_x^2. \quad (22)$$

Using (22) in (21), the non-linear partial differential equation can be expressed as in (20). Then,

$$\psi(x, y) = \frac{[c_0 + c_1(y - x)]^{2/3}}{2^{1/3}}$$

can be identified to be a real solution of (20) in accordance with Theorem 2.1.

Next, we present the second non-linear example for $n = 2$ for which the generalized Weiss operator can be written as

$$L_3 = \prod_{j=0}^2 [D + (j-1)V] = (D-V)D(D+V)$$

or

$$L_3 = D^3 + 4QD + 2DQ,$$

where Q is given by (18). The operator L_3 acting on a continuously differentiable function $\psi(x, y, z)$ leads to a partial differential equation in the form

$$D^3\psi(x, y, z) + 4Q(x, y, z)D\psi(x, y, z) + 2\psi(x, y, z)DQ(x, y, z) = 0. \quad (23)$$

Choosing, for instance,

$$D = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \psi \frac{\partial}{\partial z} \quad (24)$$

for a producing function $\phi(x, y, z) = x - y + z$, the partial differential equation in (23) becomes non-linear. One can see, for (24), that

$$D\psi = \psi_x + \psi_y + \psi\psi_z.$$

The explicit form of (23) can be obtained using

$$D\phi = \psi,$$

$$D^2\phi = D\psi = \psi_x + \psi_y + \psi\psi_z,$$

and therefore

$$V = \frac{D^2\phi}{D\phi} = \frac{\psi_x + \psi_y + \psi\psi_z}{\psi}$$

together with (18) in (23). Following from Theorem 2.1,

$$\psi(x, y, z) = \pm \sqrt{c_0 + c_1(x - y + z) + c_2(x - y + z)^2}$$

is given as a solution of (23).

3. Discussion

As we mentioned above, starting from the original work [3], the operator family L_{n+1} is usually used in the analysis of partial differential equations of solitonic type [4]. Now we can extend the class of those equations to the case of multi-dimensional spatial variables. The appropriate choice of the coefficients a_i can fit the necessary form of operator.

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