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Some results on g-frames in Hilbert spaces

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Abstract

In this paper we show that every g-frame for a Hilbert space $\mathcal{H}$ can be represented as a linear combination of two g-orthonormal bases if and only if it is a g-Riesz basis. We also show that every g-frame can be written as a sum of two tight g-frames with g-frame bounds one or a sum of a g-orthonormal basis and a g-Riesz basis for $\mathcal{H}$. We further give necessary and sufficient conditions on g-Bessel sequences $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ and $\{\Gamma_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ and operators L_1, L_2 on $\mathcal{H}$ so that $\{\Lambda_i L_1 + \Gamma_i L_2 : i \in J\}$ is a g-frame for $\mathcal{H}$. We next show that a g-frame can be added to any of its canonical dual g-frame to yield a new g-frame.

Key Words: Frame, g-frame, g-orthonormal basis, tight g-frame, g-Bessel sequence

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer ([5]), reintroduced in 1986 by Daubechies, Grossman, and Meyer ([4]), and popularized from then on. In [11], a generalization of the frame concept was introduced. Sun introduced a g-frame and a g-Riesz basis in a complex Hilbert space and discussed some properties of them. A frame of subspaces ([1], [3]) and a system of bounded quasi-projectors ([6]) are a g-frame in a complex Hilbert space. From a g-frame, we may construct a frame for a complex Hilbert space ([11]). A natural question which immediately comes to mind is, "Which properties of the frame may be extended to the g-frame for a complex Hilbert space?". G-frames and g-Riesz bases in complex Hilbert spaces have some properties similar to those of frames, Riesz bases, but not all the properties are similar (see [11]). In this paper we generalize some results in [2], [7], [10] from frame theory to g-frames.

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces and $\{\mathcal{H}_i\}_{i\in J} \subseteq \mathcal{K}$ is a sequence of separable Hilbert spaces, where J is a subset of $\mathbb{Z}$, $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$ is the collection of all bounded linear operators from $\mathcal{H}$ to $\mathcal{H}_i$. For each sequence $\{\mathcal{H}_i\}_{i\in J}$, we define the space $(\sum_{i\in J} \bigoplus \mathcal{H}_i)_{l_2}$ by

$$(\sum_{i\in J} \bigoplus \mathcal{H}_i)_{l_2} = \{\{f_i\}_{i\in J} : f_i \in \mathcal{H}_i,\ i \in J \ and \sum_{i\in J} \|f_i\|^2 < \infty\}.$$

With the inner product defined by

$$\langle \{f_i\}, \{g_i\} \rangle = \sum_{i\in J} \langle f_i, g_i \rangle,$$

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it is clear that $(\sum_{i\in J}\bigoplus\mathcal{H}_i)_{l_2}$ is a Hilbert space.

A frame for a complex Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i\in J}$ so that there are two positive constants A and B satisfying

$$A\|f\|^2\leq\sum_{i\in J}|\langle f,f_i\rangle|^2\leq B\|f\|^2, f\in\mathcal{H}.$$

The constants A and B are called lower and upper frame bounds.

A sequence $\{\Lambda_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in J\}$ is called a generalized frame, or simply a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ if there exist two positive constants A and B such that, for all $f\in\mathcal{H}$,

$$A\|f\|^2\leq\sum_{i\in J}\|\Lambda_i f\|^2\leq B\|f\|^2.$$

The constants A and B are called the lower and upper g-frame bounds, respectively. The supremum of all such A and the infimum of all such B are called the optimal bounds. If $A=B$ we call this g-frame a tight g-frame and if $A=B=1$, it is called a normalized tight g-frame. A g-frame is exact if it is ceases to be a g-frame whenever any single element is removed from $\{\Lambda_i\}_{i\in J}$. We say simply a g-frame for $\mathcal{H}$ whenever the space sequence $\mathcal{H}_i$ is clear. We say $\{\Lambda_i\}_{i\in J}$ is a g-frame sequence, if it is a g-frame for $\overline{span}\{\Lambda_i^\star(\mathcal{H}_i)\}_{i\in J}$. If we only have the upper bound, we call $\{\Lambda_i\}_{i\in J}$ a g-Bessel sequence with bound B. We say that $\{\Lambda_i\}_{i\in J}$ is g-complete, if $\{f:\Lambda_i f=0,\forall i\in J\}=\{0\}$; and is called g-orthonormal basis for $\mathcal{H}$, if

$$\langle\Lambda_i^\star g_i,\Lambda_j^\star g_j\rangle=\delta_{i,j}\langle g_i,g_j\rangle,\ i,j\in J,\ g_i\in\mathcal{H}_i,\ g_j\in\mathcal{H}_j,$$

and

$$\sum_{i\in J}\|\Lambda_i f\|^2=\|f\|^2.$$

We say that $\{\Lambda_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i):i\in J\}$ is a g- Riesz basis for $\mathcal{H}$, if it is g-complete and there exist constants $0<A\leq B<\infty$, such that for any finite subset $I\subseteq J$ and $g_i\in\mathcal{H}_i$, $i\in I$,

$$A\sum_{i\in I}\|g_i\|^2\leq\|\sum_{i\in I}\Lambda_i^\star g_i\|^2\leq B\sum_{i\in I}\|g_i\|^2$$

Recall that a unitary operator $K:\mathcal{H}\longrightarrow\mathcal{H}$ is an onto isometry, a partial isometry is an operator that is an isometry on the orthogonal complement of its kernel, a co-isometry is an operator whose adjoint is an into isometry, and a maximal partial isometry is either an isometry or a co-isometry.

In order to present the main results of this paper, we need the following Theorems and Propositions which can be found in [11], [2] and [9]

Proposition 1.1 *([2]) Let $K:\mathcal{H}\longrightarrow\mathcal{H}$ be a bounded linear operator. Then the following hold:*

i) $K=a(U_1+U_2+U_3)$, where each $U_j, j=1,2,3,$ is a unitary operator and a is a constant.

ii) If K is onto, then it can be written as a linear combination of two unitary operators if and only if K is invertible.

Theorem 1.2 *([11])Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$. The operator*

$$S : \mathcal{H} \to \mathcal{H}, Sf = \sum_{i\in J} \Lambda_i^\star \Lambda_i f,$$

is a positive invertible operator and every $f \in \mathcal{H}$ has an expansion

$$f = \sum_{i\in J} S^{-1}\Lambda_i^\star \Lambda_i f = \sum_{i\in J} \Lambda_i^\star \Lambda_i S^{-1} f.$$

So $\{\widetilde{\Lambda_i} = \Lambda_i S^{-1} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ and is called canonical dual g-frame of $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$. The operator S is called the g-frame operator of $\{\Lambda_i\}_{i\in J}$.

Definition 1.3 *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be a g-frame for $\mathcal{H}$. Then the synthesis operator for $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is the operator*

$$T : (\sum_{i\in J} \bigoplus \mathcal{H}_i)_{l_2} \longrightarrow \mathcal{H},$$

defined by

$$T(\{f_i\}_{i\in J}) = \sum_{i\in J} \Lambda_i^\star (f_i).$$

We call the adjoint $T^\star$ of the synthesis operator the analysis operator.

Proposition 1.4 *([9])Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be a g-frame for $\mathcal{H}$. Then the analysis operator for $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is the operator*

$$T^\star : \mathcal{H} \longrightarrow (\sum_{i\in J} \bigoplus \mathcal{H}_i)_{l_2},$$

defined by

$$T^\star(f) = \{\Lambda_i(f)\}_{i\in J}.$$

Proposition 1.5 *([9]) Let $\{\Lambda_i\}_{i\in J}$ be a sequence in $\mathcal{L}(\mathcal{H}, \mathcal{H}_i)$. Then the following are equivalent:*

i) $\{\Lambda_i\}_{i\in J}$ is a g-frame for $\mathcal{H}$;

ii) The operator $T : (\{f_i\}_{i\in J}) \mapsto \sum_{i\in J} \Lambda_i^\star(f_i)$ is well-defined and bounded from $(\sum_{i\in J} \bigoplus \mathcal{H}_i)_{l_2}$ onto $\mathcal{H}$;

iii) The operator $S : f \mapsto \sum_{i\in J} \Lambda_i^\star \Lambda_i f$ is well-defined and bounded from $\mathcal{H}$ onto $\mathcal{H}$.

Proposition 1.6 *([9]) Let $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$ and $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ be a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in J}$. Then there is a bounded and onto operator $K : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i K^\star$ for all $i \in J$. Furthermore, K is invertible if $\{\Lambda_i\}_{i\in J}$ is a g-Riesz basis for $\mathcal{H}$ and K is unitary if $\{\Lambda_i\}_{i\in J}$ is a g-orthonormal basis for $\mathcal{H}$.*

2. Some g-frame representations

In this section we show that every g-frame for a Hilbert space $\mathcal{H}$ can be written as a sum of three g-orthonormal bases for $\mathcal{H}$. We next show that a g-frame can be represented as a linear combination of two g-orthonormal bases if and only if it is a g-Riesz basis. We further show that every g-frame can be written as a sum of two tight g-frames with g-frame bounds one or a sum of a g-orthonormal basis and a g-Riesz basis for $\mathcal{H}$.

Proposition 2.1 *If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-frame for a Hilbert space $\mathcal{H}$, and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis for $\mathcal{H}$, there are g-orthonormal bases $\{\Upsilon_i\}$, $\{\Gamma_i\}$, $\{\Psi_i\}$ for $\mathcal{H}$ and a constant a so that $\Lambda_i = a(\Upsilon_i + \Gamma_i + \Psi_i)$ for all $i \in J$.*

Proof. By Proposition 1.6 there is a bounded and onto operator $K : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i K^\star$ and by Proposition 1.1 we have $K^\star = a(U_1 + U_2 + U_3)$, where each U_j is a unitary operator and a is a constant. So $\Lambda_i = \Theta_i K^\star = a(\Theta_i U_1 + \Theta_i U_2 + \Theta_i U_3)$. Since $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis and for each $r = 1, 2, 3, U_r$ is a unitary operator, we have

$$\langle (\Theta_i U_r)^\star g_i, (\Theta_j U_r)^\star g_j \rangle = \langle \Theta_i^\star g_i, \Theta_j^\star g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle,$$

and

$$\sum_{i \in J} \|\Theta_i U_r f\|^2 = \|U_r f\|^2 = \|f\|^2.$$

So $\{\Theta_i U_r\}_i$ is a g-orthonormal basis and the proof is complete by putting $\Upsilon_i = \Theta_i U_1, \Gamma_i = \Theta_i U_2$ and $\Psi_i = \Theta_i U_3$. □

Proposition 2.2 *([12]) For the family $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ the following two statements are equivalent:*

i) The sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-Riesz basis for $\mathcal{H}$.

ii) The sequence $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-frame for $\mathcal{H}$, and if $\sum_{i \in J} \Lambda_i^\star g_i = 0$ then $g_i = 0$ for all $i \in J$.

Proposition 2.3 *If $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis for $\mathcal{H}$ then we have a g-frame $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ which can be written as a linear combination of two g-orthonormal bases for $\mathcal{H}$ if and only if $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-Riesz basis for $\mathcal{H}$.*

Proof. If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-Riesz basis, by Proposition 1.6, there is an invertible operator K: $\mathcal{H} \longrightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i K^\star$ and by Proposition 1.1 we have $K^\star = aU_1 + bU_2$ for some constants a, b, and unitary operators U_1 and U_2. So $\Lambda_i = \Theta_i K^\star = a\Theta_i U_1 + b\Theta_i U_2$. Since $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis and U_1 and U_2 are unitary operators, for $g_i \in \mathcal{H}_i$, $g_j \in \mathcal{H}_j$ and $f \in \mathcal{H}$, we have

$$\langle (\Theta_i U_r)^\star g_i, (\Theta_j U_r)^\star g_j \rangle = \langle \Theta_i^\star g_i, \Theta_j^\star g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle$$

and

$$\sum_{i \in J} \|\Theta_i U_r f\|^2 = \|U_r f\|^2 = \|f\|^2.$$

So $\{\Theta_i U_r\}_i$ is a g-orthonormal basis.

Now suppose that there are g-orthonormal bases $\{\Upsilon_i\}$, $\{\Gamma_i\}$ for $\mathcal{H}$ and constants a, b such that $\Lambda_i = a\Upsilon_i + b\Gamma_i$ for all $i \in J$. By Proposition 1.6, there are an onto operator T, and unitary operators K and R such that $\Lambda_i = \Theta_i T^\star$, $\Gamma_i = \Theta_i K^\star$, $\Upsilon_i = \Theta_i R^\star$. Since $\Lambda_i = a\Upsilon_i + b\Gamma_i$ and $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis for $\mathcal{H}$, we have $T = aK + bR$ and so, by Proposition 1.1, T is an invertible operator. If $\sum_{i\in J} \Lambda_i^\star g_i = 0$ then $T\sum_{i\in J} \Theta_i^\star g_i = 0$, and so $\sum_{i\in J} \Theta_i^\star g_i = 0$. Therefore, by Proposition 2.2, $g_i = 0$ which implies that the family $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-Riesz basis. □

Proposition 2.4 *If K is a co-isometry on $\mathcal{H}$, and if $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis for $\mathcal{H}$, then $\{\Theta_i K^\star : i \in J\}$ is a normalized tight g-frame for $\mathcal{H}$.*

Proof. Since K is a co-isometry, $K^\star$ is an isometry. Hence, for all $f \in \mathcal{H}$,

$$\sum_{i\in J} \|\Theta_i K^\star f\|^2 = \|K^\star f\|^2 = \|f\|^2.$$

□

Every operator K on a Hilbert space can be written in the form $K = VP = \frac{\|T\|}{2} V(W + W^\star)$, where W is unitary and V is a maximal partial isometry. It follows that VW and $VW^\star$ are maximal partial isometries. That is, each of these operators is either an isometry or a co-isometry. However, if K has dense range, V must be a co-isometry (see [2]).

By using the above facts we have the following propositions.

Proposition 2.5 *If $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis for $\mathcal{H}$ then every g-frame is the sum of two normalized tight g-frames for $\mathcal{H}$.*

Proof. By Proposition 1.6, there is a bounded and onto operator $K\colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i K^\star$, and by the above explanation, we have $K = \frac{\|T\|}{2} V(W + W^\star)$, where W is unitary and V must be a co-isometry. So VW and $VW^\star$ are co-isometry. Then

$$\Lambda_i = \Theta_i K^\star = \Theta_i(\frac{\|T\|}{2}((VW)^\star + (VW^\star)^\star)),$$

and, by Proposition 2.4, $\Theta_i(VW)^\star$ and $\Theta_i(VW^\star)^\star$ are normalized tight g-frames. □

Proposition 2.6 *If $\{\Theta_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-orthonormal basis for $\mathcal{H}$, then every g-frame for a Hilbert space $\mathcal{H}$ is the sum of a g-orthonormal basis for $\mathcal{H}$ and a g-Riesz basis for $\mathcal{H}$.*

Proof. If $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ is a g-frame for a Hilbert space $\mathcal{H}$ then, by Proposition 1.6, there is a bounded and onto operator $K\colon \mathcal{H} \longrightarrow \mathcal{H}$ such that $\Lambda_i = \Theta_i K^\star$. For any $0 < \epsilon < 1$ define an operator L by

$$L = \frac{3}{4} I + \frac{1}{4}(1 - \epsilon)\frac{K^\star}{\|K^\star\|}.$$

Then we have $\|I-L\|<1$ and $\|L\|\leq 1$. So L is an invertible operator and, as in the proof of proposition 1.1, (see [2]) we can write

$$L=\frac{1}{2}(W+W^{\star}),$$

where W is a unitary operator. We also have the relation

$$\begin{aligned} K^{\star} &= \frac{4\|K^{\star}\|}{(1-\epsilon)}[\frac{1}{2}(W+W^{\star})-\frac{3}{4}I] \\ &= \frac{2\|K^{\star}\|}{(1-\epsilon)}[W+R], \end{aligned}$$

where $R=W^{\star}-\frac{3}{2}I$. Since W is unitary, $\{\Theta_i W : i\in J\}$ is a g-orthonormal basis, and $W^{\star}$ is unitary which implies that R is an isomorphism (possibly into). But, it is easily checked that R is onto, since

$$\|I-\frac{-1}{2}R\|=\|\frac{1}{4}I+\frac{1}{2}W^{\star}\|<1.$$

Thus, $\frac{-1}{2}R$ is an invertible operator and hence R is an invertible operator. We also have

$$\sum_{i\in J}(\Theta_i R)^{\star}g_i=\sum_{i\in J}R^{\star}\Theta_i^{\star}g_i=R^{\star}(\sum_{i\in J}\Theta_i^{\star}g_i).$$

Since R is an invertible operator, if $\sum_{i\in J}(\Theta_i R)^{\star}g_i=0$ then $\sum_{i\in J}\Theta_i^{\star}g_i=0$ and since $\{\Theta_i : i\in J\}$ is a g-orthonormal basis we conclude $g_i=0$ for all $i\in J$. Therefore, by Proposition 2.2, $\Theta_i R$ is a g-Riesz basis for $\mathcal{H}$. □

3. Sums of g-bessel sequences

In this section we give necessary and sufficient conditions on g-Bessel sequences $\{\Lambda_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i): i\in J\}$ and $\{\Gamma_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i): i\in J\}$ and operators L_1, L_2 on $\mathcal{H}$ so that $\{\Lambda_i L_1+\Gamma_i L_2 : i\in J\}$ is a g-frame for $\mathcal{H}$, and we show that a g-frame can be added to any of its canonical dual g-frame to yield a new g-frame.

Proposition 3.1 *Let $\{\Lambda_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i) : i\in J\}$ and $\{\Gamma_i\in\mathcal{L}(\mathcal{H},\mathcal{H}_i) : i\in J\}$ be g-Bessel sequences in $\mathcal{H}$ with analysis operators T_1, T_2 and g-frame operators S_1, S_2, respectively. For the given operators L_1, L_2 : $\mathcal{H}\longrightarrow\mathcal{H}$ the following are equivalent:*

i) $\{\Lambda_i L_1+\Gamma_i L_2 : i\in J\}$ *is a g-frame for* $\mathcal{H}$.

ii) $T_1L_1+T_2L_2$ *is a bounded and one-to-one operator on* $\mathcal{H}$.

iii) The operator $S=L_1^{\star}T_1^{\star}T_1L_1+L_1^{\star}T_1^{\star}T_2L_2+L_2^{\star}T_2^{\star}T_1L_1+L_2^{\star}T_2^{\star}T_2L_2$ *is a well-defined and bounded mapping from* $\mathcal{H}$ *onto* $\mathcal{H}$. *Moreover, in this case,* S *is the g-frame operator for* $\{\Lambda_i L_1+\Gamma_i L_2 : i\in J\}$.

Proof. The family $\{\Lambda_i L_1 + \Gamma_i L_2 : i \in J\}$ is a g-frame if and only if its analysis operator T which is defined by

$$\begin{aligned} T(f) = \{(\Lambda_i L_1 + \Gamma_i L_2)(f)\}_{i\in J} &= \{\Lambda_i L_1(f)\}_{i\in J} + \{\Gamma_i L_2(f)\}_{i\in J} \\ &= (T_1 L_1 + T_2 L_2)(f), \end{aligned}$$

is a bounded and one-to-one operator on $\mathcal{H}$, and this happens if and only if the g-frame operator for our family

$$\begin{aligned} S &= (T_1L_1 + T_2L_2)^\star (T_1L_1 + T_2L_2) \\ &= L_1^\star T_1^\star T_1 L_1 + L_1^\star T_1^\star T_2 L_2 + L_2^\star T_2^\star T_1 L_1 + L_2^\star T_2^\star T_2 L_2 \end{aligned}$$

is well defined and bounded. □

The following theorem enables one to get a g-frame from a combination of a known g-frame and a g-Bessel sequence.

Theorem 3.2 *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in J\}$ be a g-frame for a Hilbert space $\mathcal{H}$ with g-frame operator S_1 and let $\{\Gamma_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in J\}$ be a g-Bessel sequence in $\mathcal{H}$ with g-frame operator S_2. Let T_1, T_2 be the analysis operators for $\{\Lambda_i : i \in J\}$, $\{\Gamma_i : i \in J\}$, respectively, so that $rangeT_2 \subseteq rangeT_1$. If the operator $R = T_1^\star T_2$ is a positive operator, then $\{\Lambda_i + \Gamma_i : i \in J\}$ is a g-frame for $\mathcal{H}$ with g-frame operator $S_1 + R + R^\star + S_2$.*

Proof. Let T_1, T_2 be the analysis operators for $\{\Lambda_i : i \in J\}$, $\{\Gamma_i : i \in J\}$, respectively. By letting $L_1 = I = L_2$ in Proposition 3.1, we see that the g-frame operator for $\{\Lambda_i + \Gamma_i : i \in J\}$ is

$$S = T_1^\star T_1 + T_1^\star T_2 + T_2^\star T_1 + T_2^\star T_2 = S_1 + R + R^\star + S_2.$$

□

Corollary 3.3 *If $\{\Lambda_i \in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in J\}$ is a g-frame with g-frame operator S and $\{\Gamma_i :\in \mathcal{L}(\mathcal{H},\mathcal{H}_i) : i \in J\}$ is a g-Bessel sequence in $\mathcal{H}$, such that $f = \sum_{i\in J} \Lambda_i^\star \Gamma_i f$, for all $f \in \mathcal{H}$, then $\{\Lambda_i S^a + \Gamma_i S^b : i \in J\}$ is a g-frame, for all real numbers a and b.*

Proof. If T_1 and T_2 are the analysis operators for $\{\Lambda_i S^a : i \in J\}$, $\{\Gamma_i S^b : i \in J\}$, respectively, then for $R = T_1^\star T_2$ we have

$$\begin{aligned} R(f) &= T_1^\star T_2(f) \\ &= T_1^\star(\{\Gamma_i S^b f\}) \\ &= \sum_{i\in J} (\Lambda_i S^a)^\star \Gamma_i S^b f \\ &= \sum_{i\in J} S^a \Lambda_i^\star \Gamma_i S^b f \\ &= S^{a+b} f. \end{aligned}$$

Since S is invertible, $\{\Lambda_i S^a + \Gamma_i S^b : i \in J\}$ is a g-frame, by Theorem 3.2. □

Corollary 3.4 *If* $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ *is a g-frame with g-frame operator* S *and* $\{\widetilde{\Lambda_i} \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ *is a canonical dual g-frame then* $\{\Lambda_i S^a + \widetilde{\Lambda_i} S^b : i \in J\}$ *is a g-frame for all real numbers* a, b.

Proposition 3.5 *Let* $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in J\}$ *be a g-frame for a Hilbert space* $\mathcal{H}$ *with g-frame operator* S *and g-frame bounds* A *and* B. *Let* $\{I_1, I_2\}$ *be a partition of* J *and let* S_j *be the g-frame operator for the g-Bessel sequences* $\{\Lambda_i : i \in I_j\}$, $j = 1, 2$. *Then* $\{\Lambda_i + \Lambda_i S_1^a : i \in I_1\} \bigcup \{\Lambda_i + \Lambda_i S_2^b : i \in I_2\}$, *is a g-frame for any real numbers* a, b *that the operator* $S_1(I + S_1^a)^2 + S_2(I + S_2^b)^2$ *is onto.*

Proof. Note that, for each $f \in \mathcal{H}$

$$\begin{aligned}
(\sum_{i\in I_1} \|\Lambda_i f + \Lambda_i S_1^a f\|^2)^{\frac{1}{2}} &\leq (\sum_{i\in I_1} \|\Lambda_i f\|^2)^{\frac{1}{2}} + (\sum_{i\in I_1} \|\Lambda_i S_1^a f\|^2)^{\frac{1}{2}} \\
&\leq \sqrt{B}\|f\| + \sqrt{B}\|S_1^a f\| \\
&\leq \sqrt{B}(1 + \|S_1^a\|)\|f\|.
\end{aligned}$$

Similarly, we have

$$(\sum_{i\in I_2} \|\Lambda_i f + \Lambda_i S_2^b f\|^2)^{\frac{1}{2}} \leq \sqrt{B}(1 + \|S_2^b\|)\|f\|,$$

Thus

$$\{\Lambda_i + \Lambda_i S_1^a : i \in I_1\} \bigcup \{\Lambda_i + \Lambda_i S_2^b : i \in I_2\},$$

is a g-Bessel sequence. On the other hand, the frame operator for $\{\Lambda_i + \Lambda_i S_1^a : i \in I_1\}$ is

$$\begin{aligned}
\sum_{i\in I_1} (\Lambda_i + \Lambda_i S_1^a)^\star (\Lambda_i + \Lambda_i S_1^a) &= \sum_{i\in I_1} \Lambda_i^\star \Lambda_i + S_1^a \sum_{i\in I_1} \Lambda_i^\star \Lambda_i \\
&+ \sum_{i\in I_1} \Lambda_i^\star \Lambda_i S_1^a + S_1^a \sum_{i\in I_1} \Lambda_i^\star \Lambda_i S_1^a \\
&= S_1 + S_1^a S_1 + S_1 S_1^a + S_1^a S_1 S_1^a \\
&= S_1 + 2S_1^{a+1} + S_1^{2a+1} \\
&= S_1(I + S_1^a)^2,
\end{aligned}$$

Similarly for $\{\Lambda_i + \Lambda_i S_2^b : i \in I_2\}$ the frame operator is $S_2(I + S_2^a)^2$. Hence, the g-frame operator S_0 for our family is an onto and bounded operator and hence, by Proposition 1.5, $\{\Lambda_i + \Lambda_i S_1^a : i \in I_1\} \bigcup \{\Lambda_i + \Lambda_i S_2^b : i \in I_2\}$ is a g-frame. □

4. Subsequence of g-frames

A g-frame for Hilbert space $\mathcal{H}$ has been decomposed into two infinite subsequences, if one of the subsequence is a g-frame for $\mathcal{H}$ a necessary and sufficient condition under which the other subsequence is a g-frame for $\mathcal{H}$ has been given.

Theorem 4.1 *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N}\}$ be a g-frame for a Hilbert space $\mathcal{H}$ and let $\{m_k\}$ and $\{n_k\}$ be two infinite increasing sequences with $\{m_k\} \bigcup \{n_k\} = \mathbb{N}$. Also let $\{\Lambda_{m_k} : k \in \mathbb{N}\}$ be a g-frame for $\mathcal{H}$. Then $\{\Lambda_{n_k} : k \in \mathbb{N}\}$ is a g-frame for $\mathcal{H}$ if and only if there exists a bounded linear operator $U : (\sum_{k\in\mathbb{N}} \bigoplus \mathcal{H}_{n_k})_{l_2} \longrightarrow (\sum_{k\in\mathbb{N}} \bigoplus \mathcal{H}_{m_k})_{l_2}$ such that $U(\{\Lambda_{n_k} f\}_{k\in\mathbb{N}}) = \{\Lambda_{m_k} f\}_{k\in\mathbb{N}}$, $f \in \mathcal{H}$.*

Proof. Let A be a lower bound of the g-frame $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N}\}$. Since

$$\begin{aligned} \sum_{k\in\mathbb{N}} \|\Lambda_{m_k} f\|^2 &= \|U(\{\Lambda_{n_k} f\}_{k\in\mathbb{N}})\| \leq \|U\| \|\{\Lambda_{n_k} f\}_{k\in\mathbb{N}}\| \\ &= \|U\| \sum_{k\in\mathbb{N}} \|\Lambda_{n_k} f\|^2, \end{aligned}$$

we have

$$\sum_{k\in\mathbb{N}} \|\Lambda_{n_k} f\|^2 \geq \frac{\sum_{k\in\mathbb{N}} \|\Lambda_{m_k} f\|^2}{\|U\|} \geq \frac{A}{\|U\|} \|f\|^2,$$

which implies that $\{\Lambda_{n_k} : k \in \mathbb{N}\}$ is a g-frame. Conversely, let $\{\Lambda_{n_k} : k \in \mathbb{N}\}$ be a g-frame. Let T_1, T_2 be the analysis operators for $\{\Lambda_{n_k} : k \in \mathbb{N}\}$ and $\{\Lambda_{m_k} : k \in \mathbb{N}\}$, respectively. Put $U = T_2 S_1^{-1} T_1^{\star}$. Then U is a bounded linear operator with the desired properties. □

In the following Theorem, we give a sufficient condition for a g-frame of nonzero elements in terms of g-frame sequences for its exactness.

Theorem 4.2 *Let $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N}\}$ be a g-frame for a Hilbert space $\mathcal{H}$ with optimal bounds A and B such that $\Lambda_i \neq 0$, for all $i \in \mathbb{N}$. If for every infinite increasing sequence $\{n_k\}$ in $\mathbb{N}$, $\{\Lambda_{n_k} f\}_{k\in\mathbb{N}}$ is a g-frame sequence with optimal bounds A and B, then $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N}\}$ is an exact g-frame.*

Proof. Suppose $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N}\}$ is not exact. Then there exists a positive integer $m \in \mathbb{N}$ such that $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \neq m, i \in \mathbb{N}\}$ is a g-frame. Let $\{n_k\}$ be an increasing sequence given by $n_k = k, k = 1, 2, ..., m-1$ and $n_k = k+1, k = m, m+1, \ldots$. Since $\{\Lambda_{n_k} f\}_{k\in\mathbb{N}}$ is a g-frame sequence with optimal bounds A and B, we have

$$A\|f\|^2 \leq \sum_{i\neq m} \|\Lambda_i f\|^2 \leq B\|f\|^2.$$

Therefore, by g-frame inequality for the frame $\{\Lambda_i \in \mathcal{L}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{N}\}$, for all $f \in \mathcal{H}$ we have

$$\|\Lambda_m f\|^2 = 0,$$

This given $\Lambda_m = 0$, which is a contradiction. □

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