Turk J Math
36 (2012) , 113 – 120.

doi:10.3906/mat-0904-38

On the order of weighted approximation by positive linear operators

Tülin Coşkun

Abstract

An estimation of approximation of continuous functions by positive linear operators in weighted norm using the weighted modulus of continuity is established. Application of the main result to the known Gadjiyev* -Ibragimov operators is given.

Key Words: Positive linear operators, weighted spaces, weighted modulus of continuity, Korovkin type theorems, Gadjiev-Ibragimov operators

1. Introduction and preliminaries

Let m be a natural number and $B_{2m} = B_{2m}(0,\infty)$ be the space of all functions f defined on the semi-axis $[0,\infty)$ satisfying the inequality

$$|f(x)| \le M_f(1+x^{2m}),$$

where M_f is a positive constant depending on functions f only. Obviously, B_{2m} is the linear normed space with the norm

$$\|f\|_{2m} = \sup_{x\ge 0} \frac{|f(x)|}{1+x^{2m}}.$$

The subspace of all continuous functions belonging to B_{2m} will be denoted by $C_{2m} := C_{2m}(0,\infty)$, and also define

$$C^0_{2m} := \{f \in C_{2m} : \lim_{x\to\infty} \frac{f(x)}{1+x^{2m}} = K_f < \infty\}.$$

As it follows from the Gadjiev papers [9], [10] the Korovkin-type theorems for positive linear operators (p.l.o.) does not hold in whole space C_{2m} and has the following forms.

Theorem 1.1 *If the sequence of p.l.o. L_n from C_{2m} to B_{2m} satisfies conditions*

$$\lim_{n\to\infty} \|L_n(t^{\nu m},x) - x^{\nu m}\|_{2m} = 0, \quad \nu = 0,1,2,$$

2000 *AMS Mathematics Subject Classification:* Primary 41A36; Secondary 47A58.
*Gadziev=Gadjiev, see [11] [12])

then for any function $f \in C_{2m}^0$

$$\lim_{n\to\infty} \|L_n f - f\|_{2m} = 0.$$

Theorem 1.2 *For any* $m \in \mathbb{N}$ *there exists a sequence of p.l.o.* $(L_n)_{n\in\mathbb{N}}$*, satisfying the conditions of Theorem 1.1 and yet, at the same time, there exists a function* $f^* \in C_{2m}$*, for with*

$$\lim_{n\to\infty} \|L_n f^* - f^*\|_{2m} \geq 1.$$

Note that various theorems related to theorems 1.1 and 1.2 were proved in [3], [4], [5], [8], [11], [12].

In this paper the order of convergence of the sequences of p.l.o. in the weighted norm will be studied by using the weighted modulus of continuity. For a function $f \in C_{2m}^0$, the following weighted modulus of continuity is considered:

$$\omega(f,\delta)_{2m} = \sup_{x\geq 0,\, |h|\leq\delta} \frac{|f(x+h)-f(x)|}{(1+x^{2m})(1+h^{2m})}.$$

Note that this type of modulus of continuity for a function $f \in C_{2m}$ was first introduced in [1] by Achieser. In general, this modulus of continuity of function $f \in C_{2m}$ does not tend to zero as $\delta \to 0$. But, as proved in [13], $\lim_{\delta\to 0} \omega(f,\delta)_2 = 0$ for any function $f \in C_2^0$.

We will show now that for any fixed $m \in \mathbb{N}$, $\lim_{\delta\to 0} \omega(f,\delta)_{2m} = 0$ for each function $f \in C_{2m}^0$. In fact, by the definition of C_{2m}^0 given $\varepsilon > 0$, we can find $x_0 > 0$ such that

$$\left|\frac{f(x+h)}{1+(x+h)^{2m}} - K_f\right| < \varepsilon, \quad \left|\frac{f(x)}{1+x^{2m}} - K_f\right| < \varepsilon$$

for all $x > x_0$. Also, there exist $\delta > 0$ such that the inequality $\sup_{0\leq x\leq x_0,\, |h|\leq\delta} |f(x+h)-f(x)| < \varepsilon$ is satisfied for all $f \in C[0,x_0]$. Therefore

$$\begin{aligned}\omega(f,\delta)_{2m} &\leq 2\varepsilon + \sup_{x\geq x_0,\, |h|\leq\delta} \frac{|K_f(1+x^{2m})-f(x)|}{(1+x^{2m})(1+h^{2m})} \\ &\quad + K_f\delta \cdot \sup_{x\geq x_0,\, |h|\leq\delta} \frac{\sum_{j=1}^{2m}\binom{2m}{j} x^{2m-j}|h|^{j-1}}{(1+x^{2m})(1+h^{2m})} \leq 3\varepsilon + 2^{2m}K_f\cdot\delta,\end{aligned}$$

and the desired result can be obtained.

Using the equality

$$f(x+mh)-f(x) = \sum_{k=1}^{m}[f(x+kh)-f(x+(k-1)h)]$$

we obtain that the inequality

$$\omega(f,\lambda\delta)_{2m} \leq 2^{2m}(1+\lambda)(1+\delta^{2m})\omega(f,\delta)_{2m}$$

holds for any positive number λ. This inequality gives that

$$|f(t)-f(x)| \leq (1+x^{2m})(1+|t-x|^{2m})\omega(f,|t-x|)_{2m}$$

$$\leq 2^{2m}(1+x^{2m})(1+|t-x|^{2m})(1+\frac{|t-x|}{\delta})(1+\delta^{2m})\omega(f,\delta)_{2m} \tag{1}$$

for any function $f \in C^0_{2m}$, any $t, x \geq 0$ and $\delta > 0$.

Since $\omega(f,\delta)_{2m}$ tends to zero as $\delta \to 0$, it may be used for the estimate of degree of approximation of functions in C^0_{2m} by p.l.o. We will use the sequences of p.l.o. introduced in [14] called the Gadjiev-Ibragimov operators (see, for example, [16]). These operators were studied in different papers, for example, in references [2], [6], [7], [13], [15] and [16].

In particular, in [14] authors established the estimate for $\frac{|L_n(f,x)-f(x)|}{(1+x^2)^\alpha}$, where $\alpha \geq 3$, (L_n) are Gadjiev-Ibragimov operators, $f \in C^0_2$ and $x \in [0,\gamma_n]$, $\lim\limits_{n\to\infty} \gamma_n = \infty$. In [14] the estimate for the same ratio is stated as an "open problem", if $\alpha \in [1,3)$. Our main theorem, which is proved in the space C_{2m} in the last section of this paper, strengthens the above result for $m = 1$. This idea provides an answer to the open problem given above.

2. The degree of approximation

Recall that the construction of Gadjiev-Ibragimov operators, given in [14], are based on Taylor expansion of functions $K_n(x,t,u)$ of variables x,t,u, which is an entire analytic function with respect to variable u for fixed $x,t \in [0,A]$. In [13] authors consider a more general case, when $x,t \in [0,\gamma_n]$ with $\lim\limits_{n\to\infty} \gamma_n = \infty$.

We consider the following modified form of these operators. Let functions $K_n(x,t,u)$ of variables $x,t,u \in [0,\infty)$ satisfy the following conditions:

(1°) each functions $K_n(x,t,u)$ is entire analytic function with respect to variable u for fixed $x,t \in [0,\infty)$;

(2°) for any natural n and any $x \in [0,\infty)$ $K_n(x,0,0) = 1$;

(3°) $\left\{(-1)^\nu \frac{\partial^\nu K_n(x,t,u)}{\partial u^\nu}\Big|_{u=u_1;t=0}\right\} \geq 0, \qquad \nu, n \in \mathbb{N}$ for any fixed $u = u_1$;

(4°) $\frac{\partial^\nu}{\partial u^\nu}K_n(x,t,u)\Big|_{u=u_1;t=0} = -nx\left[\frac{\partial^{\nu-1}}{\partial u^{\nu-1}}K_{n+m}(x,t,u)\Big|_{u=u_1;t=0}\right.$ for any fixed $u = u_1$,
where $(n+m)$ is natural number and m is a constant independent of ν.

Moreover, let $(\varphi_n(t))_{n\in\mathbb{N}}$ and $(\psi_n(t))_{n\in\mathbb{N}}$ be sequences of continuous functions on $[0,\infty)$ such that $\varphi_n(0) = 0$, $\psi_n(t) > 0$, for all $t \in [0,\infty)$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers having the properties

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = 1, \quad \lim_{n\to\infty} \frac{1}{n^2\psi_n(0)} = 0.$$

In [14] the authors write the Taylor expansion of entire function $K_n(x,t,\varphi_n(t))$ in the powers of $(\varphi_n(t)-\alpha_n\psi_n(t))$ and taking, then, $t = 0$ obtained the expansion in the powers of $(-\alpha_n\psi_n(0))$ since $\varphi_n(0) = 0$.

Under these conditions Gadjiev-Ibrahimov operators have the form

$$L_n(f,x) = \sum_{\nu=0}^{\infty} f(\frac{\nu}{n^2\psi_n(0)})K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}, \tag{2}$$

where the notation

$$K_n^{(\nu)}(x,0,\alpha_n\psi_n(0)) = \frac{\partial^\nu K_n(x,t,u)}{\partial u^\nu}\Big|_{u=\alpha_n\psi_n(t),t=0}$$

is used. In [14] it has been shown that for different functions $\psi_n(t)$ the operators (2) generalize Bernstein polynomials, Bernstein-Cholodowsky polynomials, Szass operators, etc. Some new properties of operators (2) will be needed.

Lemma 2.1 *For any natural* N

$$L_n(t^N,x) = \sum_{k=1}^{N}(\frac{\alpha_n}{n})^k\frac{n(n+m)(n+2m)\dots(n+(k-1)m)}{n^k}\cdot\frac{C_{k,N}}{(n^2\psi_n(0))^{N-k}}x^k, \tag{3}$$

where $C_{k,N}$ *are positive constants and* $C_{N,N}=1$.

Proof. For $N=1,2$ equality (3) is true since $L_n(t,x)=\frac{\alpha_n}{n}x$

$$L_n(t^2,x) = (\frac{\alpha_n}{n})^2\frac{n+m}{n}x^2 + \frac{\alpha_n}{n}\frac{1}{n^2\psi_n(0)}x.$$

Assuming that (3) holds for natural p, we will use the induction method. Since for any natural ν there exist constants a_j such that

$$\nu^{p+1} = \nu(\nu-1)\cdots(\nu-p) + \sum_{j=1}^{p}\nu^j a_j,$$

we can write

$$(\frac{\nu}{n^2\psi_n(0)})^{p+1} = \frac{\nu(\nu-1)\dots(\nu-p)}{(n^2\psi_n(0))^{p+1}} + \sum_{j=1}^{p}(\frac{\nu}{n^2\psi_n(0)})^j\frac{a_j}{(n^2\psi_n(0))^{p+1-j}}.$$

Then taking in (2) $f(t)=t^{p+1}$, we obtain

$$\begin{aligned} L_n(t^{p+1},x) &= \sum_{\nu=0}^{\infty}(\frac{\nu}{(n^2\psi_n(0))})^{p+1}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} \\ &= \sum_{\nu=0}^{\infty}\Big[\frac{\nu(\nu-1)\dots(\nu-p)}{(n^2\psi_n(0))^{p+1}} + \sum_{j=1}^{p}(\frac{\nu}{n^2\psi_n(0)})^j\frac{a_j}{(n^2\psi_n(0))^{p+1-j}}\Big] \\ &\quad\cdot K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!} \\ &= \frac{1}{(n^2\psi_n(0))^{p+1}}\sum_{\nu=p+1}^{\infty}K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{(\nu-p-1)!} \\ &\quad+\sum_{j=1}^{p}\frac{a_j}{(n^2\psi_n(0))^{p+1-j}}L_n(t^j,x). \end{aligned}$$

Applying the properties (4°) of the function $K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))$ to the first term in the right hand side and the formula (3) to the second term, by taking into account the properties (2°), we obtain

$$\begin{aligned} L_n(t^{p+1},x) &= (-1)^{p+1}(\frac{\alpha_n}{n})^{p+1}\frac{n(n+m)(n+2m)\dots(n+pm)}{n^{p+1}}K_{n+(p+1)m}(x,0,0).x^{p+1} \\ &+\sum_{j=1}^{p}\frac{a_j}{(n^2\psi_n(0))^{p+1-j}}\Big[\sum_{k=1}^{j}(\frac{\alpha_n}{n})^k\frac{n(n+m)(n+2m)\dots(n+(k-1)m)}{n^k} \\ &\cdot\frac{C_{k,N}}{(n^2\psi_n(0))^{j-k}}x^k\Big] \\ &= (\frac{\alpha_n}{n})^{p+1}\frac{n(n+m)(n+2m)\dots(n+pm)}{n^p}x^{p+1} \\ &+\sum_{j=1}^{p}a_j\sum_{k=1}^{j}(\frac{\alpha_n}{n})^k\frac{n(n+m)(n+2m)\dots(n+(k-1)m)}{n^k}\frac{C_{k,N}}{(n^2\psi_n(0))^{p+1-k}}x^k. \end{aligned}$$

Changing the order of summation in right hand side we can write

$$L_n(t^{p+1},x)=(\frac{\alpha_n}{n})^{p+1}\frac{n(n+m)(n+2m)\dots(n+pm)}{n^{p+1}}x^{p+1}+$$

$$+\sum_{k=1}^{p}(\frac{\alpha_n}{n})^k\frac{n(n+m)(n+2m)\dots(n+(k-1)m)}{n^k}\frac{C_{k,N}}{(n^2\psi_n(0))^{p+1-k}}x^k.\sum_{j=k}^{p}a_j,$$

and denoting

$$C_{k,p+1}=\begin{cases}\sum\limits_{k=1}^{p}(\frac{\alpha_n}{n})^k\frac{n(n+m)(n+2m)\dots(n+(k-1)m)}{n^k}\frac{C_{k,N}}{(n^2\psi_n(0))^{p+1-k}}x^k.\sum\limits_{j=k}^{p}a_j, & if\ 1\le k\le p,\\ 1, & if\ k=p+1\end{cases}$$

we obtain the formula (13) with $N=p+1$. The proof is complete. □

Note that another form of the formula (3) was proved in [2].

Let N be a natural number, then

$$T_{N,n}(x)=\sum_{\nu=0}^{\infty}\Big(\frac{\nu}{n^2\psi_n(0)}-x\Big)^N K_n^{(\nu)}(x,0,\alpha_n\psi_n(0))\frac{(-\alpha_n\psi_n(0))^\nu}{\nu!}$$

is the N^{th} moments of operators (2).

Lemma 2.2 *For any $x\in[0,\infty)$ and sufficiently large n the inequality*

$$|T_{N,n}(x)|=(x+x^2+\dots+x^N)O(\frac{1}{n^2\psi_n(0)}) \tag{4}$$

holds.

Proof. From the definition of $T_{N,n}(x)$ we can write the equality

$$T_{N,n}(x) = (-x)^N + \binom{N}{1} L_n(t,x)(-x)^{N-1} + \binom{N}{2} L_n(t^2,x)(-x)^{N-2} + \cdots + \binom{N}{N-1} L_n(t^{N-1},x)(-x) + \binom{N}{N} L_n(t^N,x).$$

Now using (3) the coefficients under x^j has consequently the multipliers $(n^2\psi_n(0))^{j-N}$, $j = 1, \cdots, N-1$ and therefore tends to zero as $n \to \infty$. The term with x^N consists also the part which tends to zero as $n \to \infty$ and the part

$$x^N \Big[(-1)^N + \binom{N}{1}(-1)^{N-1} + \binom{N}{2}(-1)^{N-2} + \cdots + \binom{N}{N-1}(-1) + \binom{N}{N} \Big] = x^N \sum_{k=0}^{N} (-1)^{N-k} \binom{N}{k} = 0.$$

Since all other terms as stated above consists of the terms $(\frac{1}{n^2\psi_n(0)})^l$ for some natural l and $n^2\psi_n(0) \to \infty$, the desired result is obtained. □

Now the following main result will be proved.

Theorem 2.3 *Let $f \in C_{2m}^0$ be given and $\omega(f,\delta)_{2m}$ be its weighted modulus of continuity. Then for a sufficiently large n, the inequality*

$$\sup_{x \geq 0} \frac{|L_n(f,x) - f(x)|}{(1+x^{2m})(1+x^{2m+1})} \leq \tilde{c}_m \cdot \omega(f, \frac{1}{\sqrt{n^2\psi_n(0)}})_{2m}$$

holds.

Proof. Denoting

$$P_{\nu,n}(x) = K_n^{(\nu)}(x, 0, \alpha_n\psi_n(0)) \frac{(-\alpha_n\psi_n(0))^\nu}{\nu!},$$

and taking into account that $L_n(1,x) = 1$ and $P_{\nu,n}(x) \geq 0$, following inequality can be written

$$|L_n(f,x) - f(x)| \leq \sum_{\nu=0}^{\infty} \Big| f(\frac{\nu}{n^2\psi_n(0)}) - f(x) \Big| P_{\nu,n}(x).$$

By the properties (1) of modulus of continuity $\omega(f,\delta)_m$, the inequality

$$\Big| f(\frac{\nu}{n^2\psi_n(0)}) - f(x) \Big| \leq 2^{2m}(1+x^{2m})(1+(\frac{\nu}{n^2\psi_n(0)} - x)^{2m}) \cdot (1+\delta_n^{2m}) \Big(1 + \frac{|\frac{\nu}{n^2\psi_n(0)} - x|}{\delta_n} \Big) \omega(f,\delta_n)_{2m}$$

holds for any sequences of positive numbers δ_n. Therefore applying the Hölder inequality we obtain

$$\frac{|L_n(f,x) - f(x)|}{1+x^{2m}} \leq 2^{2m}(1+\delta_n^{2m})\omega(f,\delta_n)_{2m} \Big\{ 1 + T_{2m}(x) + \frac{1}{\delta_n} T_2^{\frac{1}{2}}(x) + \frac{1}{\delta_n} T_{2m+2}^{\frac{1}{2}}(x) T_{2m}^{\frac{1}{2}}(x) \Big\}.$$

Using (4) we have the inequality

$$\frac{|L_n(f,x)-f(x)|}{1+x^{2m}} \leq 2^{2m}(1+\delta_n^{2m})\omega(f,\delta_n)_{2m}\Big\{1+ +(x+\ldots+x^{2m})\frac{1}{n^2\psi_n(0)}+\frac{1}{\delta_n}(x+x^2)^{\frac{1}{2}}\frac{1}{(n^2\psi_n(0))^{\frac{1}{2}}} +\frac{1}{\delta_n}(x+\ldots+x^{2m+2})^{\frac{1}{2}}(x+\ldots+x^{2m})^{\frac{1}{2}}\frac{1}{n^2\psi_n(0)}\Big\}.$$

Choosing $\delta_n = \sqrt{\frac{1}{n^2\psi_n(0)}}$ we see that $\delta_n \to 0$ as $n\to\infty$ and therefore for a sufficiently large n, $\delta_n < 1$. This gives

$$\frac{|L_n(f,x)-f(x)|}{(1+x^{2m})(1+x^{2m+1})} \leq 2^{2m+1}(5+4m)\cdot\omega(f,\frac{1}{\sqrt{n^2\psi_n(0)}})_{2m}$$

and denoting $\tilde{c}_m = 2^{2m+1}(5+4m)$, the proof is complete. □

In conclusion consider the case $m=1$. In this case the inequality of Theorem 2.3 gives

$$\sup_{x\geq 0}\frac{|L_n(f,x)-f(x)|}{(1+x^2)(1+x^3)} \leq \tilde{c}_1\cdot\omega(f,\frac{1}{\sqrt{n^2\psi_n(0)}})_2.$$

From this we obtain

$$\sup_{x\in[0,\gamma_n]}\frac{|L_n(f,x)-f(x)|}{(1+x^2)(1+x^3)} \leq \tilde{c}_1\cdot\omega(f,\frac{1}{\sqrt{n^2\psi_n(0)}})_2$$

for any sequence (γ_n) such that $\lim_{n\to\infty}\gamma_n=\infty$. Since

$$(1+x^2)(1+x^3) < (1+x^2)^3,$$

the inequality tried to be proved in this study could be considered better than the inequality obtained in [13].

Acknowledgments

The author is thankful to referees for their attentive reading of the manuscript and for useful comments.

References

[1] Ahiezer, N. I.:*Lectures in the theory of approximation.* Second, revised and enlarged edition Izdat. "Nauka", Moscow, 407 pp, (1965).

[2] Aral, A.:*Approximation by Ibragimov-Gadjiyev operators in polynomial weighted space.* Proc. of IMM of NAS of Azerbaijan, Vol.XIX, 35–44, (2003).

[3] Coşkun, T. :*Some properties of linear positive operators on the spaces of weight functions.* Commun. Fac. Sci. Univ. Ank. Series. A1, V. 47, 175-181, (1998).

[4] Coşkun, T.: *Weighted approximation of continuous functions by sequences of linear positive operators.* Proc. Indian Acad. Sci. (Math. Sci.) Vol. 110, No. 4, 357-362, (2000).

[5] Coşkun, T.: *Weighted approximation of unbounded continuous functions by sequence of linear positive operators.* Indian J. Pure Appl. Math., 34(3), 477-485, (2003).

[6] Doğru, O.: *On a certain family linear positive operators.* Tr. J. Math., 21, 387-399, (1997).

[7] Doğru, O.: *On the order of approximation of unbounded functions by the family of generalized linear positive operators.* Commun Fac. Sci. Univ. Ankara, Series A1, Vol.46, 173-181, (1997).

[8] Doğru, O.: *On weighted approximation of continuous functions by linear positive operators on infinite intervals.* Mathematica, tome 41(64) N!, 39-46, (1999).

[9] Gadzhiev, A.D.: *The convergence problem for a sequence of positive linear operators on unbounded sets and theorems analogous to that of P.P. Korovkin.* English translated Sov. Math. Dokl. Vol 15, No 5, (1974).

[10] Gadzhiev, A.D.: *On P.P Korovkin type theorems.* Mathem. Zametki, Vol 20, No 5, (Engl. Transl., Math. Notes 20, 995-998, (1976).)

[11] Gadjiev, A.D., Efendiev, I., Ibikli, E.: *Generalized Bernstein-Chlodowsky polynomials.* Rocky Mount. J. Math., V. 28, No 4, 1267-1277, (1998).

[12] Gadjiev, A.D., Efendiev, I., Ibikli, E.: *On Korovkin type theorem in the space of locally integrable fuctions.* Czech. Math. J., 53(128), 45-53, (2003).

[13] Gadjiev, A.D., Ispir, N.: *On a sequence of linear positive operators in weighted spaces.* Proc. of IMM of Azerbaijan AS, Vol. XI(XIX), 45-56, (1999).

[14] Ibragimov, I. I., Gadjiev, A. D.: *On a certain Family of linear positive operators.* Soviet Math. Dokl., English Trans., 11, 1092-1095,(1970).

[15] Poğany, T.: *Some Korovkin-type theorem for stocastic processes.* Teor. Imovi Stat., No 61, 145-151, (1999). Trans. in theory Prob. Math. Stat. N61, 153–159, (2000).

[16] Radatz, P., Wood, B.: *Approximating Derivativer of Functions Unbounded on the Positive Axis with Lineare Operators.* Rev. Roum. Math. Pures et Appl., Bucarest, Tome XXIII, No 5, 771-781, (1978).

Tülin COŞKUN
Department of Mathematics,
Zonguldak Karaelmas University,
67100 Zonguldak-TURKEY
e-mail: tcoskun@karaelmas.edu.tr

Received: 21.04.2009