Banach limit and some new spaces of double sequences

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Abstract

In this paper, we define and study the Banach limit for double sequences and introduce some new spaces related to the concept of almost and strong almost convergence for double sequences. We characterize these spaces through some sublinear functionals and we also establish some inclusion relations.

Key words and phrases: Double sequences; P-convergence; Banach limit; sublinear functionals; almost convergence; strong almost convergence

1. Introduction

A double sequence $x = (x_{jk})$ is said to be *Pringsheim's convergent* (or *P-convergent*) if for given $\epsilon > 0$ there exists an integer N such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k > N$. We shall write this as

$$\lim_{j,k \to \infty} x_{jk} = \ell,$$

where j and k tending to infinity independent of each other (cf [9]). We denote by c_2, the space of P-convergent sequences.

A double sequence x is *bounded* if

$$\| x \| = \sup_{j,k \geq 0} |x_{jk}| < \infty.$$

We denote by ℓ_2^∞ the space of bounded double sequences. Note that, in contrast to the case for single sequences, a convergent double sequence need not be bounded. By c_2^∞, we denote the space of double sequences which are boundedly convergent.

In this paper, we first define the concept of Banach limit for double sequences.

Definition 1.1 *Let ℓ_2^∞ be the set of all real or complex double sequences $x = (x_{jk})$ with the norm $\|x\| = \sup_{j,k} |x_{jk}| < \infty$. A linear functional L on ℓ_2^∞ is said to be Banach limit if it has the following properties:*

(i) $L(x) \geq 0$ if $x \geq 0$ (i.e., $x_{jk} \geq 0$ for all j, k),

(ii) $L(E) = 1$, where $E = (e_{jk})$ with $e_{jk} = 1$ for all j, k, and

(iii) $L(S_{11}x) = L(x) = L(S_{10}x) = L(S_{01}x)$,

where the shift operators S_{01}, S_{10} and S_{11} are defined by

$$S_{01}x = (x_{j,k+1}), S_{10}x = (x_{j+1,k}), S_{11}x = (x_{j+1,k+1}).$$

Let $\mathcal{B}_2$ be the set of all Banach limits on ℓ_2^∞. A double sequence $x = (x_{jk})$ is said to be almost convergent to a number ℓ if $L(x) = \ell$ for all $L \in \mathcal{B}_2$.

The idea of almost convergence for single sequences was introduced by Lorentz [3] and for double sequences by Moricz-Rhoades [5] and further studied in [6]–[8].

The space f_2 of almost convergent double sequences was defined by Moricz and Rhoades [5] as

$$f_2 = \{x = (x_{jk}) \lim_{p,q\to\infty} |\tau_{pqst}(x) - \ell| = 0, \ \text{uniformly in } s, t\},$$

where

$$\tau_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+s,k+t}.$$

Note that a convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent and every almost convergent double sequence is also bounded, i.e. $c_2^\infty \subset f_2 \subset \ell_2^\infty$ and each inclusion is proper.

The idea of strong almost convergence for single sequences is due to Maddox [4] and for double sequences by Başarir [1].

A double sequence $x = (x_{jk})$ is said to be strongly almost convergent to a number ℓ if

$$\lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} |x_{j+s,k+t} - \ell| = 0,$$

uniformly in s, t. By $[f_2]$, we denote the space of all strongly almost convergent double sequences. Note that $[f_2] \subset f_2 \subset \ell_2^\infty$ and each inclusion is proper.

2. Some new spaces of double sequences

In this section, we introduce the following sequence spaces, while such spaces for single sequences were studied by Das and Sahoo [2].

$$w_2 = \left\{ x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x - \ell E) \longrightarrow 0 \text{ as } m, n \longrightarrow \infty, \right.$$

$$\left. \text{uniformly in } s, t, \text{ for some } \ell \right\},$$

$$[w_2] = \left\{ x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{pqst}(x - \ell E)| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty, \right.$$

$$\left. \text{uniformly in } s, t, \text{ for some } \ell \right\}$$

$$[w]_2 = \left\{ x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(|x - \ell E|) \longrightarrow 0 \text{ as } m, n \longrightarrow \infty, \right.$$

$$\left. \text{uniformly in } s, t, \text{ for some } \ell \right\}$$

By $(C_2, 2)$, we denote the space of *Cesàro summable* double sequences of order 2 defined by

$$(C_2, 2) = \left\{ x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{p,q,0,0}(x) \longrightarrow \ell \text{ as } m, n \longrightarrow \infty \right\}$$

and by $[C_2, 2]$, we denote the space of *strongly Cesàro summable* double sequences of order 2 defined by

$$[C_2, 2] = \left\{ x = (x_{jk}) : \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{p,q,0,0}(x) - \ell| \longrightarrow 0 \text{ as } m, n \longrightarrow \infty \right\}.$$

Remark 2.1 If $[w_2]\text{-}\lim x = \ell$, that is

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{pqst} - \ell| \longrightarrow 0$$

as $m, n \longrightarrow \infty$, uniformly in s, t; then

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \frac{1}{p+1} \sum_{j=0}^{p} \tau_{jqst} - \ell \right| \longrightarrow 0$$

and

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \frac{1}{q+1} \sum_{k=0}^{q} \tau_{pkst} - \ell \right| \longrightarrow 0.$$

3. Some sublinear functionals

Let G be any sublinear functional on ℓ_2^∞. We write $\{\ell_2^\infty, G\}$ to denote the set of all linear functionals F on ℓ_2^∞ such that $F \leq G$, i.e., $F(x) \leq G(x)$ for all $x = (x_{jk}) \in \ell_2^\infty$.

Now we define the following functionals on the space ℓ_2^∞ of real bounded double sequences:

$$\phi(x) = \limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x),$$

$$\psi(x) = \limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{pqst}(x)|,$$

$$\theta(x) = \limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(|x|),$$

$$\xi(x) = \limsup_{p,q} \sup_{s,t} \tau_{pqst}(x),$$

$$\eta(x) = \limsup_{p,q} \sup_{s,t} \tau_{pqst}(|x|),$$

where $|x| = (|x_{jk}|)_{j,k=1}^{\infty}$.

It can be easily verified that each of the above functionals are finite, well defined and sublinear on ℓ_2^{∞}.

A sublinear functional G is said to *generate* Banach limits if $F \in \{\ell_2^{\infty}, G\}$ is a Banach limit and it is said to *dominate* Banach limits if $F \in \mathcal{B}_2$ implies $F \in \{\ell_2^{\infty}, G\}$.

In the following theorem we characterize the space $\ell_2^{\infty} \cap w_2$ in terms of the sublinear functional ϕ.

Theorem 3.1 *We have the following:*

(i) *The sublinear functional ϕ both dominates and generates Banach limits, i.e. $\phi(x) = \xi(x)$, for all $x = (x_{jk}) \in \ell_2^{\infty}$.*

(ii) *$\ell_2^{\infty} \cap w_2 = \{x = (x_{jk}) \in \ell_2^{\infty} : \phi(x) = -\phi(-x)\} = f_2$.*

Proof.   (i) From definition of ξ, for given $\epsilon > 0$ there exist p_0, q_0 such that

$$\tau_{pqst}(x) < \xi(x) + \epsilon$$

for $p > p_0, q > q_0$ and for all s, t. This implies that

$$\phi(x) \le \xi(x) + \epsilon$$

for all $x = (x_{jk}) \in \ell_2^{\infty}$. Since ϵ is arbitrary, so that $\phi(x) \le \xi(x)$, for all $x = (x_{jk}) \in \ell_2^{\infty}$ and hence

$$\{\ell_2^{\infty}, \phi\} \subset \{\ell_2^{\infty}, \xi\} = \mathcal{B}_2, \tag{3.1.1}$$

i.e., ϕ generates Banach limits.

Conversely, suppose that $L \in \mathcal{B}_2$. As L is the shift invariant i.e., $L(S_{11}x) = L(x) = L(S_{10}x) = L(S_{01}x)$, we have

$$L(x) = L\left(\frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+s,k+t}\right)$$

$$= L\left(\tau_{pqst}(x)\right)$$

$$= L\left(\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x)\right). \tag{3.1.2}$$

But it follows from the definition of ϕ, that for given $\epsilon > 0$ there exist m_0, n_0, such that

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x) < \phi(x) + \epsilon \tag{3.1.3}$$

for $m > m_0, n > n_0$ and for all s, t. Hence, by (3.1.3) and properties (i) and (ii) of Banach limits, we have

$$L\left(\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x)\right) < L\left((\phi(x) + \epsilon)E\right) = \phi(x) + \epsilon, \tag{3.1.4}$$

for $m > m_0, n > n_0$ and for all s, t, where E is defined in the beginning of Section 2. Since ϵ was arbitrary, it follows from (3.1.2) and (3.1.4) that

$$L(x) \leq \phi(x), \text{ for all } x = (x_{jk}) \in \ell_2^\infty.$$

Hence

$$\mathcal{B}_2 \subset \{\ell_2^\infty, \phi\}. \tag{3.1.5}$$

That is, ϕ dominates Banach limits.

Combining (3.1.1) and (3.1.5), we get

$$\{\ell_2^\infty, \xi\} = \{\ell_2^\infty, \phi\},$$

this implies that ϕ dominates and generates Banach limits and $\phi(x) = \xi(x)$ for all $x \in \ell_2^\infty$.

(ii) As a consequence of Hahn-Banach theorem, $\{\ell_2^\infty, \phi\}$ is nonempty and a linear functional $F \in \{\ell_2^\infty, \phi\}$ is not necessarily uniquely defined at any particular value of x. This is evident in the manner the linear functionals are constructed. But in order that all the functionals $\{\ell_2^\infty, \phi\}$ coincide at $x = (x_{jk})$ it is necessary and sufficient that

$$\phi(x) = -\phi(-x); \tag{3.1.6}$$

we have

$$\limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x) = \liminf_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x). \tag{3.1.7}$$

But (3.1.7) holds if and only if

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x) \longrightarrow \ell \text{ (say) as } m, n \longrightarrow \infty,$$

uniformly in s, t. Hence, $x = (x_{jk}) \in w_2 \cap \ell_2^\infty$. But (3.1.6) is equivalent to

$$\xi(x) = -\xi(-x),$$

this holds if and only if $x = (x_{jk}) \in f_2$

This completes the proof of the theorem. $\square$

In the following theorem we characterize the spaces $[w_2] \cap \ell_2^\infty$ and $[w]_2 \cap \ell_2^\infty$ in terms of the sublinear functionals.

Theorem 3.2 *We have the following:*

(i) $[w_2] \cap \ell_2^\infty = \{x = (x_{jk}) : \psi(x - \ell E) = 0, \text{ for some } \ell\} = \{x = (x_{jk}) : F(x - \ell E) = 0, \text{ for all } F \in \{\ell_2^\infty, \psi\}, \text{ for some } \ell\}.$

(ii) $[w]_2 \cap \ell_2^\infty = \{x = (x_{jk}) : \theta(x - \ell E) = 0, \text{ for some } \ell\} = \{x = (x_{jk}) : F(x - \ell E) = 0, \text{ for all } F \in \{\ell_2^\infty, \theta\}, \text{ for some } \ell\}.$

Proof.   (i) It can be easily verified that $x = (x_{jk}) \in [w_2] \cap \ell_2^\infty$ if and only if

$$\psi(x - \ell E) = -\psi(\ell E - x). \tag{3.2.1}$$

since $\psi(x) = \psi(-x)$ then (3.2.1) reduces to

$$\psi(x - \ell E) = 0. \tag{3.2.2}$$

Now, if $F \in \{\ell_2^\infty, \psi\}$ then from (3.2.2) and linearity of F, we have

$$F(x - \ell E) = 0.$$

Conversely, suppose that $F(x - \ell E) = 0$ for all $F \in \{\ell_2^\infty, \psi\}$ and hence by Hahn-Banach theorem, there exists $F_0 \in \{\ell_2^\infty, \psi\}$ such that $F_0(x) = \psi(x)$. Hence

$$0 = F_0(x - \ell E) = \psi(x - \ell E).$$

(ii) The proof is similar as above. □

4.   Inclusion relations

We establish here some inclusion relations between the spaces defined in Section 2.

Theorem 4.1 *We have the following proper inclusions and the limit is preserved in each case:*

$$[f_2] \subset [w]_2 \subset [w_2] \subset w_2 \subset (C_2, 2).$$

Proof.   Let $x \in [f_2]$ with $[f_2]\text{-}\lim x = \ell$, say. Then

$$\tau_{pqst}(|x - \ell E|) \longrightarrow 0 \text{ as } p, q \longrightarrow \infty, \text{ uniformly in } s, t.$$

This implies that

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(|x - \ell E|) \longrightarrow 0 \text{ as } p, q \longrightarrow \infty, \text{ uniformly in } s, t.$$

This proves that $x \in [w]_2$ and $[f_2]\text{-}\lim x = [w]_2\text{-}\lim x = \ell$.

Since

$$\frac{1}{(m+1)(n+1)} \left| \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x - \ell E) \right| \leq \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{pqst}(x - \ell E)|$$

$$\leq \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(|x - \ell E|)$$

this implies that $[w]_2 \subset [w_2] \subset w_2$ and

$$[w]_2\text{-}\lim x = [w_2]\text{-}\lim x = w_2\text{-}\lim x = \ell.$$

Since

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x - \ell E)$$

converges uniformly in s, t as $m, n \longrightarrow \infty$ implies the convergence for $s = 0 = t$. It follows that $w_2 \subset (C_2, 2)$ and $w_2\text{-}\lim x = (C_2, 2)\text{-}\lim x = \ell$.

This completes the proof of the theorem. $\qquad\qquad\square$

Theorem 4.2 *We have the following proper inclusions*

$$[f_2] \subset [w]_2 \cap \ell_2^\infty \subset [w_2] \cap \ell_2^\infty \subset f_2.$$

Proof. If $x = (x_{jk})$ then

$$\phi(x) \leq \psi(x) \leq \theta(x) \leq \eta(x).$$

By sublinearity properties of these functionals, for $x = (x_{jk}) \in \ell_2^\infty$, we have

$$-\eta(-x) \leq -\theta(-x) \leq -\psi(-x) \leq -\phi(-x) \leq \phi(x) \leq \psi(x) \leq \theta(x) \leq \eta(x).$$

Hence

$$\eta(x) = -\eta(-x)$$
$$\implies \theta(x) = -\theta(-x)$$
$$\implies \psi(x) = -\psi(-x)$$
$$\implies \phi(x) = -\phi(-x). \qquad\qquad (4.2.1)$$

It is easy to see that

$$\{x = (x_{jk}) \in \ell_2^\infty : \eta(x) = -\eta(-x)\} = [f_2].$$

we have already proved in Theorem 3.2 that

$$\{x = (x_{jk}) \in \ell_2^\infty : \theta(x) = -\theta(-x)\} = \{x = (x_{jk}) : \theta(x - \ell E) = 0\} = [w]_2 \cap \ell_2^\infty$$

$$\{x = (x_{jk}) \in \ell_2^\infty : \psi(x) = -\psi(-x)\} = \{x = (x_{jk}) : \psi(x - \ell E) = 0\} = [w_2] \cap \ell_2^\infty.$$

Also from Theorem 3.1 we have

$$\{x = (x_{jk}) \in \ell_2^\infty : \phi(x) = -\phi(-x)\} = w_2 \cap \ell_2^\infty = f_2.$$

Hence the result follows from (3.2.1). $\qquad\qquad\square$

Example 4.3 $[w]_2 \cap l_\infty \subsetneqq [w_2] \cap l_\infty$ *(or* $[w]_2 \subsetneqq [w_2]$*).*

Let $x = (x_{jk})$ *be defined by*

$$x_{jk} = (-1)^k \ \text{for all } j,$$

that is

$$\begin{pmatrix} -1 & 1 & -1 & 1 & \cdot & \cdot & \cdot \\ & & & & & & \\ -1 & 1 & -1 & 1 & \cdot & \cdot & \cdot \\ & & & & & & \\ -1 & 1 & -1 & 1 & \cdot & \cdot & \cdot \\ & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$|\tau_{pqst}(x - o)| = \left| \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+s,k+t} \right|$$

$$\leq \frac{q+1}{(p+1)(q+1)} = \frac{1}{p+1} \qquad \text{uniformly for } s, t$$

Hence

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{pqst}(x - o)| \leq \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \frac{1}{p+1}$$

$$= \frac{1}{(n+1)(m+1)} \sum_{p=0}^{m} \frac{n+1}{p+1}$$

$$= \frac{1}{(m+1)} \sum_{p=0}^{m} \frac{1}{p+1} \qquad \to 0 \quad as \ m \to \infty,$$

i.e. $x = (x_{jk}) \in [w_2] \cap l_\infty$ *and hence by Theorem 4.2,* $x \in f_2$*. But* $x \notin [w]_2 \cap l_\infty$ *and hence* $x \notin [f_2]$*.*

Theorem 4.4 $[w_2]$*-*$\lim x = \ell$ *if and only if*

(i) w_2*-*$\lim x = \ell$*;*

(ii) $\frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |T_1(m, n, s, t) - \ell| \longrightarrow 0 \ (u, v \longrightarrow \infty)$ *uniformly in* s, t*;*

(iii) $\frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |T_2(m, n, s, t) - \ell| \longrightarrow 0 \ (u, v \longrightarrow \infty)$ *uniformly in* s, t*;*

(iv) $\frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |\tau_{mnst} + d_{mnst} - T_1(m, n, s, t) - T_2(m, n, s, t)| \longrightarrow 0 \ (u, v \longrightarrow \infty)$ *uniformly in* s, t*;*

where

$$d_{m,n,s,t} = d_{m,n,s,t}(x) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x),$$

$$d_{0,0,s,t}(x) = \tau_{0,0,s,t} = x_{s,t}, \; d_{-1,0,s,t}(x) = \tau_{-1,0,s,t}(x) = x_{s-1,t},$$

$$d_{0,-1,s,t}(x) = \tau_{0,-1,s,t}(x) = x_{s,t-1}, \; d_{-1,-1,s,t}(x) = \tau_{-1,-1,s,t}(x) = x_{s-1,t-1},$$

$$T_1(m,n,s,t) = \frac{1}{(m+1)} \sum_{p=0}^{m} \tau_{pnst} \; \text{ and } \; T_2(m,n,s,t) = \frac{1}{(n+1)} \sum_{q=0}^{n} \tau_{mqst}.$$

Proof. Let $[w_2]\text{-}\lim x = \ell$. Then obviously $w_2\text{-}\lim x = \ell$. From Remark 2.1, (ii) and (iii) follow immediately. Now

$$\frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |\tau_{mnst} + d_{mnst} - T_1(m,n,s,t) - T_2(m,n,s,t)|$$

$$= \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |\tau_{mnst} - \ell + d_{mnst} - \ell - T_1(m,n,s,t) + \ell - T_2(m,n,s,t) + \ell|$$

$$\leq \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} (|\tau_{mnst} - \ell| + |d_{mnst} - \ell| + |T_1(m,n,s,t) - \ell| + |T_2(m,n,s,t) - \ell|)$$

$\longrightarrow 0$ as $u, v \longrightarrow \infty$, uniformly in s, t; since

(a) $[w_2]\text{-}\lim x = \ell$ implies that the first sum tends to zero;

(b) (ii) and (iii) imply that third and fourth sums tend to zero;

(c) (i) implies that $d_{mnst} \longrightarrow \ell \; (m, n \longrightarrow \infty)$ uniformly in s, t; and so the second sum tends to zero.

Conversely, suppose that the conditions hold. Now

$$\frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |\tau_{mnst} - \ell|$$

$$\leq \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |\tau_{mnst} + d_{mnst} - T_1(m,n,s,t) - T_2(m,n,s,t)| + \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |d_{mnst} - \ell|$$

$$+ \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |T_1(m,n,s,t) - \ell| + \frac{1}{uv} \sum_{m=1}^{u} \sum_{n=1}^{v} |T_2(m,n,s,t) - \ell|$$

$\longrightarrow 0$ as $u, v \longrightarrow \infty$, uniformly in s, t.

This completes the proof of the theorem. □

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