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C^*-convexity and C^*-faces in $*$-rings

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Abstract

Existence of rich algebraic, geometric and topological structures on self-adjoint operator algebras raises the general question that, for a particular theorem which of these structures have made the result work. The present paper is an effort toward the answer of this question, by investigating the role of algebraic structure in the subject of C^*-convexity.

In this paper, we extend the notions of C^*-convexity, C^*-extreme point and C^*-face to $*$-rings and we study some of their properties.

We introduce the notion of C^*-convex map on C^*-convex subsets of a $*$-ring. Moreover we identify optimal points of some unital $*$-homomorphisms on some C^*-convex sets.

Key Words: $*$-ring, C^*-convexity, C^*-extreme point, C^*-convex map, C^*-face

1. Introduction

The term non-commutative convexity refers to any one of the various forms of convexity in which convex coefficients need not commute among themselves. Formal study of C^*-convexity as a form of non-commutative convexity, was initiated by Loebl and Paulsen in [12], where the notion of C^*-extreme point, as a non-commutative analog of extreme point was also studied. However it was not determined there whether C^*-extremeness is distinct from linear extremeness. This distinction was shown in [11] by Hopenwasser, Moore and Paulsen. The later group also obtained geometrical and algebraic characterizations of these sets. Farenick in [5] developed a Caratheodory-type theorem for C^*-convex hulles of compact sets of matrices and applied it to the theory of matricial ranges. It was conjectured in [12] that a variant of the Krein-Milman theorem should hold for compact C^*-convex sets. For subsets of M_n such a theorem was established by Morenz [16] using some previous work of Farenick and Morenz (see [5], [6] and [8]). In [16] Morenz extended the notion of face from linear convexity to C^*-face on the C^*-convex subsets of a C^*-algebra. Farenick and Morenz studied C^*-extreme points of the generalised state spaces $S_H(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ in [9]. In [10] Farenick and Zhou continued this work by providing a precise description of C^*-extreme points of $S_H(\mathcal{A})$ for a finite dimensional Hilbert space H. In [13] Magajna extended the notion of C^*-convexity to operator modules and proved some separation theorems. For every element a in a von Neumann algebra [respectively a C^*-algebra] $\mathcal{A}$, Magajna identified all normal elements in the w^*-closure [respectively the norm closure] of the C^*-convex

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hull of a in [14]. Also he proved the existence of C^*-extreme points in the w^*-compact C^*-convex subsets of a von Neumann algebra. But these extreme points were not sufficient to produce the original C^*-convex set. So he defined a special kind of C^*-extreme points which he called $\mathcal{R}$-extreme points for the unital C^*-algebra $\mathcal{R}$ ([15]) and used them to prove a Krein-Milman type theorem for hyperfinite factors (and in particular for $B(H)$ where H is a separable Hilbert space) in [15].

Our main motivation for the present paper and [3] is the following general question. Operator algebras are equipped with rich algebraic, geometric and topological structures such that one naturally asks: which of these structures have made a particular theorem work. In the algebraic direction this question has led to evolution of the algebraic theory of operator algebras. See [1] and [2] for fundamentals and history of this ever growing subject. In [3] we studied matricial range from algebraic point of view. Here we investigate the role of algebraic structure in the subject of C^*-convexity. Indeed we define the notions of C^*-convexity, C^*-extreme point, C^*-face and C^*-convex map in $*$-rings and investigate some of their properties.

Remainder of this paper is organized as follows. In section 2 we define the notions of C^*-convexity and C^*-extreme point in $*$-rings. Some illustrative examples of C^*-convex subsets of $*$-rings are discussed. In section 3 we extend the notion of C^*-face to $*$-rings and we study some of their properties. In section 4 we introduce the notion of C^*-convex map on C^*-convex subsets of a $*$-ring. Then we prove the correspondence between C^*-convex maps on a $*$-ring $\mathcal{R}$ and diag-C^*-convex subsets of $\mathcal{R} \oplus \mathcal{R}$. Also, we identify some C^*-convex subsets of $*$-rings by applying C^*-convex maps. Moreover we identify optimal points of some unital $*$-homomorphisms on some C^*-convex sets.

2. C^*-convexity in $*$-rings

Throughout $\mathcal{R}$ is a unital $*$-ring, that is, a ring with an involution which has an identity element.

Definition 2.1 *A subset K of $\mathcal{R}$ is called C^*-convex, if*

$$\sum_{i=1}^{n} a_i^* x_i a_i \in K,$$

whenever $x_i \in K$, $a_i \in \mathcal{R}$ for all i and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.

The C^*-convex hull of a subset $N \subseteq \mathcal{R}$ is the smallest C^*-convex set containing N and is denoted by C^*-$Co(N)$. Indeed, C^*-$Co(N)$ is the intersection of all C^*-convex subsets of $\mathcal{R}$ containing N.

An element x in $\mathcal{R}$ is called positive, written $x \geq 0$, if $x = y_1^* y_1 + \cdots + y_n^* y_n$ for some $y_1, \cdots, y_n \in \mathcal{R}$. For a pair of self-adjoint elements $x, y \in \mathcal{R}$ we define $x \leq y$ if $y - x \geq 0$.

Example 2.2 *The following sets are C^*-convex.*

1. $\mathcal{R}^+ = \{x \in \mathcal{R} : x \geq 0\}$.
2. $\mathcal{P} = \{x \in \mathcal{R} : 0 \leq x \leq 1_{\mathcal{R}}\}$.
3. $\mathcal{R}_{sa} = \{x \in \mathcal{R} : x^* = x\}$.
4. $\{x\}$ *when $x \in Z(\mathcal{R})$, where $Z(\mathcal{R})$ is the center of $\mathcal{R}$.*

5. *Every two-sided ideal in* $\mathcal{R}$*.*

Definition 2.3 *For* $x, y \in \mathcal{R}$*, the segment connecting* x *and* y *is defined to be the set*

$$\{\sum_{i=1}^{n} a_i^* x a_i + \sum_{j=1}^{m} b_j^* y b_j \mid \sum_{i=1}^{n} a_i^* a_i + \sum_{j=1}^{m} b_j^* b_j = 1_{\mathcal{R}}\}.$$

For instance, the segment connecting 0 *and* $1_{\mathcal{R}}$ *is equal to the set*

$$\mathcal{P} = \{x \in \mathcal{R} : 0 \leq x \leq 1_{\mathcal{R}}\}.$$

We recall the following definition from [1].

Definition 2.4 *An element* x *in* $\mathcal{R}$ *is called bounded whenever there exists a positive integer* k *such that* $x^* x \leq k1_{\mathcal{R}}$*. The set of all bounded elements of* $\mathcal{R}$ *is denoted by* $\mathcal{R}_0$*.*

Remark 2.5 Unlike linear convexity subspaces, subalgebras are not necessarily C^*-convex. To see this, let $\mathcal{R}$ be a unital $*$-algebra over $\mathbb{C}$ and $P = \begin{pmatrix} 1_{\mathcal{R}} & 0 \\ 0 & 0 \end{pmatrix}$, then $P\mathcal{R}P = \left\{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} : x \in \mathcal{R} \right\}$ is a $*$-subalgebra of $\mathcal{R}$ which is not C^*-convex. To see this let a and b be non zero distinct elements of $\mathcal{R}$. Then the C^*-convex combination

$$\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

is not in $P\mathcal{R}P$. However $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in P\mathcal{R}P$.

The first part of the following proposition is an extension of [12, Lemma 12].

Proposition 2.6 *(i) For* $x, y \in \mathcal{R}$*, the segment connecting* x *and* y *is a* C^**-convex set that contains both of* x *and* y*.*

(ii) Suppose $n1_{\mathcal{R}}$ *is invertible in* $\mathcal{R}$ *for every positive integer* n*. Then* $\mathcal{R}_0$ *is a* C^**-convex* $*$*-subring of* $\mathcal{R}$*.*

Proof. (i) Let $x_1, \ldots, x_n$ be elements of the segment connecting x and y. Then

$$x_k = \sum_{i=1}^{m_k} (a_i^k)^* x a_i^k + \sum_{j=1}^{n_k} (b_j^k)^* y b_j^k,$$

where

$$\sum_{i=1}^{m_k} (a_i^k)^* a_i^k + \sum_{j=1}^{n_k} (b_j^k)^* b_j^k = 1_{\mathcal{R}}.$$

If $\sum_{k=1}^{n} t_k^* t_k = 1_{\mathcal{R}}$, then

$$\sum_{k=1}^{n} t_k^* x_k t_k = \sum_{k=1}^{n} t_k^* [\sum_{i=1}^{m_k} (a_i^k)^* x a_i^k + \sum_{j=1}^{n_k} (b_j^k)^* y b_j^k] t_k$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{m_k} t_k^*(a_i^k)^* x a_i^k t_k + \sum_{k=1}^{n}\sum_{j=1}^{n_k} t_k^*(b_j^k)^* y b_j^k t_k.$$

By rearranging the coefficients, we have

$$\sum_{k=1}^{n}\sum_{i=1}^{m_k} t_k^*(a_i^k)^* a_i^k t_k + \sum_{k=1}^{n}\sum_{j=1}^{n_k} t_k^*(b_j^k)^* b_j^k t_k = \sum_{k=1}^{n} t_k^*[\sum_{i=1}^{m_k}(a_i^k)^* a_i^k + \sum_{j=1}^{n_k}(b_j^k)^* b_j^k] t_k$$

$$= \sum_{k=1}^{n} t_k^*(1_{\mathcal{R}})t_k = 1_{\mathcal{R}}.$$

So the segment connecting x and y is C^*-convex. Clearly x and y belong to this segment.

(ii) $\mathcal{R}_0$ is a $*$-subring of $\mathcal{R}$ by [1, proposition 1, page 243]. So it is enough to show that $\mathcal{R}_0$ is closed under C^*-convex combinations. Suppose $a \in \mathcal{R}, a^*a \leq 1_{\mathcal{R}}$, and $x \in \mathcal{R}_0$. Then $a \in \mathcal{R}_0$, and hence $a^* \in \mathcal{R}_0$. So $aa^* \leq m1_{\mathcal{R}}$ for some $m \in \mathbb{N}$. On the other hand the assumption $x \in \mathcal{R}_0$ implies that $x^*x \leq k1_{\mathcal{R}}$ for some $k \in \mathbb{N}$. Thus

$$\begin{aligned}(a^*xa)^*(a^*xa) &= (a^*x^*a)(a^*xa) = a^*x^*(aa^*)xa\\ &\leq a^*x^*(m1_{\mathcal{R}})xa = ma^*(x^*x)a\\ &\leq ma^*(k1_{\mathcal{R}})a = mk(a^*a)\\ &\leq mk1_{\mathcal{R}}.\end{aligned}$$

Therefore $a^*xa \in \mathcal{R}_0$, and hence $\mathcal{R}_0$ is C^*-convex as it is a ring. □

Example 2.7 *Every unital $*$-algebra on $\mathbb{C}$ or $\mathbb{R}$ or $\mathbb{Q}$ satisfies the conditions of the above proposition.*

Definition 2.8 *If K is a C^*-convex subset of $\mathcal{R}$, then $x \in K$ is called a C^*-extreme point for K if the condition*

$$x = \sum_{i=1}^{n} a_i^* x_i a_i, \ \sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}, x_i \in K, a_i \textit{ is invertible in } \mathcal{R}, n \in \mathbb{N} \quad (1)$$

implies that all x_i are unitarily equivalent to x in $\mathcal{R}$, that is, there exist unitaries $u_i \in \mathcal{R}$ such that $x_i = u_i^* x u_i$ for all i.

The set of all C^*-extreme points of K is denoted by C^*-$ext(K)$.

If condition (1) holds, then we say that x is a proper C^*-convex combination of $x_1, \ldots, x_n$.

Remark 2.9 (i) If x is a C^*-extreme point of the C^*-convex set $K \subseteq \mathcal{R}$ then $-x$ and x^* are C^*-extreme points of $-K$ and K^*, respectively.

(ii) Let K be a C^*-convex subset of $\mathcal{R}$. Then K is a C^*-convex subset of $\mathcal{R}'$ for every $*$-subring $\mathcal{R}'$ of $\mathcal{R}$ containing $1_{\mathcal{R}}$ such that $K \subseteq \mathcal{R}'$.

(iii) C^*-$Co(K^*) = (C^*$-$Co(K))^*$. This is immediate from the identity

$$\sum_{i=1}^{n} a_i^* x_i^* a_i = (\sum_{i=1}^{n} a_i^* x_i a_i)^*.$$

Definition 2.10 *Let K be a C^*-convex subset of $\mathcal{R}$ and $x \in \mathcal{R}$. Then the set of all elements of K which are unitarily equivalent to x is called the unitary orbit of x and is denoted by $U(x)$, that is, $U(x) = \{u^*xu : u^*u = uu^* = 1_{\mathcal{R}}\}$. Every C^*-convex set contains the unitary orbits of its elements.*

Example 2.11 *The segment $[0, a] = \{x : 0 \leq x \leq a\}$ is not, in general, C^*-convex. For example, if we consider the $*$-ring of all 2×2 complex matrices with usual involution, and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $[0, A]$ is not C^*-convex since $B = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is unitarily equivalent to A, but $B \nleq A$.*

In the next theorem we extend some basic facts of classical convexity to C^*-convexity.

Theorem 2.12 *Let K be a C^*-convex subset of $\mathcal{R}$.*

(i) If $x \in$ C^*-$ext(K)$, then $U(x) \subseteq C^*$-$ext(K)$.

(ii)$K \setminus U(x)$ is a C^*-convex subset of $\mathcal{R}$ if and only if for every finite subset $E = \{x_1, ..., x_n\}$ of K, the identity $x \in C^*$-$Co(E)$ implies that $x \sim x_i$ for some i $(1 \leq i \leq n)$.

Proof. (i) Let $y \in U(x)$. Then there exists a unitary $u \in \mathcal{R}$ such that $y = u^*xu$. If $y = \sum_{i=1}^{n} a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in K$, then

$$x = uyu^* = u(\sum_{i=1}^{n} a_i^* x_i a_i)u^* = \sum_{i=1}^{n}(ua_i^*)x_i(a_iu^*) = \sum_{i=1}^{n}(a_iu^*)^* x_i(a_iu^*)$$

and

$$\sum_{i=1}^{n}(a_iu^*)^*(a_iu^*) = \sum_{i=1}^{n} ua_i^*a_iu^* = u(\sum_{i=1}^{n} a_i^*a_i)u^* = uu^* = 1_{\mathcal{R}}.$$

But $x \in C^*$-$ext(K)$. So $x \sim x_i$ for all i $(1 \leq i \leq n)$, and since $y \sim x$ it follows that $y \sim x_i$ for $i = 1, \ldots, n$. Therefore $y \in C^*$-$ext(K)$.

(ii) Let $B = K \setminus U(x)$ be a C^*-convex subset of $\mathcal{R}$ and $x = \sum_{i=1}^{n} a_i^* x_i a_i$ be a C^*-convex combination of elements $x_i \in K$ such that $x \nsim x_i$ for all i. Then $x_i \in B$ for all i. Hence $x \in B$, since B is a C^*-convex set. Thus $x \notin U(x)$ which is a contradiction.

Conversely, suppose whenever x is written as a C^*-convex combination of elements $x_i \in K$, then necessarily $x \sim x_i$ for some i $(1 \leq i \leq n)$. Let $y = \sum_{i=1}^{n} a_i^* y_i a_i$ be a C^*-convex combination of elements $y_i \in B$. We must show that $y \in B$. since $y_i \in K$ for all i, and K is C^*-convex, then $y \in K$. If $y \in U(x)$ then there exists a unitary $u \in \mathcal{R}$ such that $y = u^*xu$, and hence

$$x = uyu^* = u(\sum_{i=1}^{n} a_i^* y_i a_i)u^* = \sum_{i=1}^{n} ua_i^* y_i a_i u^*.$$

So x is a C^*-convex combination of elements $y_i \in K$, and by assumption $x \sim y_i$ for some i which is a contradiction, as $y_i \in B$. So $y \notin U(x)$ and hence $y \in B$. Therefore B is a C^*-convex set. □

Proposition 2.13 *Assume that K_1 and K_2 are C^*-convex subsets of $*$-rings $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Then $K_1 \oplus K_2$ with pointwise operations is a C^*-convex subset of $\mathcal{R}_1 \oplus \mathcal{R}_2$ and*

$$C^* - ext(K_1) \oplus C^* - ext(K_2) \subseteq C^* - ext(K_1 \oplus K_2).$$

If for every positive integer n, $n1_{\mathcal{R}_1}$ and $n1_{\mathcal{R}_2}$ have invertible positive square roots, then the reverse inclusion holds too. (In particular, it holds when $\mathcal{R}_1$ and $\mathcal{R}_2$ are real or complex $*$-algebras.)

Proof. Suppose $(x_i, y_i) \in K_1 \oplus K_2$, $(a_i, b_i) \in \mathcal{R}_1 \oplus \mathcal{R}_2$, and $\sum_{i=1}^n (a_i, b_i)^*(a_i, b_i) = (1_{\mathcal{R}_1}, 1_{\mathcal{R}_2})$. We must show that $\sum_{i=1}^n (a_i, b_i)^*(x_i, y_i)(a_i, b_i) \in K_1 \oplus K_2$ that is $(\sum_{i=1}^n a_i^* x_i a_i, \sum_{i=1}^n b_i^* y_i b_i) \in K_1 \oplus K_2$. Since $\sum_{i=1}^n (a_i, b_i)^*(a_i, b_i) = (1_{\mathcal{R}_1}, 1_{\mathcal{R}_2})$, then we have $(\sum_{i=1}^n a_i^* a_i, \sum_{i=1}^n b_i^* b_i) = (1_{\mathcal{R}_1}, 1_{\mathcal{R}_2})$. Hence $\sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}_1}$ and $\sum_{i=1}^n b_i^* b_i = 1_{\mathcal{R}_2}$. Since K_1 and K_2 are C^*-convex sets in $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, then $\sum_{i=1}^n a_i^* x_i a_i \in K_1$ and $\sum_{i=1}^n b_i^* y_i b_i \in K_2$.

To prove the last part we assume that $x \in C^*\text{-}ext(K_1)$, $y \in C^*\text{-}ext(K_2)$ and $(x, y) = \sum_{i=1}^n (a_i, b_i)^*(x_i, y_i)$ (a_i, b_i) is a proper C^*-convex combination of elements $(x_i, y_i) \in K_1 \oplus K_2$. So

$$x = \sum_{i=1}^n a_i^* x_i a_i, \sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}_1}, y = \sum_{i=1}^n b_i^* y_i b_i, \sum_{i=1}^n b_i^* b_i = 1_{\mathcal{R}_2}$$

and a_i and b_i are invertible in $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively. Thus $x \sim x_i$ and $y \sim y_i$ $(i = 1, 2, \ldots, n)$ and hence $(x, y) \sim (x_i, y_i)$ $(i = 1, 2, \ldots, n)$. Therefore

$$(x, y) \in C^* - ext(K_1 \oplus K_2).$$

Conversely suppose $(x, y) \in C^*\text{-}ext(K_1 \oplus K_2)$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ is a C^*-convex combination of elements $x_i \in K_1$. Then

$$\begin{aligned}(x, y) &= (\sum_{i=1}^n a_i^* x_i a_i, \sum_{i=1}^n ((n1_{\mathcal{R}_2})^{-1/2})^* y (n1_{\mathcal{R}_2})^{-1/2}) \\ &= \sum_{i=1}^n (a_i, (n1_{\mathcal{R}_2})^{-1/2})^* (x_i, y)(a_i, (n1_{\mathcal{R}_2})^{-1/2})\end{aligned}$$

and

$$\sum_{i=1}^n (a_i, (n1_{\mathcal{R}_2})^{-1/2})^* (a_i, (n1_{\mathcal{R}_2})^{-1/2}) = (\sum_{i=1}^n a_i^* a_i, \sum_{i=1}^n (n1_{\mathcal{R}_2})^{-1}) = (1_{\mathcal{R}_1}, 1_{\mathcal{R}_2}).$$

Thus $(x, y) \sim (x_i, y)$ and hence $x \sim x_i$ for all i. Therefore $x \in C^*\text{-}ext(K_1)$. Similarly, $y \in C^*\text{-}ext(K_2)$. □

Recall that a Rickart $*$-ring is a $*$-ring $\mathcal{R}$ such that, for each $x \in \mathcal{R}$ there is a projection g which generates the right annihilator $ran(\{x\})$. (Note that such a projection is unique.)

Let $\mathcal{R}$ be a Rickart $*$-ring and $x \in \mathcal{R}$. Then there exists a unique projection e such that $(1) xe = x$ and $(2) xy = 0$ if and only if $ey = 0$. Similarly, there exists a unique projection f such that $(3) fx = x$ and $(4) yx = 0$ if and only if $yf = 0$. Explicitly, $ran(\{x\}) = (1_{\mathcal{R}} - e)\mathcal{R}$ and $lan(\{x\}) = \mathcal{R}(1_{\mathcal{R}} - f)$.

Definition 2.14 *Using the above notations, we write*

$$e = RP(x), \quad f = LP(x).$$

We call e *and* f *the right projection and the left projection of* x *respectively.*

Definition 2.15 $\mathcal{R}$ *is said to satisfy the [unique] positive square root axiom briefly, the PSR-axiom [UPSR-axiom] if, for every* $x \geq 0$, *there exists a [unique] element* y *such that* $y \geq 0$ *and* $x = y^2$.

The following theorem is a generalization of [12, Proposition 27] to Rickart $*$-rings.

Theorem 2.16 *Let* $\mathcal{R}$ *be a Rickart* $*$*-ring such that* $2(1_{\mathcal{R}})$ *is invertible and has a positive square root. Then* $T = \{x \in \mathcal{R} : \ -1_{\mathcal{R}} \leq x \leq 1_{\mathcal{R}}\}$ *is a* C^**-convex set, and the positive and negative* C^**-extreme points of* T *belong to* $\{2P - 1_{\mathcal{R}} : P \geq \ 0 \ \text{ is a projection}\}$.

Proof. It is easy to see that T is C^*-convex. Let $x \in T$ be positive (that is $0 \leq x \leq 1_{\mathcal{R}}$) and $f = LP(x)$. Then $x = fx$ and since $x = x^*$ we have $x = xf$. So $x = fxf$. On the other hand, $0 \leq f \leq 1_{\mathcal{R}}$, as f is a projection. Suppose $y = 2x - 2f + 1_{\mathcal{R}}$. We show that $y \in T$. We have

$$0 \leq x \leq 1_{\mathcal{R}} \Rightarrow -1_{\mathcal{R}} \leq x - 1_{\mathcal{R}} \leq 0 \Rightarrow -f \leq f(x - 1_{\mathcal{R}})f \leq 0 \Rightarrow$$

$$1_{\mathcal{R}} - 2f \leq 2f(x - 1_{\mathcal{R}})f + 1_{\mathcal{R}} \leq 1_{\mathcal{R}} \Rightarrow 1_{\mathcal{R}} - 2f \leq y \leq 1_{\mathcal{R}}.$$

Furthermore $-1_{\mathcal{R}} \leq 1_{\mathcal{R}} - 2f$. Therefore $y = 2x - 2f + 1_{\mathcal{R}} \in T$. Now if $t = (2(1_{\mathcal{R}}))^{-1/2}$, then $x = tyt + t(2f - 1_{\mathcal{R}})t$. If x is C^*-extreme in T, then x is unitarily equivalent to $2f - 1_{\mathcal{R}}$. So $x = 2P - 1_{\mathcal{R}}$ where $P \geq 0$ is a projection. In the case that $-1_{\mathcal{R}} \leq x \leq 0$, the proof is similar. □

Remark 2.17 Suppose K is a C^*-convex subset of $\mathcal{R}$, $0 \in K$, and $\mathcal{R}$ satisfies the positive square root axiom. Then for all $x \in K$ and $a \in \mathcal{R}$ which $a^*a \leq 1_{\mathcal{R}}$, we have $a^*xa \in K$.

3. C^*-faces in $*$-rings

Morenz extended the notion of face from linear convexity to C^*-face of C^*-convex subsets of a C^*-algebra [16, pages 1015–1017]. In this section we extend this concept to C^*-convex subsets of a $*$-ring.

Definition 3.1 *A* C^**-face* $\mathcal{F}$ *of a* C^**-convex set* $K \subseteq \mathcal{R}$ *is a nonempty subset* $\mathcal{F}$ *of* K *such that if* $x \in \mathcal{F}$ *and* $x = \sum_{i=1}^n a_i^* x_i a_i$ *is a proper* C^**-convex combination of elements* $x_i \in K$, *then necessarily* $x_i \in \mathcal{F}$ *for all* i.

Example 3.2 *(i) Let* K *be a* C^**-convex subset of* $\mathcal{R}$. *Then* K *is a* C^**-face of* K. *Thus* C^**-faces exist.*

(ii) The set of all C^**-extreme points of* K *is a* C^**-face of* K.

Proof. (ii) Suppose $x \in C^*\text{-}ext(K)$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in K$. Then $x \sim x_i$ for all i $(1 \leq i \leq n)$ by definition of C^*-extreme points. Also $C^*\text{-}ext(K)$ is closed under unitary orbit of its elements. So $x_i \in C^*\text{-}ext(K)$ for all i. Therefore $C^*\text{-}ext(K)$ is a C^*-face of K. □

In [16] Morenz introduced the notion of C^*-summand of x in compact C^*-convex subsets of $M_n(\mathbb{C})$ to prove the Krein-Milman theorem. We extend this concept for every element x of C^*-convex subsets of $\mathcal{R}$.

Definition 3.3 *Suppose K is a C^*-convex subset of $\mathcal{R}$ and $x=\sum_{i=1}^{n} a_i^* x_i a_i$ is a C^*-convex combination of elements $x_i \in K$ with $a_i \neq 0$. Then each x_i is called a C^*-summand of x. We denote the set of all C^*-summands of x by C^*-summ(x).*

Note that the restriction $a_i \neq 0$ in the definition of C^*-summ(x) keeps the set of C^*-summands proper.

Proposition 3.4 *Let K be a C^*-convex subset of $\mathcal{R}$ and $x \in K$. Then C^*-summ(x) is a C^*-face of K.*

Proof. Let $y_1 \in C^*$-*summ*(x) and $y_1=\sum_{i=1}^{m} b_i^* z_i b_i$ be a representation of y_1 as a proper C^*-convex combination of elements $z_i \in K$. Then there exist non-zero elements $a_i \in \mathcal{R}$ and $y_i \in K$ $(2 \leq i \leq n)$ such that $x=\sum_{i=1}^{n} a_i^* y_i a_i$ and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$. So

$$x = a_1^* y_1 a_1 + \sum_{i=2}^{n} a_i^* y_i a_i = a_1^*(\sum_{i=1}^{m} b_i^* z_i b_i) a_1 + \sum_{i=2}^{n} a_i^* y_i a_i$$

and hence $x=\sum_{i=1}^{m}(b_i a_1)^* z_i (b_i a_1) + \sum_{i=2}^{n} a_i^* y_i a_i$.

But $b_i a_1 \neq 0$ as $a_1 \neq 0$ and b_i is invertible for all i $(1 \leq i \leq m)$. On the other hand

$$\sum_{i=1}^{m}(b_i a_1)^*(b_i a_1) + \sum_{i=2}^{n} a_i^* a_i = \sum_{i=1}^{m} a_1^* b_i^* b_i a_1 + \sum_{i=2}^{n} a_i^* a_i$$

$$= a_1^*(\sum_{i=1}^{m} b_i^* b_i) a_1 + \sum_{i=2}^{n} a_i^* a_i = a_1^* 1_{\mathcal{R}} a_1 + \sum_{i=2}^{n} a_i^* a_i = \sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}.$$

Thus each z_i is a C^*-summand of x. Therefore C^*-*summ*(x) is a C^*-face of K. □

Since the definition of a C^*-face requires the coefficients to be invertible, and the definition of C^*-*summ*(x) does not, C^*-*summ*(x) need not be the minimal C^*-face containing x.

Remark 3.5 (i) Unlike faces of ordinary convex sets in a $*$-algebra, C^*-faces are not usually C^*-convex, or even convex in a $*$-algebra. For example suppose K is a C^*-convex subset of $\mathcal{R}$ and $x \in K$, then C^*-*summ*(x) is a C^*-face of K which is not convex in general (see [16, Remarks 3.2.2]).

(ii) If $\mathcal{F}$ is a C^*-face and $x \in \mathcal{F}$ then $U(x) \subseteq \mathcal{F}$.

(iii) The intersection of a family of C^*-faces is a C^*-face, provided that it is nonempty.

(iv) Let $\mathcal{R}$ be a topological $*$-ring and K be a C^*-convex subset of $\mathcal{R}$. Then using Zorn's lemma, one can see that every compact C^*-face of K contains a minimal compact C^*-face of K.

(v) Unlike ordinary convexity, $y \in C^*$-*summ*(x) and $z \in C^*$-*summ*(y) does not imply $z \in C^*$-*summ*(x). For example, suppose that $\mathcal{R}$ is a $*$-ring such that $n1_{\mathcal{R}}$ is invertible for each $n \in \mathbb{N}$, $K=\{A \in M_2(\mathcal{R}) : 0 \leq A \leq I_2\}$ and

$$x=\begin{pmatrix} 1_{\mathcal{R}} & 0 \\ 0 & 0 \end{pmatrix}, y=\begin{pmatrix} 1_{\mathcal{R}} & 0 \\ 0 & (2(1_{\mathcal{R}}))^{-1} \end{pmatrix}, z=\begin{pmatrix} (2(1_{\mathcal{R}}))^{-1} & 0 \\ 0 & (4(1_{\mathcal{R}}))^{-1} \end{pmatrix}.$$

Then $y \in C^*$-$summ(x)$ and $z \in C^*$-$summ(y)$, but $z \notin C^*$-$summ(x)$.

Theorem 3.6 *Let $\mathcal{F}$ be a C^*-face of a C^*-convex set K and $x \in \mathcal{F}$. Then*

$$x \in C^* - ext(C^* - Co(\mathcal{F})) \iff x \in C^* - ext(K) \ \ i.e.$$

$$\mathcal{F} \cap C^* - ext(C^* - Co(\mathcal{F})) = \mathcal{F} \cap C^* - ext(K).$$

Proof. Let $x \in \mathcal{F} \cap C^*$-$ext(C^*$-$Co(\mathcal{F}))$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ be a representation of x as a proper C^*-convex combination of elements $x_i \in K$. Then $x_i \in \mathcal{F}$ for each i $(1 \leq i \leq n)$ since $\mathcal{F}$ is a C^*-face of K. So $x_i \in C^*$-$Co(\mathcal{F})$ for each i. But $x \in C^*$-$ext(C^*$-$Co(\mathcal{F}))$ implies that $x \sim x_i$ for each i $(1 \leq i \leq n)$. Therefore $x \in C^*$-$ext(K)$.

Conversely, suppose $x \in \mathcal{F} \cap C^*$-$ext(K)$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in C^*$-$Co(\mathcal{F})$. Since C^*-$Co(\mathcal{F}) \subseteq K$ and $x \in C^*$-$ext(K)$, then $x \sim x_i$ for all i $(1 \leq i \leq n)$. Therefore $x \in C^*$-$ext(C^*$-$Co(\mathcal{F}))$. □

Theorem 3.7 *Suppose K_1 and K_2 are C^*-convex subsets of $\mathcal{R}$ and $\mathcal{F}_1$, $\mathcal{F}_2$ are C^*-faces of K_1 and K_2 respectively. Then,*

(i) $\mathcal{F}_1 \cap \mathcal{F}_2$ is a C^*-face of $K_1 \cap K_2$ provided that $\mathcal{F}_1 \cap \mathcal{F}_2 \neq \emptyset$.

(ii) If $K_1 \subseteq K_2$ then $\mathcal{F}_2 \cap K_1$ is a C^*-face of K_1 provided that it is nonempty.

(iii) If $\mathcal{F} \subset \mathcal{F}_1$ and $\mathcal{F}$ is a C^*-face of C^*-$Co(\mathcal{F}_1)$, then $\mathcal{F}$ is a C^*-face of K_1.

(iv) If $K_1 \subseteq K_2$ then, $K_1 \cap C^*$-$ext(K_2) \subseteq C^*$-$ext(K_1)$.

Proof. (i) Suppose $x \in \mathcal{F}_1 \cap \mathcal{F}_2$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in K_1 \cap K_2$. Since $x \in \mathcal{F}_1$, $x_i \in K_1$ $(i = 1, \cdots, n)$ and $\mathcal{F}_1$ is a C^*-face of K_1, then $x_i \in \mathcal{F}_1$ for all i $(i = 1, \cdots, n)$. Similarly $x_i \in \mathcal{F}_2$ for all i $(i = 1, \cdots, n)$. Hence $x_i \in \mathcal{F}_1 \cap \mathcal{F}_2$ for $i = 1, \cdots, n$. Therefore $\mathcal{F}_1 \cap \mathcal{F}_2$ is a C^*-face of $K_1 \cap K_2$.

(ii) K_1 is a C^*-face of K_1 and $\mathcal{F}_2$ is a C^*-face of K_2. So by part (i), $K_1 \cap \mathcal{F}$ is a C^*-face of $K_1 \cap K_2 = K_1$, provided that it is nonempty.

(iii) Let $x \in \mathcal{F}$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ be a proper C^*-convex combination of elements $x_i \in K_1$. Since $x \in \mathcal{F}_1$ and $\mathcal{F}_1$ is a C^*-face of K_1, then $x_i \in \mathcal{F}_1$ for all i $(1 \leq i \leq n)$. So $x \in \mathcal{F}$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in \mathcal{F}_1 \subseteq C^*$-$co(\mathcal{F}_1)$. Thus $x_i \in \mathcal{F}$ for all i $(1 \leq i \leq n)$. Therefore $\mathcal{F}$ is a C^*-face of K_1.

(iv) Let $x \in K_1 \cap C^*$-$ext(K_2)$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ be a proper C^*-convex combination of elements $x_i \in K_1$. The inclusion $K_1 \subseteq K_2$ implies that $x = \sum_{i=1}^n a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in K_2$. Since $x \in C^*$-$ext(K_2)$, we conclude that $x \sim x_i$ for all i $(1 \leq i \leq n)$. Thus $x \in C^*$-$ext(K_1)$. □

Corollary 3.8 *(i) Let $\{K_i\}_{i \in I}$ be a collection of compact C^*-convex subsets of a topological $*$-ring $\mathcal{R}$, and $\mathcal{F}_i$ be a compact C^*-face of K_i for each $i \in I$. Then $\cap_{i \in I} \mathcal{F}_i$ is a compact C^*-face of the compact C^*-convex set $\cap_{i \in I} K_i$.*

(ii) Let K_1 and K_2 be C^*-convex subsets of $\mathcal{R}$ such that $K_1 \subseteq K_2$ and $\mathcal{F}_2$ be a C^*-face of K_2 which is contained in K_1. Then $\mathcal{F}_2$ is also a C^*-face of K_1.

Remark 3.9 (i) Note that the inclusion in the part (iv) of the above theorem is proper. For example suppose $\mathcal{R} = \mathbb{C}$ (the ring of all complex numbers with complex conjugation as involution), $K_1 = [0,1]$ and $K_2 = [-1,1]$. It is easy to see that 0 is a C^*-extreme point of K_1 which does not belong to $K_1 \cap C^*\text{-}\ ext(K_2)$.

(ii) Let $\mathcal{F}$ be a C^*-face of the C^*-convex set K in $\mathcal{R}$ such that $\mathcal{P} = \{x \in \mathcal{R} : 0 \leq x \leq 1_{\mathcal{R}}\} \subseteq K$, and $(\mathcal{F} \cap G(\mathcal{P})) \setminus 1_{\mathcal{R}} \neq \emptyset$ (where $G(\mathcal{P})$ is the set of all invertible elements of $\mathcal{P}$) and every element of $\mathcal{P}$ has a positive square root. Then $\mathcal{F}$ contains 0 and $1_{\mathcal{R}}$. To see this let $x \in \mathcal{F} \cap G(\mathcal{P})$ and $x \neq 1_{\mathcal{R}}$. Then

$$x = x^{1/2}\ 1_{\mathcal{R}} x^{1/2} + (1_{\mathcal{R}} - x)^{1/2}\ 0 (1_{\mathcal{R}} - x)^{1/2}$$

is a proper C^*-convex combination of 0 and $1_{\mathcal{R}}$ in K. Also, $x \in \mathcal{F}$ and $\mathcal{F}$ is a C^*-face of K. Therefore $0, 1_{\mathcal{R}} \in \mathcal{F}$.

Proposition 3.10 *Let K be a compact C^*-convex subset of a topological $*$-ring $\mathcal{R}$, and $C^*\text{-}\mathcal{F}(K)$ be the collection of all compact C^*-faces of K which is partially ordered by inclusion. Then $C^*\text{-}\mathcal{F}(K)$ is a complete lattice.*

Proof. Every set of elements of $C^*\text{-}\mathcal{F}(K)$ has a greatest lower bound in the partial ordering (namely the intersection of its elements). Also every subset of $C^*\text{-}\mathcal{F}(K)$ has a least upper bound since the set of all its upper bounds has a greatest lower bound. Thus $C^*\text{-}\mathcal{F}(K)$ is a complete lattice. □

Theorem 3.11 *Suppose K is a non-empty C^*-convex compact set in a topological $*$-ring $\mathcal{R}$, $\varphi : \mathcal{R} \to \mathbb{C}$ is a continuous $*$-homomorphism and $M = \sup_{x \in K} Re(\varphi(x))$. Then the set $\mathcal{F}$ of all $x \in K$ such that $Re(\varphi(x)) = M$ is a compact C^*-face of K.*

Proof. The set $\mathcal{F}$ is non-empty, since compactness of K implies that there is a point $x_0 \in K$ such that $M = Re(\varphi(x_0))$. Since φ is continuous, then $\mathcal{F}$ is closed in K and hence $\mathcal{F}$ is compact. Suppose $x \in \mathcal{F}$ and $x = \sum_{i=1}^n a_i^* x_i a_i$ is a proper C^*-convex combination of elements $x_i \in K$. Since $\sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}}$, then

$$1 = \varphi(1_{\mathcal{R}}) = \varphi(\sum_{i=1}^n a_i^* a_i) = \sum_{i=1}^n \overline{\varphi(a_i)} \varphi(a_i) = \sum_{i=1}^n \mid \varphi(a_i) \mid^2, \qquad (1)$$

Also,

$$M = Re(\varphi(x)) = Re(\varphi(\sum_{i=1}^n a_i^* x_i a_i)) = Re(\sum_{i=1}^n \overline{\varphi(a_i)} \varphi(x_i) \varphi(a_i))$$

$$= Re(\sum_{i=1}^n \mid \varphi(a_i) \mid^2 \varphi(x_i)) = \sum_{i=1}^n \mid \varphi(a_i) \mid^2 Re(\varphi(x_i)).$$

Thus

$$M = \sum_{i=1}^n \mid \varphi(a_i) \mid^2 Re(\varphi(x_i)). \qquad (2)$$

If $x_i \notin \mathcal{F}$ for some i $(1 \leq i \leq n)$, then $Re(\varphi(x_i)) < M$. So (1) and (2) imply that $M < M$ which is a contradiction. Therefore $x_i \in \mathcal{F}$ for all i $(1 \leq i \leq n)$. □

Proof of the following proposition is not difficult and is left to the reader.

Proposition 3.12 *Suppose $\mathcal{R}_1$ and $\mathcal{R}_2$ are $*$-rings and $g:\mathcal{R}_1 \to \mathcal{R}_2$ is a $*$-isomorphism. Then*

(i) K is a C^*-convex subset of $\mathcal{R}_1$ if and only if $g(K)$ is a C^*-convex subset of $\mathcal{R}_2$.

(ii) $\mathcal{F}$ is a C^*-face of a C^*-convex set K if and only if $g(\mathcal{F})$ is a C^*-face of $g(K)$.

4. C^*-convex maps

In this section we introduce the notion of C^*-convex maps on C^*-convex subsets of a $*$-ring. The results of this section are mostly extensions of their analogs from linear convexity.

Definition 4.1 *Let K be a C^*-convex subset of $\mathcal{R}$. We say that a map $f: K \to K$ is C^*-convex if*

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i \qquad (1)$$

where $x_i \in K$, $a_i \in \mathcal{R}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$. If the inequality (1) is strict, then we say that f is strictly C^*-convex. If $-f$ is C^*-convex, we say that f is C^*-concave.

Note that if $\mathcal{R}$ is a $*$-algebra then every C^*-convex (C^*-concave) map is convex (concave) map in the classical sense.

Example 4.2 *The following maps are C^*-convex maps on $\mathcal{R}$, which are not strictly C^*-convex.*

(i) $f(x) = mx$ where $m \in \mathbb{N}$.

(ii) $f(x) = x^*$.

(iii) $f(x) = \alpha x$ where $\alpha \in \mathbb{C}$ and $\mathcal{R}$ is a $*$-algebra.

(iv) $f(x) = \alpha x + b$ where $\alpha \in \mathbb{C}$, $b \in Z(\mathcal{R})$ and $\mathcal{R}$ is a $*$-algebra.

Remark 4.3 Every increasing C^*-convex map of a C^*-convex map, is C^*-convex (Note that $f: K \to K$ is called increasing if $a \leq b$ implies that $f(a) \leq f(b)$).

To see this let f be C^*-convex and g be an increasing C^*-convex map on a C^*-convex set K in $\mathcal{R}$. Then for every C^*-convex combination of elements $x_i \in K$ we have

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i.$$

So

$$g(f(\sum_{i=1}^{n} a_i^* x_i a_i)) \leq g(\sum_{i=1}^{n} a_i^* f(x_i) a_i) \leq \sum_{i=1}^{n} a_i^* g(f(x_i)) a_i.$$

Therefore $g \circ f$ is a C^*-convex map.

Definition 4.4 *The graph of a map $f: \mathcal{R} \to \mathcal{R}$ is the set*

$$\{(x, y) : x \in \mathcal{R}, y = f(x)\} \subseteq \mathcal{R} \oplus \mathcal{R},$$

and the epi-graph of f which we denote by $epi(f)$ is the set

$$\{(x, y) : x \in \mathcal{R}, f(x) \leq y\} \subseteq \mathcal{R} \oplus \mathcal{R}.$$

Definition 4.5 *We say that $K \subseteq \mathcal{R} \oplus \mathcal{R}$ is a diag-C^*-convex subset of $\mathcal{R} \oplus \mathcal{R}$ if K is closed under C^*-convex combinations with diagonal coefficients, that is, $\sum_{i=1}^{n}(a_i, a_i)^*(x_i, y_i)(a_i, a_i) \in K$ whenever $(x_i, y_i) \in K$, $a_i \in \mathcal{R}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$.*

Theorem 4.6 *Let $K \subseteq \mathcal{R}$. A map f on K is C^*-convex if and only if $epi(f)$ is a diag-C^*-convex subset of $\mathcal{R} \oplus \mathcal{R}$.*

Proof. Let f be a C^*-convex map on $K \subseteq \mathcal{R}$ and let $(x_i, y_i) \in epi(f)$, $a_i \in \mathcal{R}$, and $\sum_{i=1}^{n} a_i^* a_i = 1_{\mathcal{R}}$. We must show that $\sum_{i=1}^{n}(a_i, a_i)^*(x_i, y_i)(a_i, a_i)$ belongs to $epi(f)$. Since $(x_i, y_i) \in epi(f)$, then $y_i \geq f(x_i)$, and hence $a_i^* y_i a_i \geq a_i^* f(x_i) a_i$. Thus

$$\sum_{i=1}^{n} a_i^* y_i a_i \geq \sum_{i=1}^{n} a_i^* f(x_i) a_i \geq f(\sum_{i=1}^{n} a_i^* x_i a_i)$$

as f is a C^*-convex map. Therefore

$$(\sum_{i=1}^{n} a_i^* x_i a_i, \sum_{i=1}^{n} a_i^* y_i a_i) \in epi(f).$$

Conversely, suppose that $epi(f)$ is a diag-C^*-convex subset of $\mathcal{R} \oplus \mathcal{R}$ and $\sum_{i=1}^{n} a_i^* x_i a_i$ is a C^*-convex combination of elements $x_i \in K$. We must show that

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i.$$

Our assumption, together with the fact that $(x_i, f(x_i)) \in epi(f)$, implies that

$$\sum_{i=1}^{n}(a_i, a_i)^*(x_i, y_i)(a_i, a_i) \in epi(f)$$

and hence

$$(\sum_{i=1}^{n} a_i^* x_i a_i, \sum_{i=1}^{n} a_i^* f(x_i) a_i) \in epi(f).$$

Therefore

$$f(\sum_{i=1}^{n} a_i^* x_i a_i) \leq \sum_{i=1}^{n} a_i^* f(x_i) a_i.$$

□

Theorem 4.7 *Let $f:\mathcal{R}\to\mathcal{R}$ be a C^*-convex map on a unital $*$-algebra $\mathcal{R}$, and $\alpha\in\mathbb{C}$. Then each of the following sets is a C^*-convex subset of $\mathcal{R}$.*

(i) $K=\{x\in\mathcal{R}\mid f(x)\leq\alpha 1_{\mathcal{R}}\}$.

(ii) $M=\{x\in\mathcal{R}: f(x)\leq x\}$.

A similar result holds when f is C^*-concave and $M=\{x\in\mathcal{R}: f(x)\geq x\}$.

(iii) $g^{-1}(\{\alpha\})$, *where $g:\mathcal{R}\to\mathbb{C}$ is a $*$-homomorphism.*

Proof. (i) Let $x_i\in K$, $a_i\in\mathcal{R}$ and $\sum_{i=1}^n a_i^*a_i=1_{\mathcal{R}}$. Then,

$$f(\sum_{i=1}^n a_i^*x_ia_i)\leq\sum_{i=1}^n a_i^*f(x_i)a_i\leq\sum_{i=1}^n a_i^*\alpha 1_{\mathcal{R}}a_i=\alpha(\sum_{i=1}^n a_i^*a_i)=\alpha(1_{\mathcal{R}}).$$

So $\sum_{i=1}^n a_i^*x_ia_i\in K$ and hence K is a C^*-convex set.

(ii) Let $x_i\in M$, $a_i\in\mathcal{R}$ and $\sum_{i=1}^n a_i^*a_i=1_{\mathcal{R}}$. Since $f(x_i)\leq x_i$ then

$$\sum_{i=1}^n a_i^*f(x_i)a_i\leq\sum_{i=1}^n a_i^*x_ia_i.$$

On the other hand $f(\sum_{i=1}^n a_i^*x_ia_i)\leq\sum_{i=1}^n a_i^*f(x_i)a_i$ since f is a C^*-convex map. So we conclude that

$$f(\sum_{i=1}^n a_i^*x_ia_i)\leq\sum_{i=1}^n a_i^*x_ia_i$$

and hence $\sum_{i=1}^n a_i^*x_ia_i\in M$. Therefore M is a C^*-convex subset of $\mathcal{R}$.

(iii) Suppose $\sum_{i=1}^n a_i^*x_ia_i$ is a C^*-convex combination of elements $x_i\in g^{-1}(\{\alpha\})$. Since $g(x_i)=\alpha$ for each i $(i=1,2,\cdots,n)$ and g is a $*$-homomorphism, then

$$g(\sum_{i=1}^n a_i^*x_ia_i)=\sum_{i=1}^n g(a_i^*)g(x_i)g(a_i)=\sum_{i=1}^n g(\bar{a_i})\alpha g(a_i)=\alpha\sum_{i=1}^n|g(a_i)|^2=\alpha.$$

$$\sum_{i=1}^n|g(a_i)|^2=\sum_{i=1}^n g(\bar{a_i})g(a_i)=\sum_{i=1}^n g(a_i^*)g(a_i)=g(\sum_{i=1}^n a_i^*a_i)=1_{\mathcal{R}}.$$

Hence $\sum_{i=1}^n a_i^*x_ia_i\in g^{-1}(\{\alpha\})$. Therefore $g^{-1}(\{\alpha\})$ is a C^*-convex set in $\mathcal{R}$. □

Theorem 4.8 *Suppose $\mathcal{R}$ is a topological $*$-ring, C^*-ext(K) is closed and S is a compact subset of $\overline{C^*-Co}(C^*$-ext$(K))$ containing C^*-ext(K). Then every continuous unital homomorphism $f:\mathcal{R}\to\mathbb{R}$ attains its maximum and minimum on S at C^*-extreme points of K. Moreover, maximum and minimum of f on S is equal with its maximum and minimum on C^*-ext(K) respectively.*

Proof. Suppose f admits its maximum on S at a point $x\in S$. Then there exists a net $(x_\lambda)\subseteq C^*$-$Co(C^*$-$ext(K))$ such that (x_λ) converges to x. But

$$x_\lambda=\sum_{i=1}^{n(\lambda)} a_{\lambda,i}^*x_{\lambda,i}a_{\lambda,i},$$

where $n(\lambda)$ is a positive integer and $x_{\lambda,i} \in C^*\text{-}ext(K)$ for $i = 1, \ldots, n(\lambda)$ and $a_{\lambda,i} \in \mathcal{R}$ satisfies $\sum_{i=1}^{n(\lambda)} a_{\lambda,i}^* a_{\lambda,i} = 1_{\mathcal{R}}$. Thus

$$\begin{aligned} f(x_\lambda) &= f(\sum_{i=1}^{n(\lambda)} a_{\lambda,i}^* x_{\lambda,i} a_{\lambda,i}) = \sum_{i=1}^{n(\lambda)} f(a_{\lambda,i}^* x_{\lambda,i} a_{\lambda,i}) \\ &= \sum_{i=1}^{n(\lambda)} f(a_{\lambda,i}^*) f(x_{\lambda,i}) f(a_{\lambda,i}) \leq \max_{1\leq i\leq n(\lambda)} f(x_{\lambda,i}) \sum_{i=1}^{n(\lambda)} f(a_{\lambda,i}^*) f(a_{\lambda,i}) \\ &= \max_{1\leq i\leq n(\lambda)} f(x_{\lambda,i}) f(\sum_{i=1}^{n(\lambda)} a_{\lambda,i}^* a_{\lambda,i}) = \max_{1\leq i\leq n(\lambda)} f(x_{\lambda,i}). \end{aligned}$$

So

$$f(x_\lambda) \leq \max_{1\leq i\leq n(\lambda)} f(x_{\lambda,i}) = f(x_{\lambda,i_\lambda}).$$

Therefore,

$$f(x) = f(\lim_{\lambda\to\infty} x_\lambda) = \lim_{\lambda\to\infty} f(x_\lambda) \leq \lim_{\lambda\to\infty} f(x_{\lambda,i_\lambda}) = f(\lim_{\lambda\to\infty} x_{\lambda,i_\lambda}). \tag{1}$$

Since S is compact and $(x_{\lambda,i_\lambda}) \subseteq S$, then $\lim_{\lambda\to\infty} x_{\lambda,i_\lambda} \in S$. On the other hand $f(x)$ is maximal. So (1) implies that $f(x) = f(\lim_{\lambda\to\infty} x_{\lambda,i_\lambda})$. Therefore f takes its maximum at the point $\lim_{\lambda\to\infty} x_{\lambda,i_\lambda}$ which is contained in $C^*\text{-}ext(K)$ (Since $C^*\text{-}ext(K)$ is closed). The statement for the minimum of f can be proved with a similar argument. □

Corollary 4.9 *If $S \subseteq M_n$ is compact, C^*-convex, and the set of all C^*-extreme points of S is closed, then every continuous unital homomorphism $f : M_n \to \mathbb{R}$, attains its maximum and minimum on S at C^*-extreme points of S.*

Proof. Let $K = S$ and use theorem 4.5. of [16]. □

Note that the same conclusion holds everywhere that a Krein-Milman type theorem exists. For example in the generalized state space of a C^*-algebra with bounded-weak topology such a conclusion holds.

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